Dimers and M-Curves. Limit Shapes from Riemann Surfaces

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joint with Nikolai Bobenko and Yuri Suris

SFB TRR 109 Discretization in Geometry and Dynamics

Papers

- Dimers and M-Curves [B-Bobenko-Suris, 2024]
- Dimers and M-Curves. Limit Shapes from Riemann Surfaces [B-Bobenko, 2024+]

Setup

The Dimer Model

- Measure on dimer configurations (perfect matchings):

$$\mathbb{P}(D) = \frac{1}{Z} \prod_{e \in D} \nu(e).$$

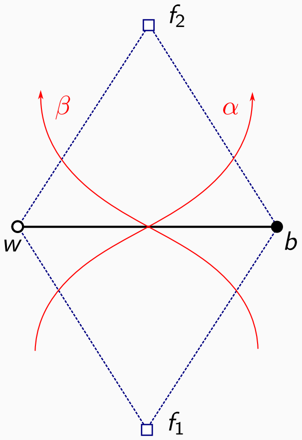
- Physical weights are face weights:

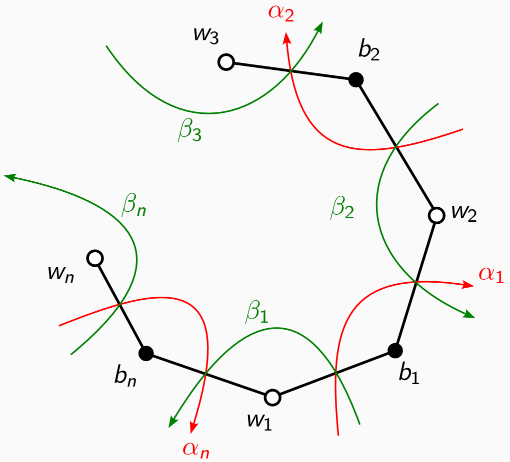
$$W_f = \frac{\nu(e_1)\nu(e_3)\ldots\nu(e_{2n-1})}{\nu(e_2)\nu(e_4)\ldots\nu(e_{2n})}.$$

- Kasteleyn condition:

$$\text{sign}(W_f) = (-1)^{(n+1)}.$$

Quad Graph and Train Tracks





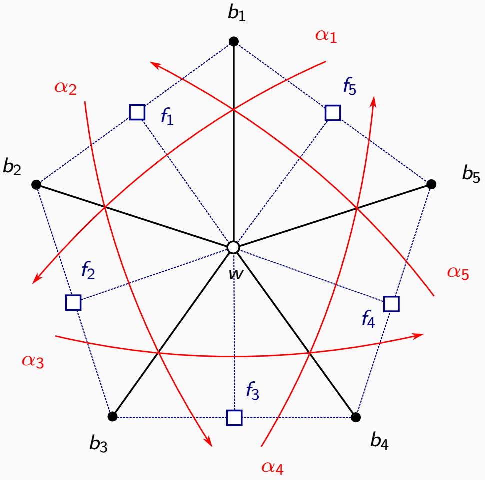
Direct and inverse problems

Planar bipartite periodic graph G.

- **Direct problem**: Weights $\Rightarrow$ Kasteleyn matrix $\Rightarrow$ Spectral curve
- **Inverse problem**: Spectral curve (Riemann surface) $\Rightarrow$ Weights

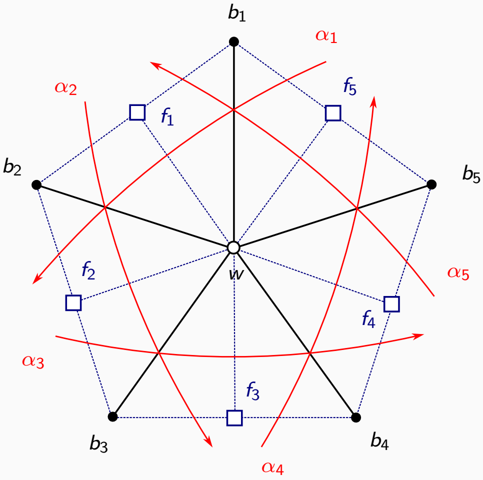
Inverse problem. Fock weights

- **Data:** Compact Riemann surface $\mathcal{R}$, train track parameters $\alpha_i \in \mathcal{R}$.



Inverse problem. Fock weights

- **Data:** Compact Riemann surface $\mathcal{R}$, train track parameters $\alpha_i \in \mathcal{R}$.
- Meromorphic function ψ on $\mathcal{R}$ from integrable systems theory (BA-function). Function $\psi_b : \mathcal{R} \to \mathbb{C}$ on every black vertex b.
- ψ picks up a zero or a pole at $\alpha \in \mathcal{R}$ whenever crossing a train track.



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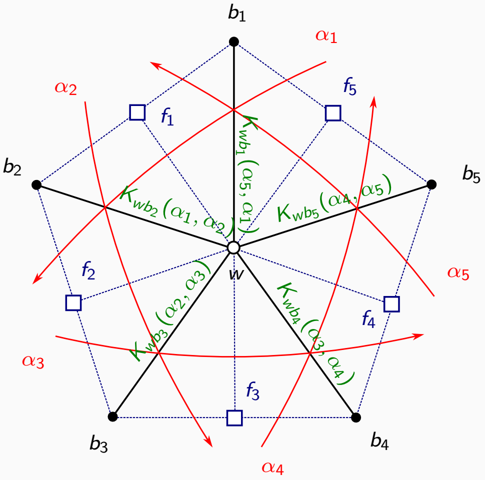
- ψ picks up a zero or a pole at $\alpha \in \mathcal{R}$ whenever crossing a train track.
- Dirac equation

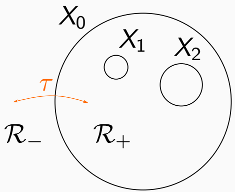
$$\sum_{k=1}^{n} K_{wb_k}(\alpha_{k-1}, \alpha_k)\psi_{b_k}(P) = 0,$$

- Fock weights K_{wb_k} [Fock '15]



Reality conditions

- $\mathcal{R}$ M-curve with antiholomorphic involution τ and fixed ovals $X_0, \ldots, X_g$.



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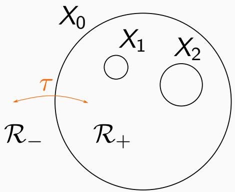

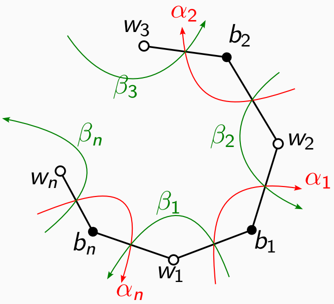

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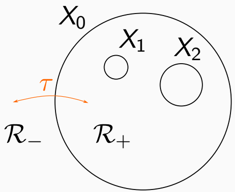
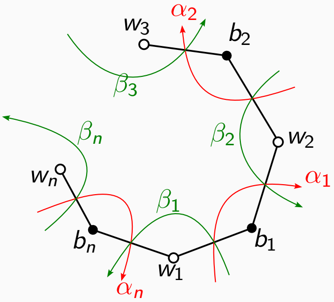
In general notion of minimal graphs [Boutillier, Cimasoni, de Tilière - '20, '21].





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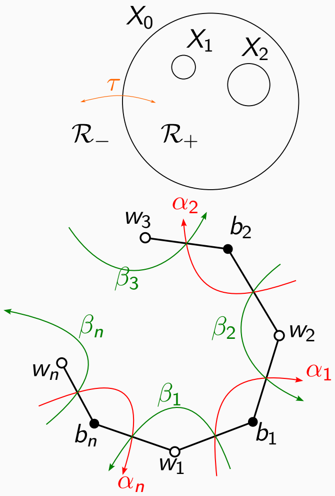
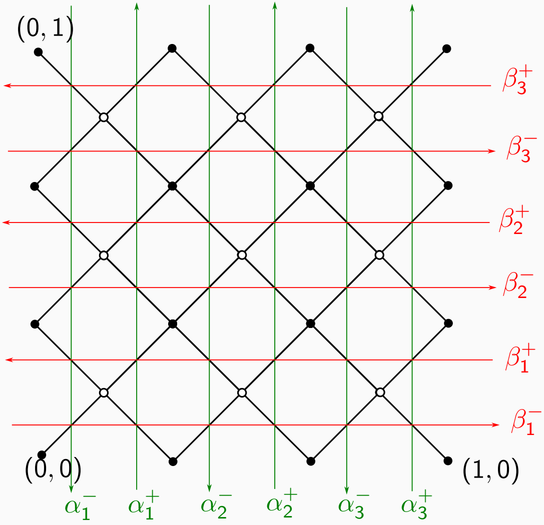
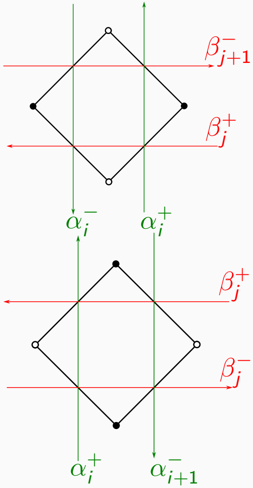
- For $g = 0$ these are isoradial weights. [Kenyon '02, Kenyon-Okounkov '03]



Kasteleyn weights: square grid





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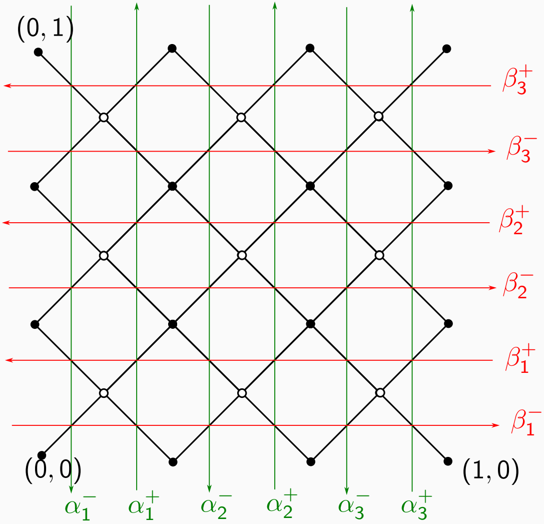
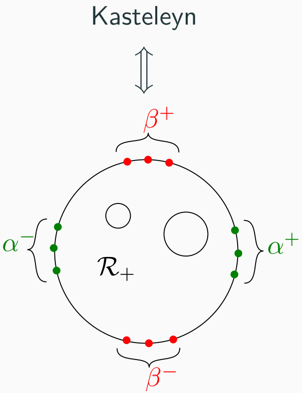
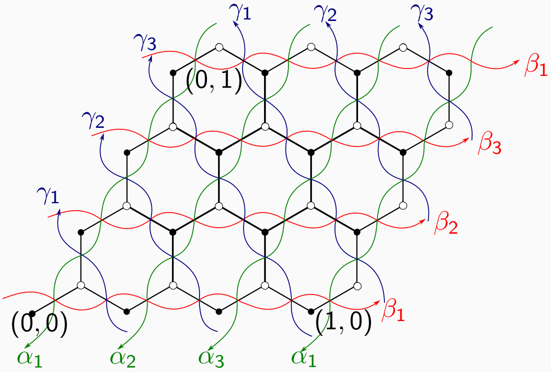
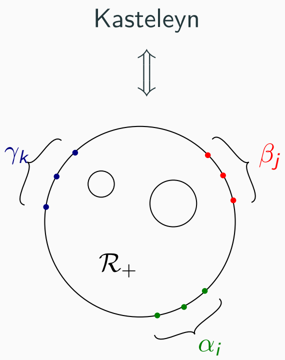


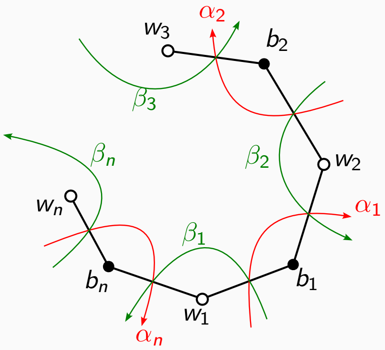


Kasteleyn weights: hex grid



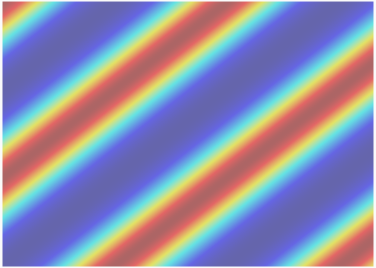


Fock weights. Formulas

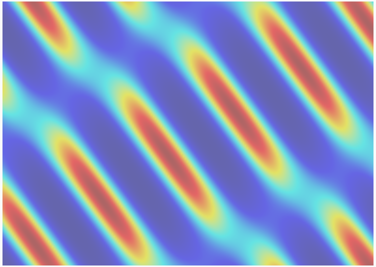


$$W_f = \prod_{i=1}^{n} \frac{\theta[\Delta]\left(\int_{\alpha_i}^{\beta_i} \boldsymbol{\omega}\right)}{\theta[\Delta]\left(\int_{\beta_i}^{\alpha_{i+1}} \boldsymbol{\omega}\right)} \frac{\theta\left(\eta(f_{2i}) + D\right)}{\theta\left(\eta(f_{2i-1}) + D\right)}$$

Quasi-periodic weights



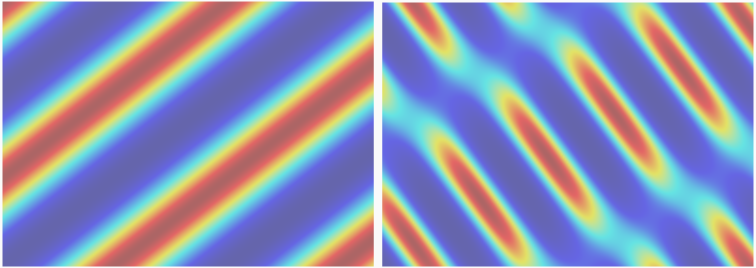
(a) W_f for $g = 1$



(b) W_f for $g = 2$

$g = 0$: isoradial weights, rational functions

Quasi-periodic weights



(a) W_f for $g = 1$ **(b)** W_f for $g = 2$

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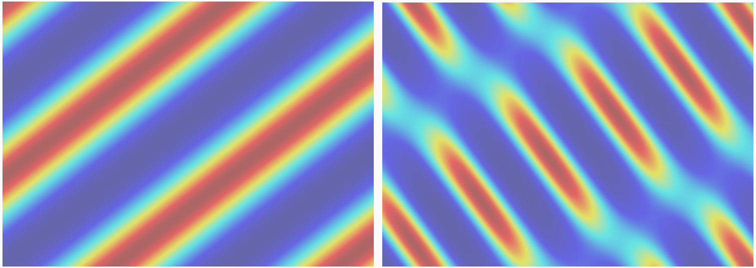

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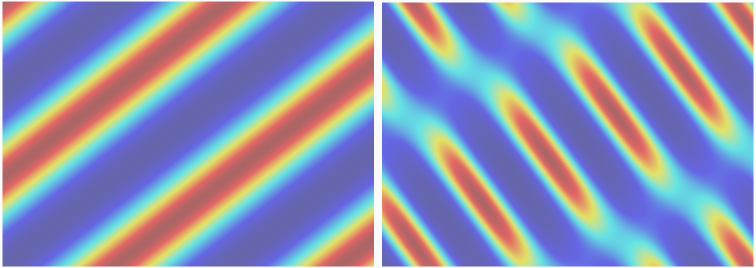

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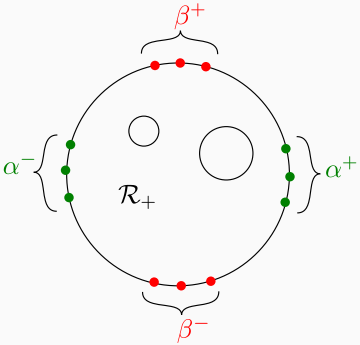
Weights periodic iff $\sum_i(\alpha_i^- - \alpha_i^+), \sum_i(\beta_i^- - \beta_i^+)$ principal divisors.
$g > 0$: Train track parameters repeat periodically $\not\Rightarrow$ weights K_{wb}, W_f periodic.

Algebro-geometric description

Amoeba and polygon map

- **Goal**: Describe limiting objects algebro-geometrically.
- **Data** $\mathcal{S}$: M-curve $\mathcal{R}$ with antiholomorphic involution τ. $\{\alpha_i^{\pm}, \beta_j^{\pm}\} \in X_0$ with clustering condition.



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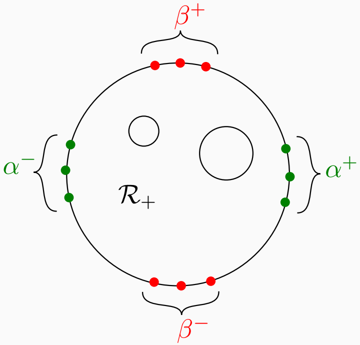
- $d\zeta_1, (d\zeta_2)$ normalized meromorphic differentials with residues ∓ 1 at $\alpha_i^{\pm}, (\beta_i^{\pm})$ and zero a periods.
- $\zeta_k(P) = \int^P d\zeta_k = x_k + iy_k$ well defined on $\mathcal{R}_+$.



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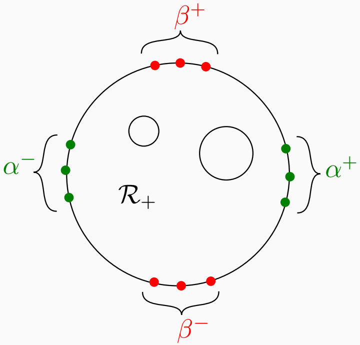
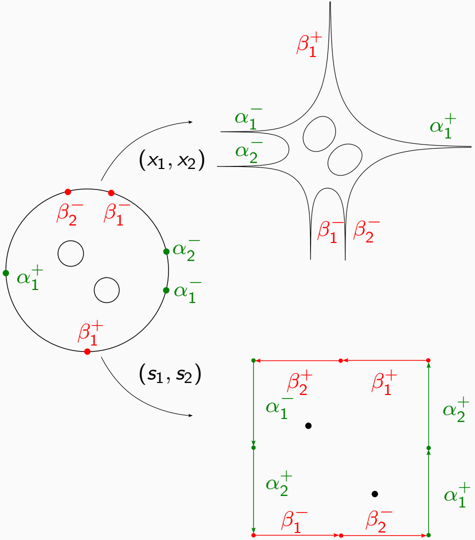
- $\zeta_k(P) = \int^P d\zeta_k = x_k + iy_k$ well defined on $\mathcal{R}_+$.
- **Proposition**: $(x_1, x_2), (y_1, y_2)$ coordinates of $\mathcal{R}_+^\circ$. [Krichever '14]



$(s_1, s_2) = \frac{1}{\pi}(y_2, -y_1)$.

Amoeba and polygon map

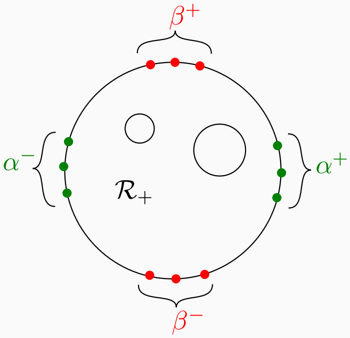


Ronkin function aand surface tension

- Ronkin function and surface tension

$$\rho(P) = \rho(x_1, x_2) = -\frac{1}{\pi} \operatorname{Im} \int_\ell \zeta_2 d\zeta_1 + x_2 s_2$$

$$\sigma(P) = \sigma(s_1, s_2) = \frac{1}{\pi} \operatorname{Im} \int_\ell \zeta_2 d\zeta_1 - x_1 s_1$$



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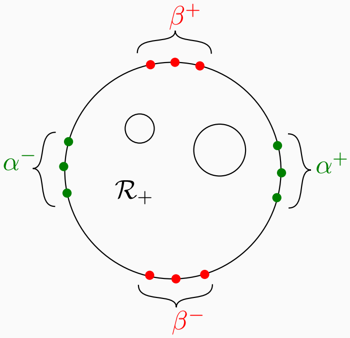
- Legendre dual

$$\nabla\sigma(s_1, s_2) = (x_1, x_2), \quad \nabla\rho(x_1, x_2) = (s_1, s_2).$$



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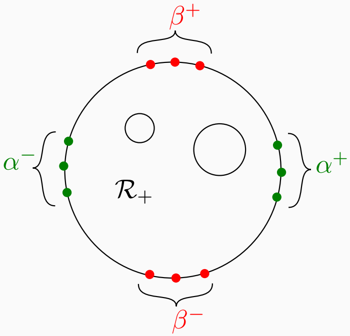
- Definitions agree with algebraic ones in doubly periodic case.



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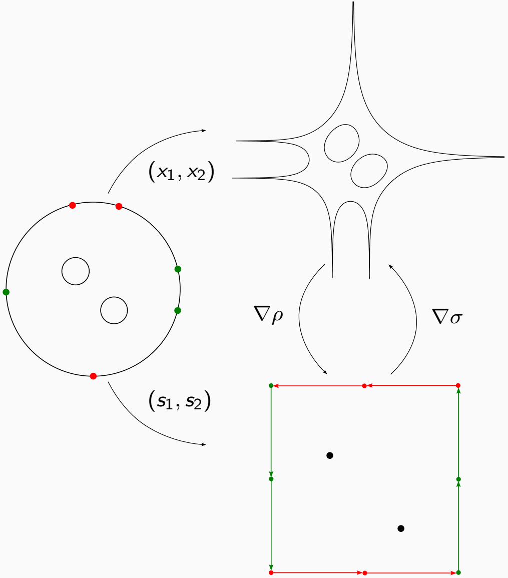
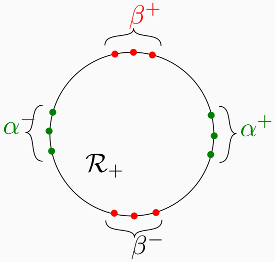


Dictionary

Object	doubly periodic	our setup
Spectral curve	$\det(P(z,w)) = 0$	$\mathcal{R}$
Monodromies	(z, w)	$(\frac{\psi_{(1,0)}}{\psi_{(0,0)}}, \frac{\psi_{(0,1)}}{\psi_{(0,0)}})$
Main differentials	$(\frac{dz}{z}, \frac{dw}{w})$	$(d\zeta_1, d\zeta_2)$
Amoeba map	$(\log\lvert z\rvert, \log\lvert w\rvert)$	$(\mathrm{Re}\,\zeta_1, \mathrm{Re}\,\zeta_2)$

Surface tension and free energy via hyperbolic volumes.

$g = 0$ case: $|\alpha_i^{\pm}| = |\beta_j^{\pm}| = 1$



$$\zeta_1(z) = \sum_i \log\left(\frac{z - \alpha_i^-}{z - \alpha_i^+}\right), \zeta_2(z) = \sum_j \log\left(\frac{z - \beta_j^-}{z - \beta_j^+}\right).$$

Bloch-Wigner function

$$D(z) := \operatorname{Im} \mathrm{Li}_2(z) + \arg(1 - z) \log |z|,$$

Treat as a function of the cross-ratio of four points:

$$\tilde{D}(z_1, z_2, z_3, z_4) := D(z), \quad \text{where } z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

The Bloch-Wigner function is the volume V of the ideal hyperbolic tetrahedron $\Delta(z_1, z_2, z_3, z_4)$ with the vertices z_1, z_2, z_3, z_4:

$$\tilde{D}(z_1, z_2, z_3, z_4) = V(\Delta(z_1, z_2, z_3, z_4)).$$

Proposition: The surface tension function is given by

$$\sigma(z) \quad = \tfrac{1}{\pi}\sum_{e\in E(G)} D(\tfrac{z-\alpha}{z-\beta}) + \log|\alpha-\beta| \arg\tfrac{z-\alpha}{z-\beta} = \\ -\tfrac{1}{\pi}\sum_{e\in E(G)} V(z,\alpha,\beta,\infty) + \phi(e)\lambda(e),$$

where $\alpha,\beta\in\mathbb{C}, |\alpha|=|\beta|=1$ are the train track parameters of the edge e, $D(z)$ is the Bloch-Wigner function, V is the volume of the corresponding ideal hyperbolic tetrahedron $\Delta(z,\alpha,\beta,\infty)$ with the dihedral angle $\phi(e)$ at the edge (z,∞), and $\lambda(e)=\log|\alpha-\beta|$ is the logarithmic length of the corresponding edge.

- For $z=0$ we recover the formula [Kenyon 2002] for the normalized determinant of the discrete Dirac operator for isoradial embeddings

$$\sigma(e,z=0)=\frac{2}{\pi}(L(\theta)+\theta\log 2\sin\theta),$$

  where $2\theta=\beta-\alpha$ and $L(\theta)=-\int_0^\theta \log 2\sin t dt$ is the Lobachevsky function.
- The functional σ coincides with the functionals describing discrete conformal mappings [B-Pinkall-Springborn 2015].

Height function

- Discrete height function $h : G^* \to \mathbb{Z}$ in bijection to dimer configuration.

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- In [BB]: purely variational proof, more general boundary conditions.

Complex structure on dimers

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Res	α_i^-	β_i^-	α_i^+	β_i^+
$d\zeta_1$	1	0	-1	0
$d\zeta_2$	0	1	0	-1
$d\zeta_3$	1	-1	1	-1

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- $\#$ zeros $=$ $\#$ poles $+\ 2g - 2$
- For any $(u, v) \in (-1, 1)^2$ have:
  - 2 zeros on any inner oval
  - 1 zero between $\alpha_i^{\pm}, \alpha_{i+1}^{\pm}$, also $\beta_i^{\pm}, \beta_{i+1}^{\pm}$.
  - 2 free zeros.

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- For any $(u,v) \in (-1,1)^2$ have:
  - 2 zeros on any inner oval
  - 1 zero between $\alpha_i^{\pm}, \alpha_{i+1}^{\pm}$, also $\beta_i^{\pm}, \beta_{i+1}^{\pm}$.
  - 2 free zeros.
- **Definition**: Conjugated free zeros $(P, \tau P)$, $P \in \mathcal{R}_+^{\circ}$. Then $(u,v) \in \mathcal{F}_{\mathcal{S}}$ liquid region.

Res	α_i^-	β_i^-	α_i^+	β_i^+
$d\zeta_1$	1	0	-1	0
$d\zeta_2$	0	1	0	-1
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Complex structure on dimers

- First definition of complex structure on dimers: [Kenyon-Okounkov '05].
- Define $d\zeta = d\zeta_{(u,v)} := -ud\zeta_2 + vd\zeta_1 + d\zeta_3$ [Kenyon-Prause '23, Berggren-Borodin '23] for Aztec diamond.
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- **Proposition**: $\mathcal{F} : P \in \mathcal{R}_+^\circ \mapsto (u, v) \in \mathcal{F}_\mathcal{S}$ is diffeomorphism. [Berggren-Borodin '23]

Res	α_i^-	β_i^-	α_i^+	β_i^+
$d\zeta_1$	1	0	-1	0
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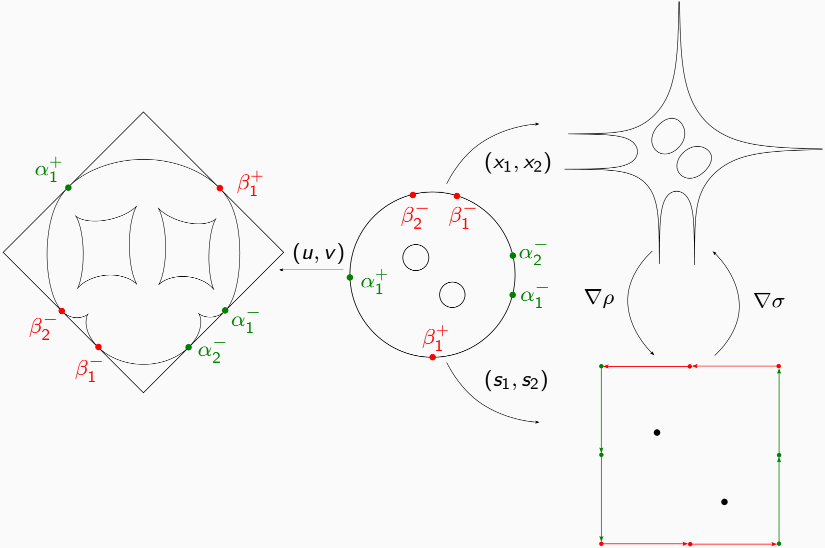
All Maps



α_1^+
β_1^+
β_2^-
α_1^-
β_1^-
α_2^-
(u, v)
β_2^-
β_1^-
α_2^-
α_1^+
α_1^-
β_1^+
(x_1, x_2)
(s_1, s_2)
$\nabla\rho$
$\nabla\sigma$

Arctic curve, isoradial case ($g = 0$)

$d\zeta_i(z) = f_i(z)dz.$

$$\begin{aligned} f_1(z) &= \sum_i \frac{1}{z-\alpha_i^-} - \frac{1}{z-\alpha_i^+}, \\ f_2(z) &= \sum_i \frac{1}{z-\beta_i^-} - \frac{1}{z-\beta_i^+}, \\ f_3(z) &= \sum_i \frac{1}{z-\alpha_i^-} + \frac{1}{z-\alpha_i^+} - \frac{1}{z-\beta_i^-} - \frac{1}{z-\beta_i^+} \end{aligned}$$

imply

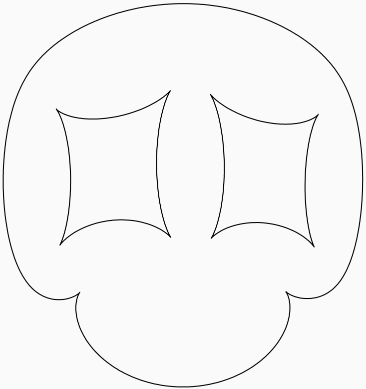
$$u = \frac{W(f_1, f_3)}{W(f_1, f_2)}, \quad v = \frac{W(f_1, f_3)}{W(f_1, f_2)},$$

where W is the Wronskian

$$W(f_i, f_j) = \begin{vmatrix} f_i & f_j \\ f_i' & f_j' \end{vmatrix}.$$

Geometric properties

- $\operatorname{div}_{(u,v)}(x_1, x_2) = \operatorname{div}_{(u,v)}(y_1, y_2) = 0.$



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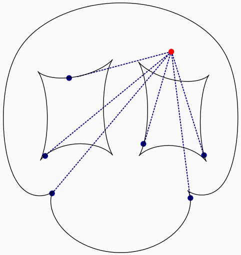
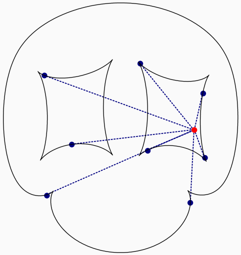
- (u, v) on tangent through $(u_0, v_0) \implies (u_0, v_0)$ zero of $d\zeta_{(u,v)}$.
    - These are all (non-free) zeros.





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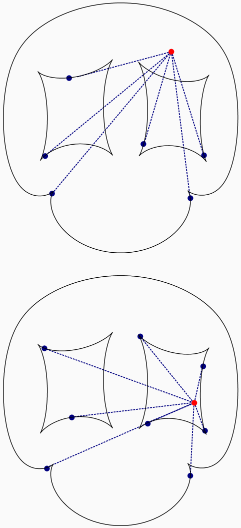

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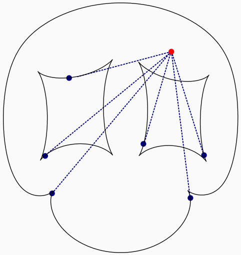
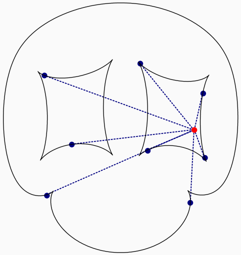
- Paralellity on real ovals: $(u', v') || (x_1', x_2')$.





The complex height function

- Define height function $H(u,v) = \int^{P} d\zeta_{(u,v)}$.
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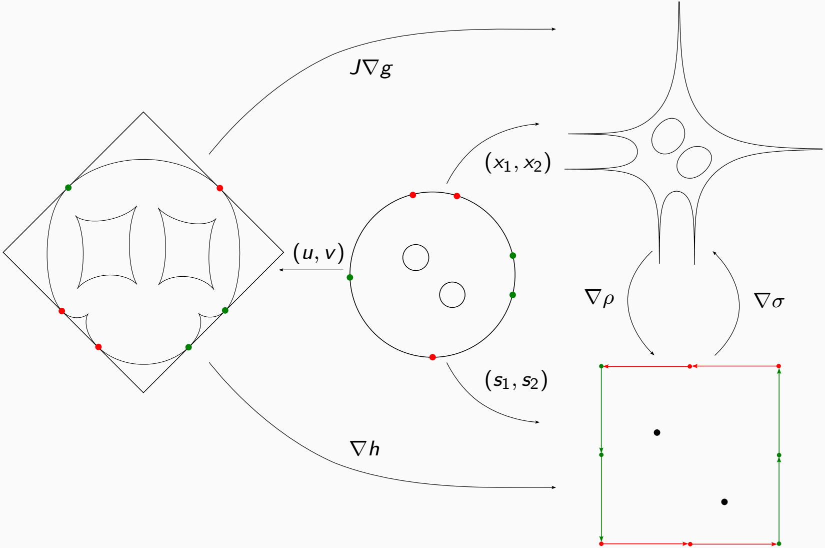
- **Theorem**: g is magnetic tension minimizer:

$$g = \arg\min_f \int_{[-1,1]^2} \rho(\nabla f).$$

All Maps

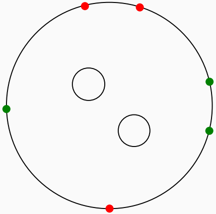


Computation

- All these formulas can be efficiently computed via Schottky uniformization,
- Schottky group is a free group G generated by inversions in circles X_i,
- differentials are given by Poincarè theta series

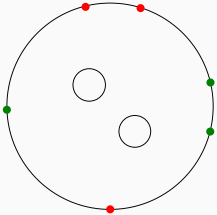
$$d\zeta_1(z) := \sum_{g\in G}\sum_i \left(\frac{1}{z-g(\alpha_i^-)} - \frac{1}{z-g(\alpha_i^+)}\right) dz,$$

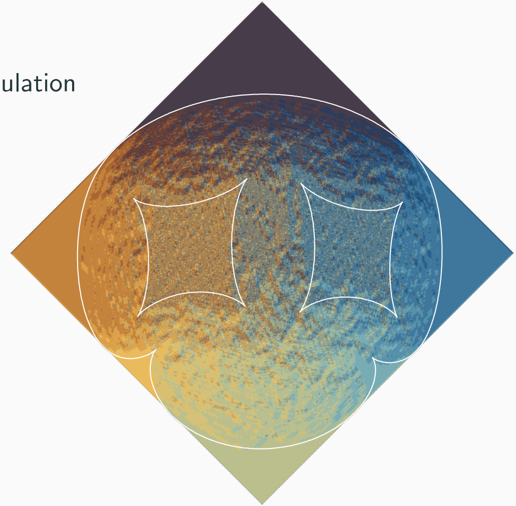
$$d\zeta_2(z) := \sum_{g\in G}\sum_i \left(\frac{1}{z-g(\beta_i^-)} - \frac{1}{z-g(\beta_i^+)}\right) dz.$$



Computation

- Pictures shown are actual computations, not just illustrations. github.com/nikolaibobenko/FockDimerSimulation
- Theoretical predictions match simulations on practical scales.





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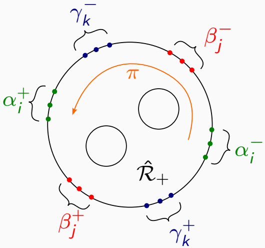
- Have non-differentiabilities in $\sigma \implies$ generalized Euler-Lagrange in terms of subgradients.
- $g = 0$: [Astala, Duse, Prause, Zhong, '20].
- In general follows from existence of extension of g to gas bubbles and frozen regions.

Hexagonal case

- $\hat{\mathcal{R}}$ double cover of $\mathcal{R}$ with symmetry $\hat{\mathcal{R}}/\pi = \mathcal{R}$.



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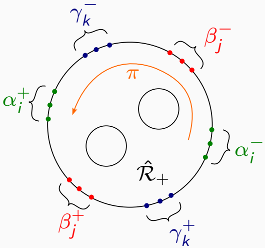

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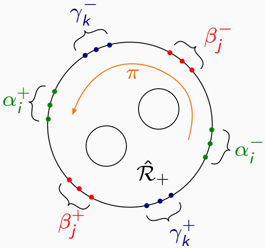

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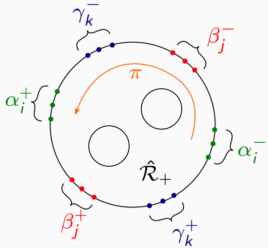
- All same properties as before
- Can extend h to hexagon, satisfies hexagon boundary conditions.



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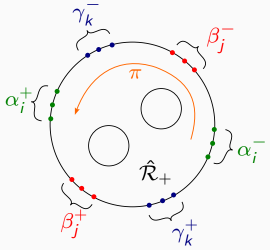

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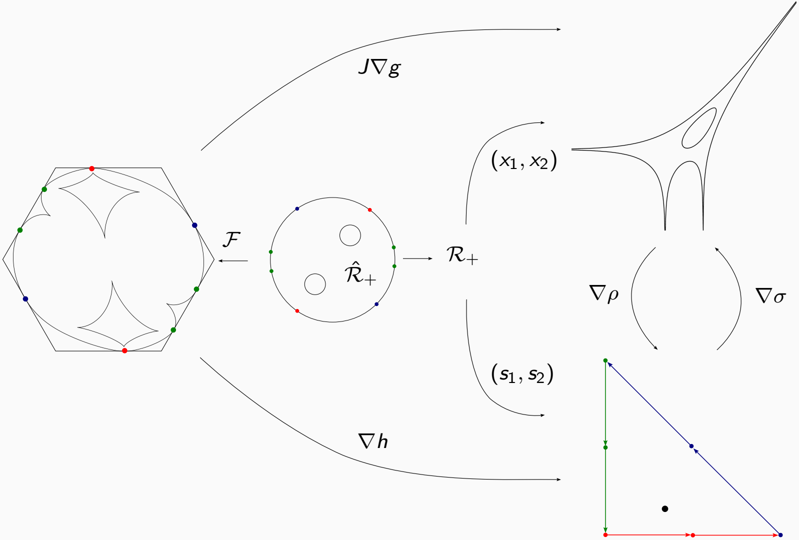
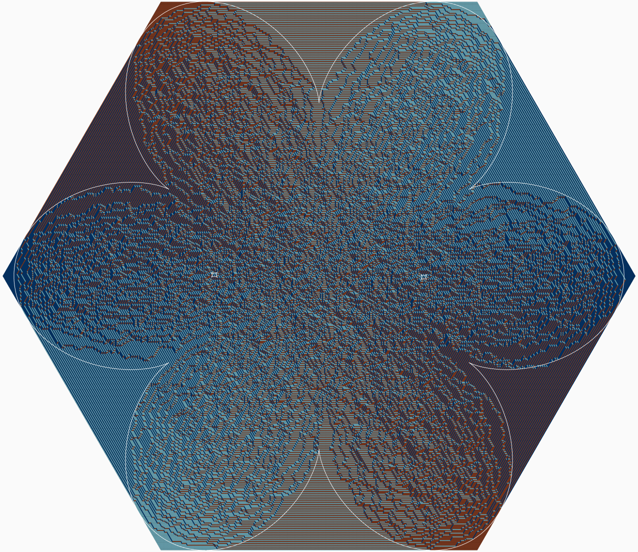
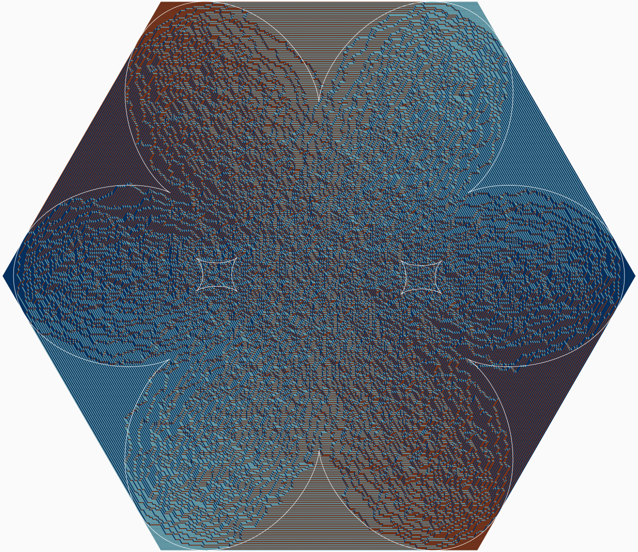


Gallery



Gallery



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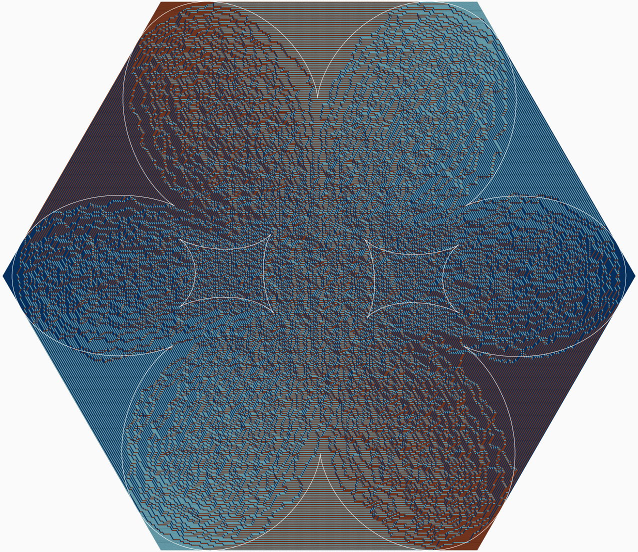

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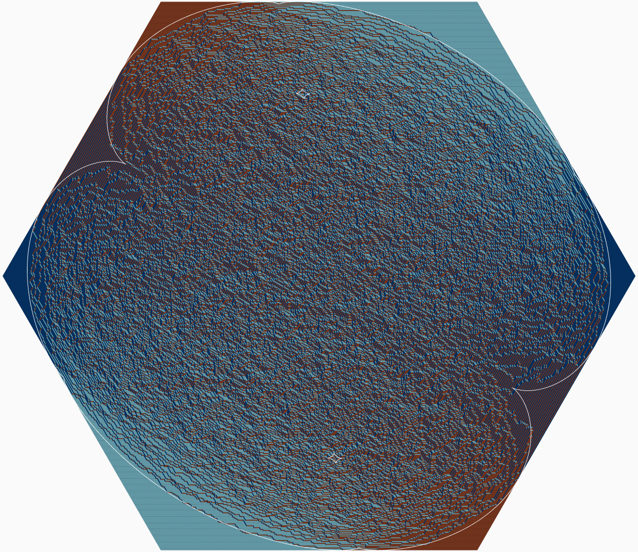

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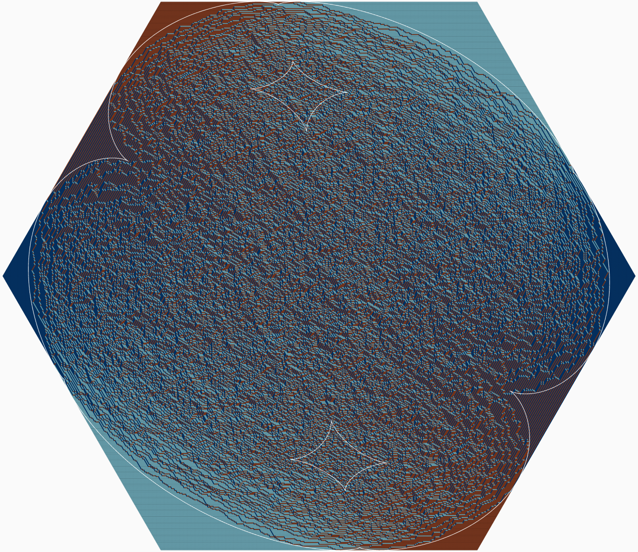

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