Geometric bounds for spanning tree entropy of planar lattices

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MARCH 25 - 29, 2024
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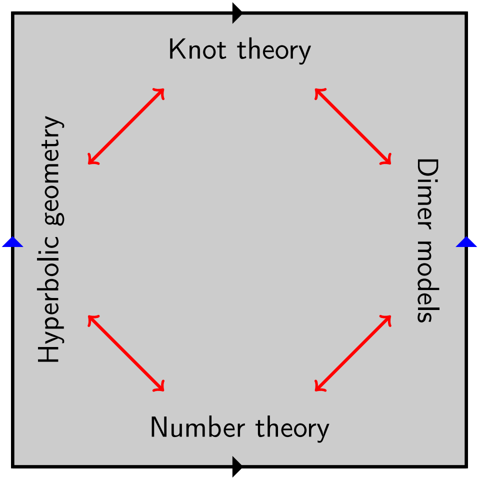
joint work with Ilya Kofman (CUNY)

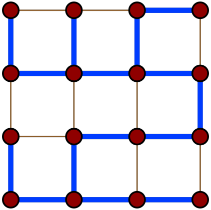
Theme of today's talk

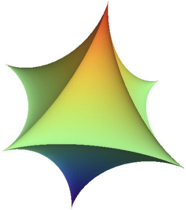


Plan

- Motivation
- Geometry and finite planar graphs
- Geometry and planar lattice graphs
- Infinite families
- Applications to finite case

Motivation 1 - Spanning tree or Dimer entropy and hyperbolic volume



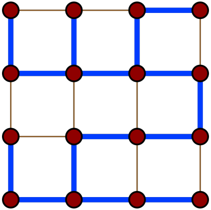


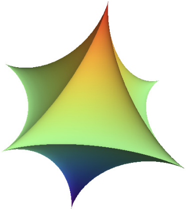
Temperley (1974)

$$\lim_{m,n\to\infty} \frac{\pi \log \tau(G_{m\times n})}{m\cdot n} = \frac{1}{\pi}\int_0^{\pi}\int_0^{\pi} \log|4-2\cos\theta-2\cos\phi|\,d\theta\,d\phi = 4\mathrm{C}$$

where C is Catalan's constant, $\mathrm{C} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \approx 0.916$

Motivation 1 - Spanning tree or Dimer entropy and hyperbolic volume





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where C is Catalan's constant, $\mathrm{C} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \approx 0.916$

$$4\mathrm{C} = v_{oct} = \text{ volume of regular ideal octahedron } \approx 3.66386$$

Motivation 2 - Mahler measure

Mahler measure of polynomial $p(z)$ is defined as

$$m(p(z)) := \frac{1}{2\pi i}\int_{S^1} \log|p(z)| \frac{dz}{z} \stackrel{\text{Jensen}}{=} \sum_{\substack{\alpha_i \text{ roots of } p \\ |\alpha_i|\geq 1}} \log|\alpha_i|$$

2-variable Mahler measure:

$$\mathrm{m}(p(z,w)) := \frac{1}{(2\pi i)^2}\int_{S^1\times S^1} \log|p(z,w)| \frac{dz}{z}\frac{dw}{w}$$

2-variable Mahler measures are related to hyperbolic volume because they can be often computed using the dilograrithm $Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

Mahler measure of 2-variable polynomials

vol() $= 2v_{tet} = 2$ volume of regular ideal tetrahedron ≈ 2.0298

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(Smyth '1981) vol() $= 2\pi\, \mathrm{m}(1 + x + y) = \dfrac{3\sqrt{3}}{2} L(\chi_{-3}, 2)$

(Boyd '2000)
A−polynomials

vol() $= \pi\, \mathrm{m}(A(L, M))$

$$= \pi\, \mathrm{m}\big(M^4 + L(1 - M^2 - 2M^4 - M^6 + M^8) - L^2 M^4\big)$$

(Kenyon '2000)
Dimer entropy

vol() $= \dfrac{2\pi}{5}\, \mathrm{m}(p(z, w))$

$$= \frac{2\pi}{5} \mathrm{m}\left(6 - w - \frac{1}{w} - z - \frac{1}{z} - \frac{w}{z} - \frac{z}{w}\right)$$

Geometric bounds for the number of spanning trees

Conjecture 1 Let Γ be a finite connected planar graph. If $\mathrm{vol}(\Gamma) > 0$ then

$$\mathrm{vol}(\Gamma) < 2\pi \log \tau(\Gamma) < |E\Gamma| v_{oct}.$$

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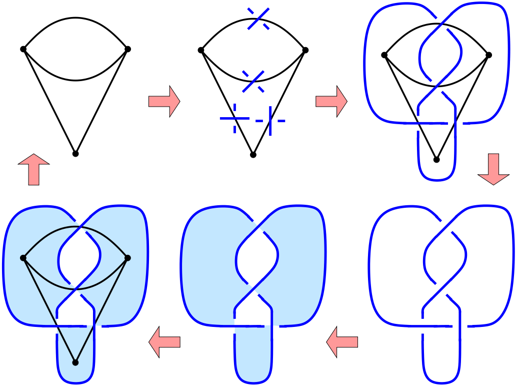
- $\tau(\Gamma)$ = number of spanning trees of Γ.
- $|E\Gamma|$ = number of edges of Γ.
- v_{oct} = the hyperbolic volume of a regular ideal octahedron = $4C \approx 3.66386$ (C is the Catalan's constant)
- $\mathrm{vol}(\Gamma)$ = hyperbolic volume of a graph defined below.

Alternating links and planar graphs

We can recover an alternating knot or link diagram (up to mirror image) from its Tait graph:



The other checkerboard coloring gives the planar dual of the Tait graph.

Knot or link determinant

The **link determinant $\det(K)$** was one of the first computable knot invariants (computable means not of the form "minimize something over all diagrams").

For alternating link K with Tait graph G_K:

$$\det(K) = \# \text{ spanning trees } \tau(G_K)$$

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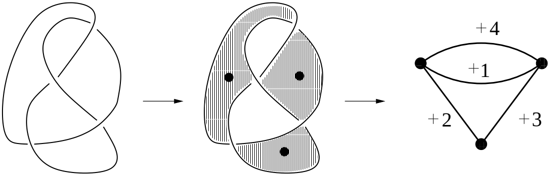

det() = 5

Hyperbolic knots and links

A link K is hyperbolic if $S^3 - K$ is a finite-volume hyperbolic 3-manifold i.e. $\pi_1(S^3 - K)$ acts properly discontinuously on $\mathbb{H}^3$ by isometries i.e. $\pi_1(S^3 - K) \subset \mathrm{Isom}^+(\mathbb{H}^3)$.

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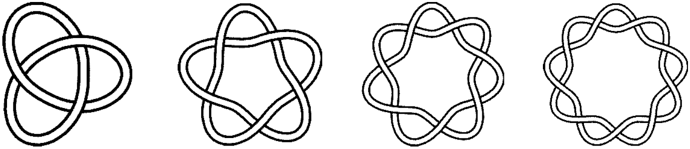
Hyperbolic alternating links are easy to spot.

Theorem (Menasco) If K has a connected prime alternating link diagram, except the standard $(2, q)$-torus diagram, then K is hyperbolic.

$(2, q)$–torus diagrams:



Knot invariants from geometry

Hyperbolic structure on $S^3 - K$ is unique (Mostow-Prasad Rigidity)
$\Longrightarrow$ geometric invariants are topological invariants !
For e.g. $\boxed{\text{volume } \mathrm{vol}(K) = \mathrm{vol}(S^3 - K)}$.

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For e.g. $\boxed{\text{volume } \mathrm{vol}(K) = \mathrm{vol}(S^3 - K)}$.

Examples

- $S^3 -$  can be decomposed into two regular hyperbolic ideal tetrahedra $\implies$ $\mathrm{vol}($  $) = 2v_{tet} = 2.0298\ldots$
- $S^3 -$  can be decomposed into two regular hyperbolic ideal octahedra $\implies$ $\mathrm{vol}($  $) = 2v_{oct} = 7.3277\ldots$

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- $|E\Gamma|$ = number of edges of Γ.
- v_{oct} = the hyperbolic volume of a regular ideal octahedron = $4C \approx 3.66386$ (C is the Catalan's constant)
- $\mathrm{vol}(\Gamma) = \mathrm{vol}(K)$ where K is the alternating link whose Tait graph is Γ.

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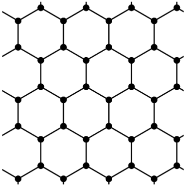
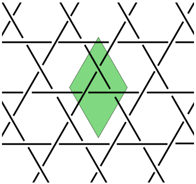
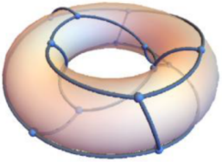
- The lower bound is the Vol-Det Conjecture by C-Kofman-Purcell in 2016.
- Conjecture is verified for Tait graphs of all prime alternating knots up to 16 crossings.
- (C-Kofman-Purcell '2016) 2π is sharp !

Volume of a toroidal graph







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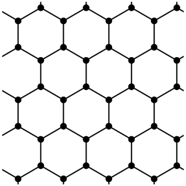
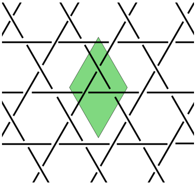
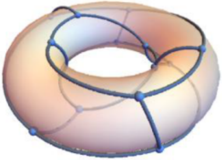
  

$\mathcal{G}$ = planar lattice graph, preserved by 2-dim lattice Λ acting on $\mathbb{R}^2$

$\mathcal{L}$ = biperiodic alternating link in $\mathbb{R}^2 \times I$ with Tait graph $\mathcal{G}$

L = link in $T^2 \times I$, such that $L = \mathcal{L}/\Lambda$.

$G = \mathcal{G}/\Lambda$, graph on T^2, which is Tait graph of L.

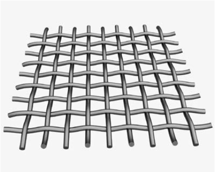
Define the volume of G and of $\mathcal{G}$ as

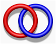
$$\mathrm{vol}(G) = \mathrm{vol}(T^2 \times I - L) \quad \text{and} \quad \mathrm{vol}(\mathcal{G}) = \frac{\mathrm{vol}(G)}{|VG|}.$$

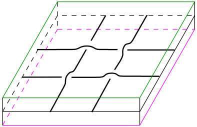
Geometry of biperiodic alternating links

For a biperiodic link $\mathcal{L}$, the $\mathbb{Z}^2$-quotient of $\mathbb{R}^3 - \mathcal{L}$ is a link complement in a thickened torus:

$$T^2 \times I - L.$$



$T^2 \times I \cong S^3 -$ , so it's also the complement of a link $L \cup H$ in S^3 with a Hopf sublink H.



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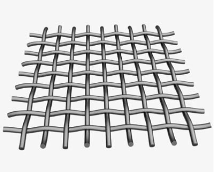
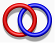
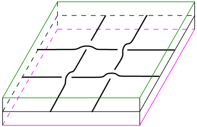
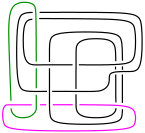


$T^2 \times I \cong S^3 -$ , so it's also the complement of a link $L \cup H$ in S^3 with a Hopf sublink H.



Example: $\mathcal{L} = \mathcal{W}$ the infinite square weave, $S^3 - (W \cup H)$ has a complete hyperbolic structure with four regular ideal octahedra. 



Generally, $\mathrm{vol}(T^2 \times I - W) = c(W) \cdot v_{oct}$

Hyperbolic structure on $T^2 \times I - W$

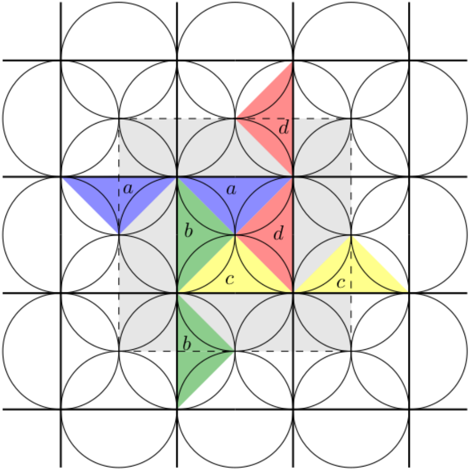


FIGURE 7. Face pairings for a fundamental domain $\mathcal{P}_{\mathcal{W}}$ of $\mathbb{R}^3 - \mathcal{W}$. The shaded part indicates the fundamental domain of $S^3 - L$.

Geometric bounds for spanning tree entropy

$\mathcal{G}$ = planar lattice graph, preserved by 2-dim lattice Λ acting on $\mathbb{R}^2$
$\Gamma_n = \mathcal{G} \cap (n\Lambda)$, exhaustive nested sequence of finite planar graphs
Define the spanning tree entropy of $\mathcal{G}$:

$$z_{\mathcal{G}} = \lim_{n\to\infty} \frac{\log \tau(\Gamma_n)}{|V\Gamma_n|}.$$

This is related to the dimer entropy for the overlaid bipartite graph.

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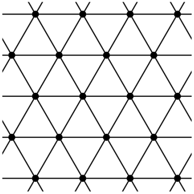
Conjecture 2 (C-Kofman 2023)

$$\mathrm{vol}(\mathcal{G}) \ \leq \ 2\pi\, z_{\mathcal{G}} \ \leq \ \overline{\nu}(\mathcal{G}).$$

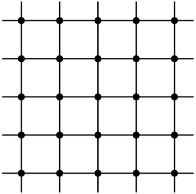
where

$$\mathrm{vol}(\mathcal{G}) = \frac{\mathrm{vol}(G)}{|VG|} \quad \text{and} \quad \overline{\nu}(\mathcal{G}) = \frac{|EG|v_{oct}}{|VG|}.$$

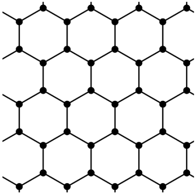
Examples: Regular lattice graphs



$\mathcal{G}_\triangle$



$\mathcal{G}_\square$



$\mathcal{G}_{\hexagon}$

$|VG_\triangle| = 1$ $|VG_\square| = 1$ $|VG_{\hexagon}| = 2$

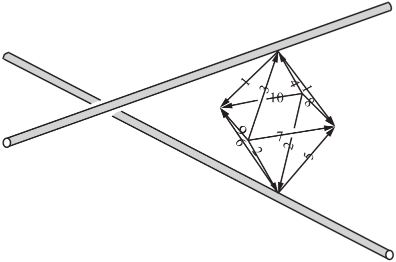
$2\pi\, z_{\mathcal{G}_\triangle} = 10 v_{tet}$ $2\pi\, z_{\mathcal{G}_\square} = 2 v_{oct}$ $2\pi\, z_{\mathcal{G}_{\hexagon}} = 5 v_{tet}$

For these regular lattice graphs,

$$2\pi\, z_{\mathcal{G}} = \mathrm{vol}(\mathcal{G}).$$

Motivation for upper bound

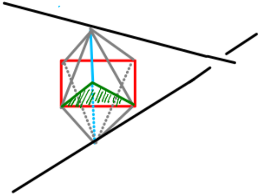
Where is the upper bound in Conjecture 1 and 2 coming from ?

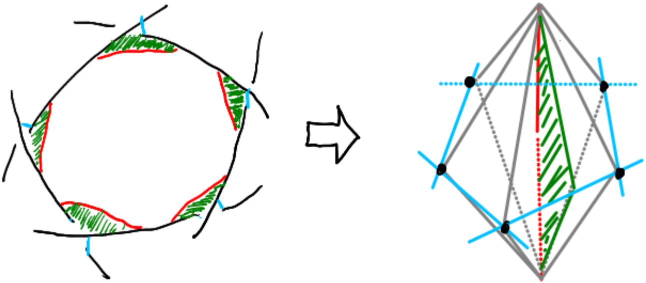


We can decompose $S^3 - K$ into octahedra, one octahedron at each crossing:

$$\Longrightarrow \quad \mathrm{vol}(K) < c(K) v_{oct}$$

Motivation for bipyramid volume





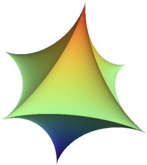
Bipyramid volume

Let B_n be the hyperbolic regular ideal bipyramid over a regular n–gon.

$$\mathrm{vol}(B_n) = 2n\,Л(\pi/n), \quad \text{for } Л(\theta) = -\int_0^{\theta} \log|2\sin t|\,dt,$$

where $Л(\theta)$ is the Lobachevsky function.

e.g. B_4 = regular ideal octahedron



Bipyramid volume

Theorem (Adams) $\mathrm{vol}(B_n) < 2\pi \log(\frac{n}{2})$ and $\mathrm{vol}(B_n) \underset{n\to\infty}{\sim} 2\pi \log(\frac{n}{2})$.

n	$\mathrm{vol}(B_n)$
2	0
3	2.02988
4	3.66386
5	4.98677
6	6.08965
7	7.03257
8	7.85498
9	8.58367
10	9.23755
11	9.83040
12	10.37255
13	10.87192
14	11.33474
15	11.76597
20	13.56682
100	23.67095
1,000	38.13817
1,000,000	81.5409

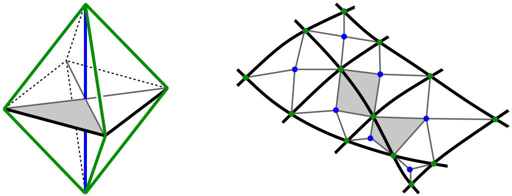
Bipyramid volume of a toroidal link

Define the bipyramid volume of L as

$$\mathrm{vol}^{\lozenge}(L) = \sum_{f \in \{\text{faces of } L\}} \mathrm{vol}\left(B_{|f|}\right).$$

Theorem (C-Kofman-Purcell '2019)

$$0 < \mathrm{vol}(T^2 \times I - L) \leq \mathrm{vol}^{\lozenge}(L)$$



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Theorem (C-Kofman-Purcell '2019)

$$0 \; < \; \mathrm{vol}(T^2 \times I - L) \; \leq \; \mathrm{vol}^{\lozenge}(L)$$

Let G, G^* be dual 2-connected graphs with disk faces on T^2.
Define the bipyramid volume of G as

$$\mathrm{vol}^{\lozenge}(G) = \sum_{f \in FG} \mathrm{vol}\left(B_{|f|}\right).$$

Theorem (C-Kofman-Purcell '2019) For Tait graph G of L on T^2,

$$0 \; < \; \mathrm{vol}(G) \; \leq \; \mathrm{vol}^{\lozenge}(G) + \mathrm{vol}^{\lozenge}(G^*).$$

Lower bound using the bipyramid volume

Define the bipyramid volume of the lattice graph $\mathcal{G}$

$$\nu^{\Diamond}(\mathcal{G}) = \frac{\mathrm{vol}^{\Diamond}(G) + \mathrm{vol}^{\Diamond}(G^*)}{|VG|}.$$

We can prove Conjecture 2 when $z_{\mathcal{G}}$ satisfies the inequality:

$$\mathrm{vol}(\mathcal{G}) \;\le\; \nu^{\Diamond}(\mathcal{G}) \;\le\; 2\pi\, z_{\mathcal{G}} \;\le\; \overline{\nu}(\mathcal{G}).$$

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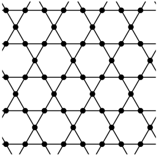
$$\nu^{\Diamond}(\mathcal{G}_{\triangle}) = 2\pi\, z_{\mathcal{G}_{\triangle}}, \qquad \nu^{\Diamond}(\mathcal{G}_{\square}) = 2\pi\, z_{\mathcal{G}_{\square}}, \qquad \nu^{\Diamond}(\mathcal{G}_{\hexagon}) = 2\pi\, z_{\mathcal{G}_{\hexagon}}$$

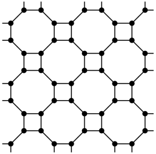
Question: Is there a $\mathcal{G}$, other than $\mathcal{G}_{\triangle}$, $\mathcal{G}_{\square}$, $\mathcal{G}_{\hexagon}$, for which $\nu^{\Diamond}(\mathcal{G}) = 2\pi\, z_{\mathcal{G}}$?

Often, $\nu^{\Diamond}(\mathcal{G}) \approx 2\pi\, z_{\mathcal{G}}$ are numerically very close!

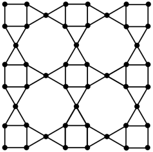
	Lattice graph $\mathcal{G}$	$\lvert VG\rvert$	$\nu^{\Diamond}(\mathcal{G})/2\pi$	$z_{\mathcal{G}}$	$\overline{\nu}(\mathcal{G})/2\pi$
1.	Triangular $\mathcal{G}_{\triangle}$ (3^6)	1	1.61533	1.61533	1.74937
2.	Square $\mathcal{G}_{\square}$ (4^4)	1	1.16624	1.16624	1.16624
3.	Hexagonal $\mathcal{G}_{\hexagon}$ (6^3)	2	0.80766	0.80766	0.87468
4.	Kagome (3-6-3-6)	3	1.12157	1.13570	1.16624
5.	Square-octagon (4-8-8)	4	0.78139	0.78668	0.87468
6.	Medial(4-8-8)	6	1.10405	1.12171	1.16624
7.	3-12-12	6	0.70590	0.72056	0.87468
8.	3-4-6-4	6	1.14390	1.14480	1.16624
9.	4-6-12	12	0.76795	0.77780	0.87468
10.	Cairo pentagonal lattice	6	0.93886	0.94057	0.97187
11.	Lattice 11	9	0.84361	0.84744	0.90708
12.	Lattice 12	2	1.39079	1.39928	1.74937
13.	3^2-4-3-4	4	1.40830	1.41086	1.45780
14.	4^4; 3^3-4^2	3	1.32761	1.32774	1.36062
15.	3^6; 3^3-4^2	3	1.47731	1.47739	1.55499



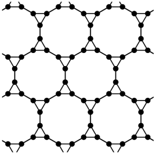
Lattice graph #4



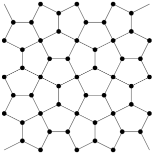
Lattice graph #5



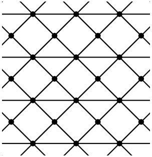
Lattice graph #6



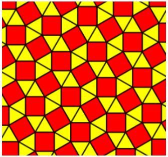
Lattice graph #7



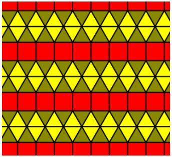
Lattice graph #10



Lattice graph #12



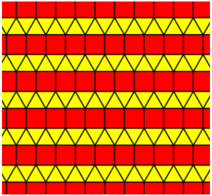
Lattice graph #13



Lattice graph #15

Lower bound using the bipyramid volume – a non-example

Lattice graph $\mathcal{G}$	$\|VG\|$
3^3-4^2	2



In this case,

$$\mathrm{vol}(\mathcal{G}) \quad < \quad 2\pi\, z_{\mathcal{G}} \quad < \quad \nu^{\lozenge}(\mathcal{G}) \quad < \quad \overline{\nu}(\mathcal{G})$$

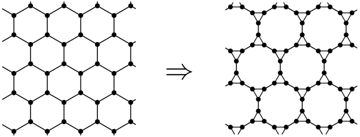
$$2\pi * \left(\quad 1.39717 \quad < \quad 1.40693 \quad < \quad 1.40830 \quad < \quad 1.45780 \quad \right)$$

Question: For which other planar lattice graphs is $2\pi\, z_{\mathcal{G}} < \nu^{\lozenge}(\mathcal{G})$?

Infinitely many cases for Conjecture 2

Theorem (C-Kofman '2023)
(1) Parallel edges: Replace every edge of $\mathcal{G}$ by $s \geq 2$ parallel edges to get $\mathcal{G}_s$.
If $\nu^{\Diamond}(\mathcal{G}) \leq 2\pi\, z_{\mathcal{G}} \leq \overline{\nu}(\mathcal{G})$, then $\nu^{\Diamond}(\mathcal{G}_s) < 2\pi\, z_{\mathcal{G}_s} < \overline{\nu}(\mathcal{G}_s)$.

(2) Truncating 3-regular lattice graph: If $\mathcal{G}$ is 3-regular, replace every vertex of $\mathcal{G}$ by K_3 to get $\mathcal{G}'$. If $\nu^{\Diamond}(\mathcal{G}) \leq 2\pi\, z_{\mathcal{G}} \leq \overline{\nu}(\mathcal{G})$, then $\nu^{\Diamond}(\mathcal{G}') < 2\pi\, z_{\mathcal{G}'} < \overline{\nu}(\mathcal{G}')$.



Infinitely many cases for Conjecture 2

(3) Medial graph of 3-regular lattice graph: Let $\mathcal{G}_n$ be 3-regular lattice graphs by truncating $\mathcal{G}_0 = \mathcal{G}_{\hexagon}$. Let $\mathcal{G}'_n$ be 4-regular medial graph of $\mathcal{G}_n$. Then $\nu^{\lozenge}(\mathcal{G}'_n) < 2\pi \, z_{\mathcal{G}'_n} < \overline{\nu}(\mathcal{G}'_n)$ for all $n \geq 0$.

Sketch of Proof:

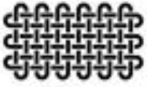
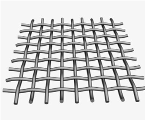
- (Teufl-Wagner '2010) Compute growth rate of $z_{\mathcal{G}}$ under above operations.
- Growth rate of bipyramid volume is similar to that of $z_{\mathcal{G}}$.

Application to Conjecture 1 - Diagrammatic convergence

$K_n \xrightarrow{\mathrm{F}} \mathcal{L}$ denotes $\{K_n\}$ *Følner converges almost everywhere* to $\mathcal{L}$.

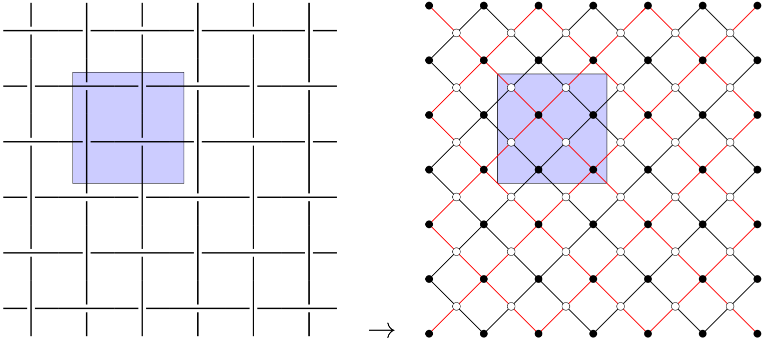
This means the alternating links K_n satisfy:

1. K_n contain increasing subsets of $\mathcal{L}$ which exhaust $\mathcal{L}$:
   $\exists\ G_n \subset G(K_n)$ such that $G_n \subset G_{n+1}$, and $\bigcup G_n = \mathcal{G}(\mathcal{L})$,
2. Følner condition for $G_n \subset \mathcal{G}(\mathcal{L})$: $\displaystyle\lim_{n\to\infty} \frac{|\partial G_n|}{|G_n|} = 0$,
3. The K_n do not have too many other crossings:
   $\displaystyle\lim_{n\to\infty} \frac{|G_n|}{c(K_n)} = 1.$

 ,  ,  ,  , $\ldots \xrightarrow{\mathrm{F}}$ 

Biperiodic overlaid graph

Biperiodic alternating link $\mathcal{L} \to$ Biperiodic bipartite graph $\mathcal{G}^b_{\mathcal{L}}$.



Determinant density convergence

Theorem (Kenyon-Okounkov-Sheffield '2006)
If $G_n^b = \mathcal{G}^b/n\Lambda$ is a toroidal exhaustion of $\mathcal{G}^b$, then

$$\lim_{n\to\infty} \frac{\log Z(G_n^b)}{n^2} = \mathrm{m}(p(z,w)).$$

where $p(z,w)$ is characteristic polynomial for toroidal dimer model on $\mathcal{G}^b$.

Corollary If $G = \mathcal{G}/\Lambda$, $\quad z_G^{\mathrm{fd}} = |VG| z_{\mathcal{G}} = \mathrm{m}(p(z,w)).$

Determinant density convergence

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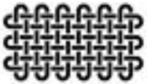
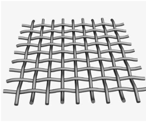
Corollary If $G = \mathcal{G}/\Lambda$, $\quad z_G^{\mathrm{fd}} = |VG| z_{\mathcal{G}} = \mathrm{m}(p(z,w)).$

Theorem (C-Kofman '2016)

$$K_n \xrightarrow{\mathrm{F}} \mathcal{L} \implies \lim_{n\to\infty} \frac{\log \det(K_n)}{c(K_n)} = \frac{\mathrm{m}(p(z,w))}{c(L)}.$$

 ,  ,  ,  , $\ldots$ $\xrightarrow{\mathrm{F}}$ 

Infinitely many cases for Conjecture 1

Theorem (C-Kofman '2023) Let $\mathcal{G}$ be a planar lattice that satisfies

$$\mathrm{vol}^{\diamondsuit}(G) + \mathrm{vol}^{\diamondsuit}(G^*) \; < \; 2\pi \, z_G^{\mathrm{fd}} \; < \; |EG| v_{oct}.$$

Let Γ_n be a sequence of connected planar graphs with bounded average degree that Folner converges to $\mathcal{G}$ almost everywhere. Then for all but finitely many n,

$$\mathrm{vol}(\Gamma_n) \; < \; 2\pi \log \tau(\Gamma_n) \; < \; |E\Gamma_n| v_{oct}.$$

Sketch of Proof:

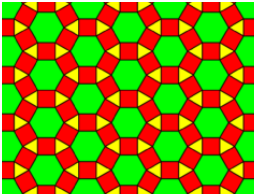
- (C-Kofman-Lalin '2019) $\nu^{\diamondsuit}(\Gamma)$ behaves well under Folner convergence namely,

$$\Gamma_n \to \mathcal{G} \quad \Rightarrow \quad \lim_{n\to\infty} \nu^{\diamondsuit}(\Gamma_n) = \nu^{\diamondsuit}(\mathcal{G}).$$

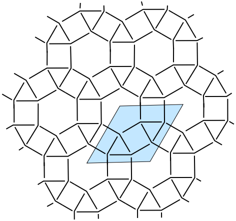
- Determinant Density convergence.
- For the upper bound, the convergence is similar: $|E\Gamma_n|/|V\Gamma_n| \to |EG|/|VG|$.

□

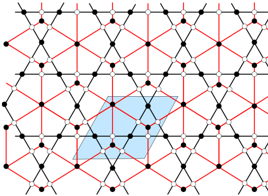
Rhombitrihexagonal link $\mathcal{R}$



$\mathcal{G}(\mathcal{R})$



$\mathcal{R}$ & R



$\mathcal{G}^b_{\mathcal{R}}$ & G^b_R

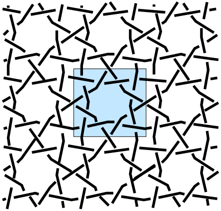
$$\mathrm{vol}(T^2 \times I - R) = \mathrm{vol}^{\diamond}(L) = 10v_{tet} + 3v_{oct} = 21.14100\ldots$$

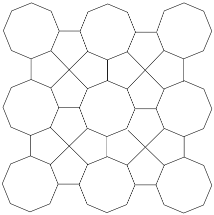
$$p(z,w) = 6\,(6 - w - 1/w - z - 1/z - w/z - z/w)$$

$$2\pi\,\mathrm{m}(p(z,w)) = 10v_{tet} + 2\pi\log 6 = 21.40737\ldots$$

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\,\mathrm{m}(p(z,w))}{c(R)} > \frac{\mathrm{vol}(T^2 \times I - R)}{c(R)}$$

A typical biperiodic alternating link (Lattice graph #11)





Faces of $L = FG \cup FG^*$:

1 octagon

4 pentagons

1 square

8 triangles

$$\mathrm{vol}(G) = \mathrm{vol}((T^2 \times I) - L) \quad \approx 47.644829$$

$$\mathrm{vol}^{\diamondsuit}(L) = \mathrm{vol}(B_8) + 4\mathrm{vol}(B_5) + v_{oct} + 16 v_{tet} \quad \approx 47.704628$$

$$\mathrm{p(z,w)} = wz^2 + z^3 - 2wz + 104z^2 - 2z^3/w + w + 510z + 510z^2/w + z^3/w^2 - 2456z/w + 104z^2/w^2$$
$$+510/w + 1/z + 510z/w^2 + z^2/w^3 + 104/w^2 - 2/(wz) - 2z/w^3 + 1/w^3 + 1/(w^2 z) + 104$$

Numerically, $2\pi\,\mathrm{m}(p(z,w)) \approx 47.9214$

So $\mathcal{G}$ satisfies the Conjecture 1 inequality within a range of 0.6%,

$$\mathrm{vol}((T^2 \times I) - L) < \mathrm{vol}^{\diamondsuit}(L) < 2\pi\,\mathrm{m}(p(z,w))$$

$$\mathrm{vol}(\mathcal{G}) < \nu^{\diamondsuit}(\mathcal{G}) < 2\pi\, z_{\mathcal{G}}.$$

Right-angled volume of planar lattice graphs

$\mathcal{G}, \mathcal{G}^*$ (resp. G, G^*) are orthogonally dual lattice graphs if inscribed circles in their faces can form an orthogonal circle pattern.

On $\partial\mathbb{H}^3$, this circle pattern defines a right-angled ideal hyperbolic polyhedron $\mathcal{P}$ in $\mathbb{H}^3$ and let $P = \mathcal{P}/\Lambda$.

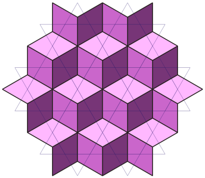
If G and G^* are orthogonally dual graphs, define the right-angled volume:

$$\mathrm{vol}^{\perp}(G) = 2\mathrm{vol}(P) = \sum_{e \in EG^b} 2Л(\theta_e) \quad \text{and} \quad \mathrm{vol}^{\perp}(\mathcal{G}) = \frac{\mathrm{vol}^{\perp}(G)}{|VG|}.$$

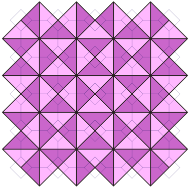
Theorem (C-Kofman-Purcell '2022) If G and G^* are dual simple 3–connected graphs on T^2 with disk faces, then G and G^* admit an orthogonally dual embedding on T^2, unique up to Möbius transformations, such that

$$\mathrm{vol}^{\perp}(G) \;\leq\; \mathrm{vol}(G) \;\leq\; \mathrm{vol}^{\lozenge}(G) + \mathrm{vol}^{\lozenge}(G^*).$$

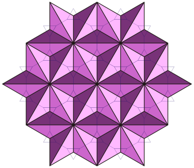
Many well-known planar lattice graphs $\mathcal{G}$ satisfy orthogonal duality:



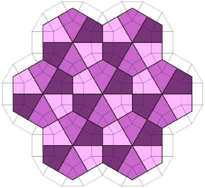
Lattice graph #4



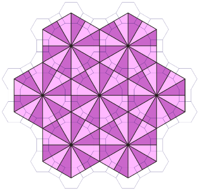
Lattice graph #5



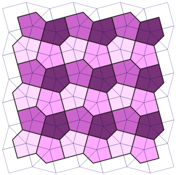
Lattice graph #7



Lattice graph #8



Lattice graph #9



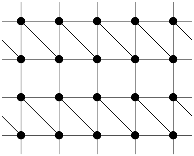
Lattice graphs #10 & 14

For all of the lattice graphs shown, $\mathrm{vol}^{\perp}(\mathcal{G}) \;\leq\; \mathrm{vol}(\mathcal{G}) \;\leq\; 2\pi\, z_{\mathcal{G}}$.

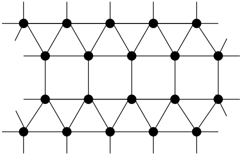
3^3-4^2 lattice

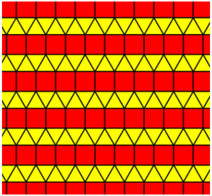
Like $\nu^\Diamond(\mathcal{G})$, $\mathrm{vol}^\perp(\mathcal{G})$ is exactly computable using the local geometry of G. If $\mathcal{G}$ satisfies orthogonal duality, the weaker lower bound should always hold:

$$\mathrm{vol}^\perp(\mathcal{G}) \;\leq\; 2\pi\, z_{\mathcal{G}}$$



$$\Longrightarrow$$





In this case,

$$\mathrm{vol}^\perp(\mathcal{G}) \quad < \quad \mathrm{vol}(\mathcal{G}) \quad < \quad 2\pi\, z_{\mathcal{G}} \quad < \quad \nu^\Diamond(\mathcal{G})$$

$$2\pi * (\quad 1.39079 \quad < \quad 1.39717 \quad < \quad 1.40693 \quad < \quad 1.40830 \quad)$$

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2. *Mahler Measure and the Vol-Det Conjecture* (joint with Ilya Kofman and Matilde Lalin), J. London Mathematical Society Volume 99, Issue 3, June 2019, Pages: 872-900.

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5. *Determinant density and biperiodic alternating links* (joint with Ilya Kofman), New York J. Math. 22 (2016) 891-906.


Thank You