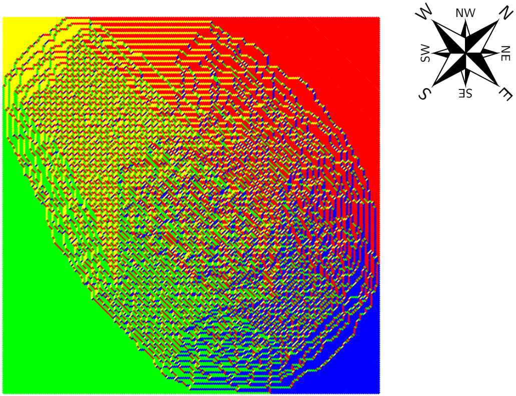
# Geometry of the doubly periodic Aztec dimer model

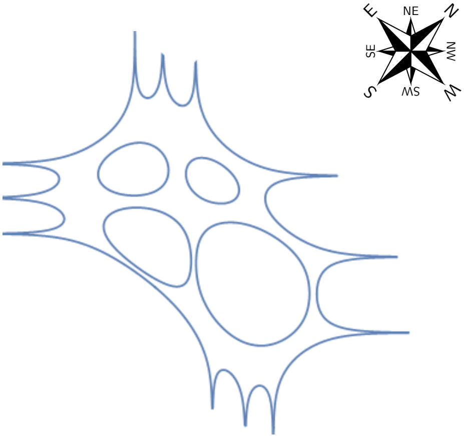
*Tomas Berggren*
*Department of Mathematics, MIT*

*Workshop I: Statistical Mechanics and Discrete Geometry, IPAM,*
*March 25, 2024*

*Joint work with Alexei Borodin*
*arXiv:2306.07482*




Random domino tilings of the Aztec diamond with periodic edge weights.




The amoeba of the spectral curve.

# Outline of the talk

The dimer model

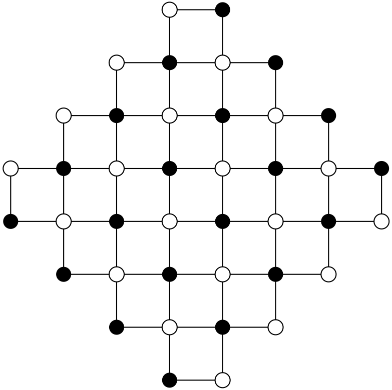
Spectral curves and their amoebas

The action function

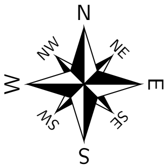
Main results

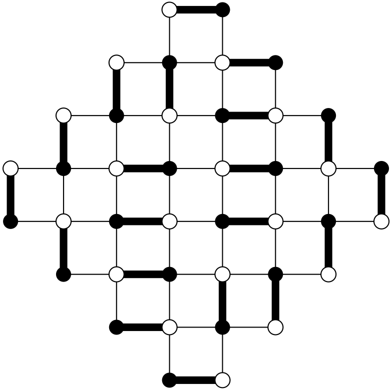
# The dimer model

# Dimer coverings of the Aztec diamond



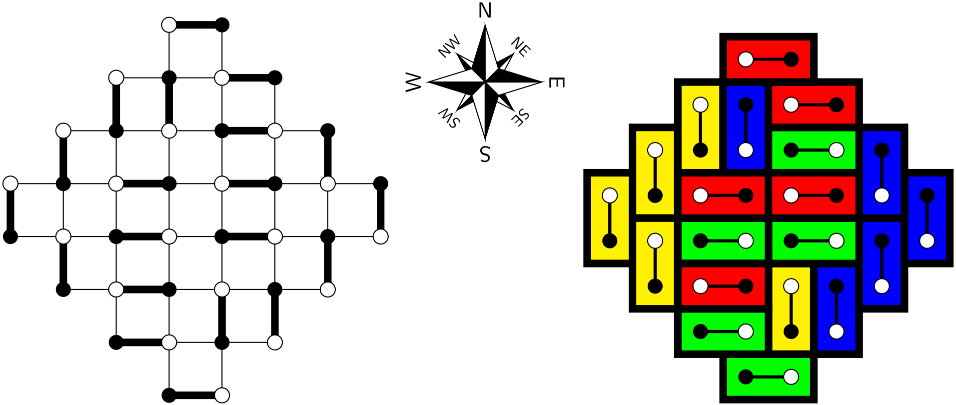
The Aztec diamond of size 4.



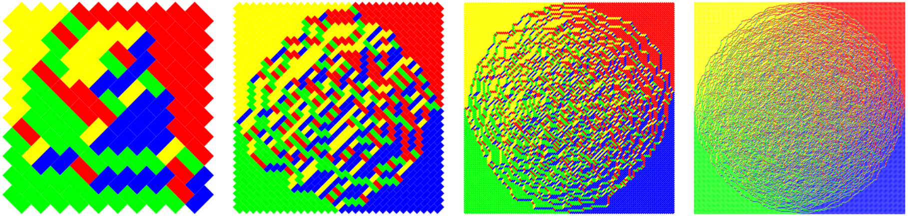



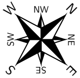
A dimer cover of the Aztec diamond.

# Dimer coverings and domino tilings of the Aztec diamond



# Uniformly distributed domino tilings of the Aztec diamond






The tiling pictures are generated using programs that were kindly provided by **S. Chhita** and **C. Charlier**.

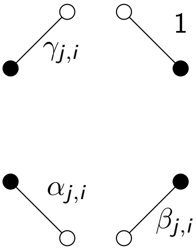
**Elkies–Kuperberg–Larsen–Propp '92**, **Jockusch–Propp–Shor '95**,
**Cohn–Elkies–Propp '96**, **Johansson '02, '05**,...

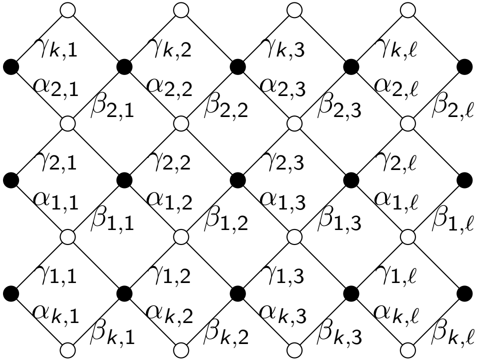
# The fundamental domain and the probability measure

We fix $k, \ell \in \mathbb{Z}_{>0}$ and edge weights $\alpha_{j,i}, \beta_{j,i}, \gamma_{j,i} > 0$ for $i = 1, \dots, \ell$, $j = 1, \dots, k$.
The probability measure on the set of all dimer coverings of the Aztec diamond of size $k\ell N$ is defined by

$$\mathbb{P}(M) = \frac{1}{Z} \prod_{e \in M} w(e), \quad \text{where} \quad Z = \sum_{M'} \prod_{e \in M'} w(e),$$

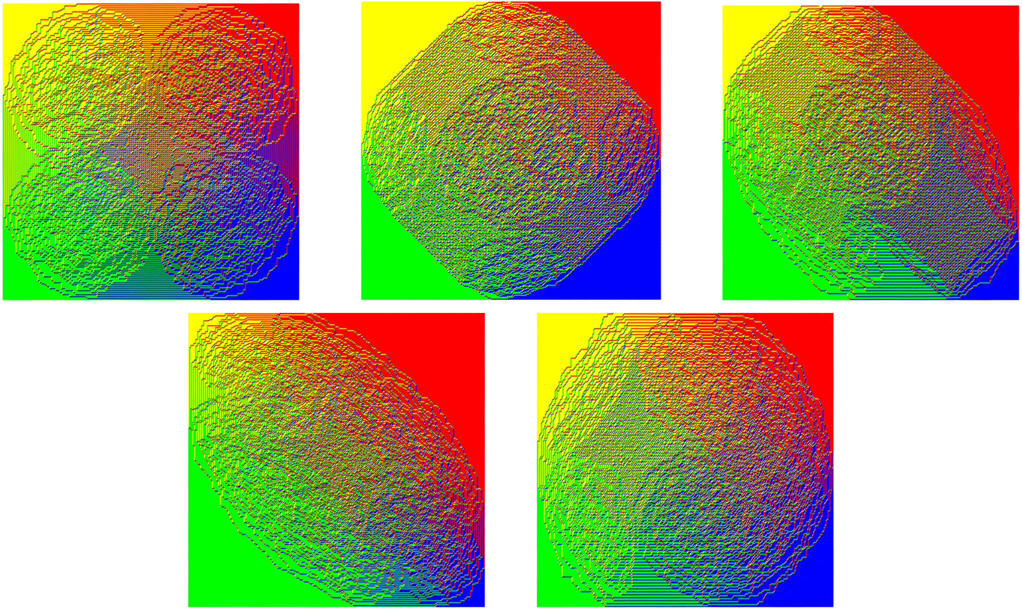
and $w(e) \in \{\alpha_{j,i}, \beta_{j,i}, \gamma_{j,i}\}$.



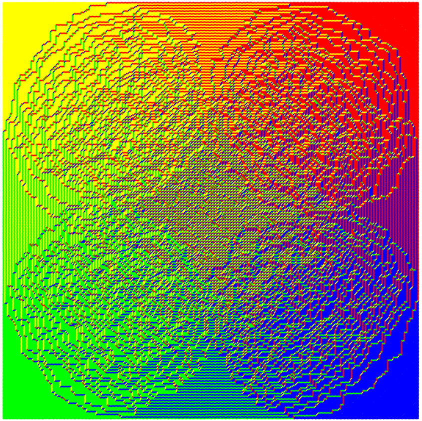

Uniform: $k = \ell = 1$ and $\alpha_{j,i} = \beta_{j,i} = \gamma_{j,i} = 1$.

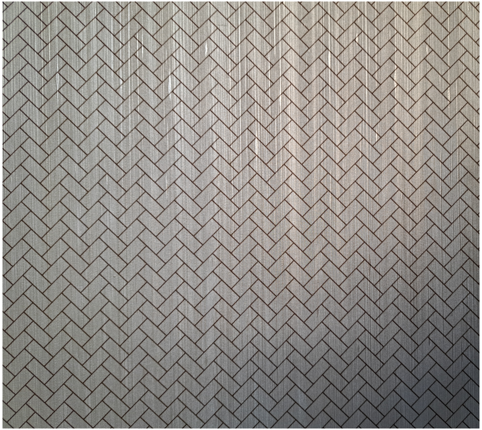
# Doubly periodic edge weights



3D figures

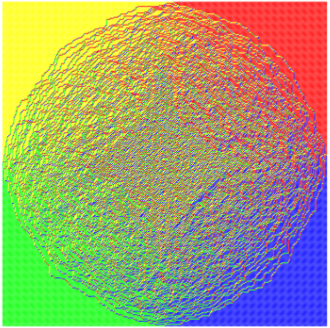
# Wallpaper in Philadelphia





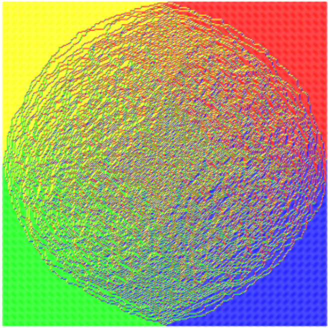
# Previously studied doubly periodic Aztec diamond models

The two-periodic Aztec diamond



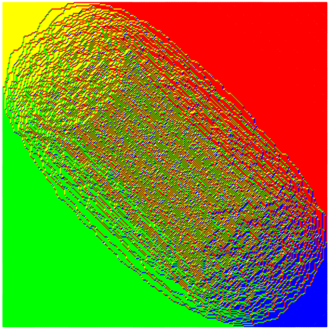
Chhita–Young '14, Chhita–Johansson '16, Beffara–Chhita–Johansson '18 '20, Duits–Kuijlaars '17, Johansson–Mason '21 '23, Bain '22 '23

The $2 \times \ell$-periodic Aztec diamond



Di Francesco–Soto-Garrido '14, Berggren '21

Biased $2 \times 2$-periodic Aztec diamond

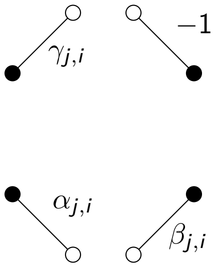


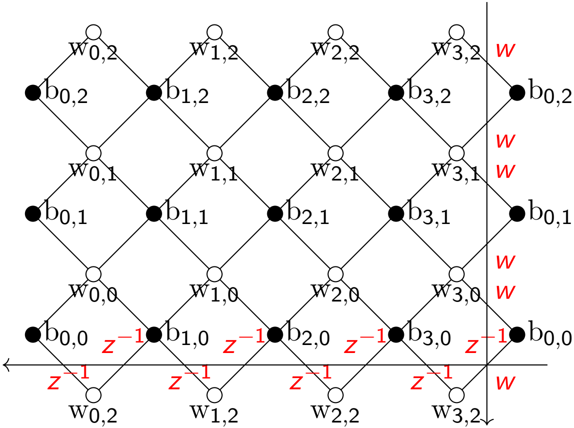
Borodin–Duits '23

In all previously asymptotically studied doubly periodic models $k = 2$ and the edge weights are at so-called torsion points (repeatedly applying the domino shuffling recovers the initial edge weights).

# Spectral curves and their amoebas

# The magnetically altered Kasteleyn matrix




We define the magnetically altered Kasteleyn matrix $K_{G_1}(z, w)$, as defined in Kenyon–Okounkov–Sheffield '06. That is, the adjacency matrix with the rows indexed by the white vertices and the columns by the black vertices of the above graph.

# The spectral curve and its amoeba

The characteristic polynomial is defined by

$$P(z, w) = \det K_{G_1}(z, w).$$

It is a degree $k$ polynomials in $z^{-1}$ and a degree $\ell$ polynomial in $w$.
The spectral curve is the zero set

$$\{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\},$$

and the amoeba is the image of the spectral curve under the map

$$\mathrm{Log}(z, w) = (\log |z|, \log |w|) = (r_1, r_2) \in \mathbb{R}^2.$$

# Uniform weights

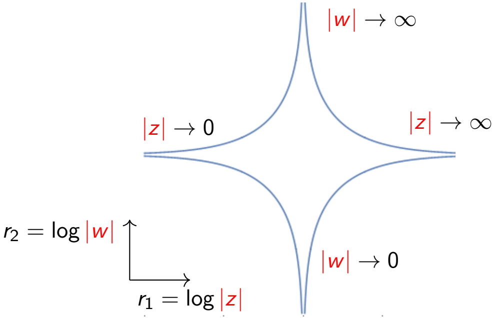
Uniform: $k = \ell = 1$ and $\alpha_{j,i} = \beta_{j,i} = \gamma_{j,i} = 1$.
Characteristic polynomial:

$$P(z, w) = 1 + z^{-1} - w + z^{-1} w.$$

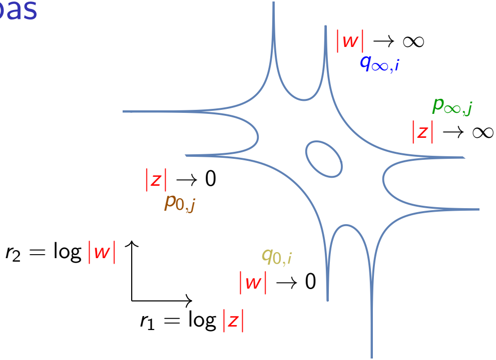
Spectral curve:

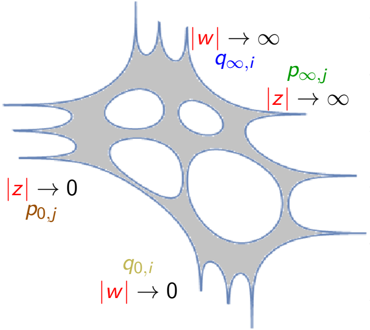
$$\left\{ (z, w) \in \mathbb{C}^2 : w = \frac{1 + z^{-1}}{1 - z^{-1}} \right\}.$$

Amoeba:



# Amoebas

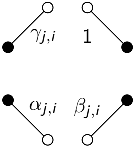



$$q_{0,i} = ((-1)^k \alpha_i^v / \gamma_i^v, 0), \quad q_{\infty,i} = (\beta_i^v, \infty), \quad i = 1, \ldots, \ell,$$

$$p_{0,j} = (0, (-1)^\ell \alpha_j^h / \beta_j^h), \quad p_{\infty,j} = (\infty, \gamma_j^h), \quad j = 1, \ldots, k,$$

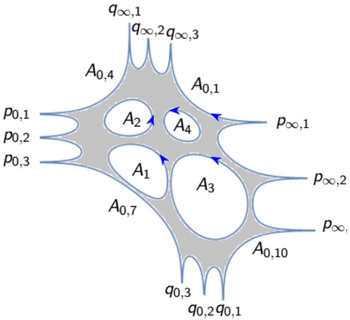
$$\frac{\alpha_i^v}{\gamma_i^v} = \frac{\prod_{j=1}^k \alpha_{j,i}}{\prod_{j=1}^k \gamma_{j,i}}, \quad \beta_i^v = \prod_{j=1}^k \beta_{j,i}, \quad \frac{\alpha_j^h}{\beta_j^h} = \frac{\prod_{i=1}^\ell \alpha_{j,i}}{\prod_{i=1}^\ell \beta_{j,i}}, \text{ and } \gamma_j^h = \prod_{i=1}^\ell \gamma_{j,i}.$$

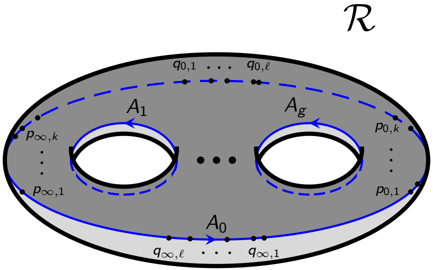



Generically there are $2k$ horizontal tentacles, $2\ell$ vertical tentacles and $(k-1)(\ell-1)$ compact ovals.

# Harnack curves

Kenyon–Okounkov–Sheffield '06 proved that the spectral curve is a Harnack curve. This means that the map $\mathrm{Log}(z, w) = (\log|z|, \log|w|)$ is at most 2-to-1.




The spectral curve can be thought of as gluing together two copies of the amoeba along their boundaries.

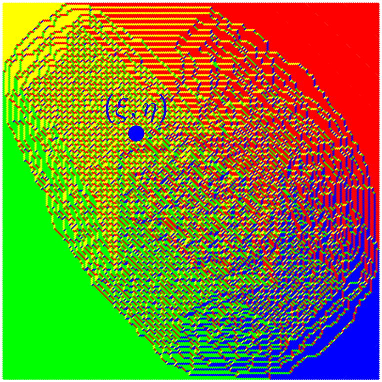
# The action function

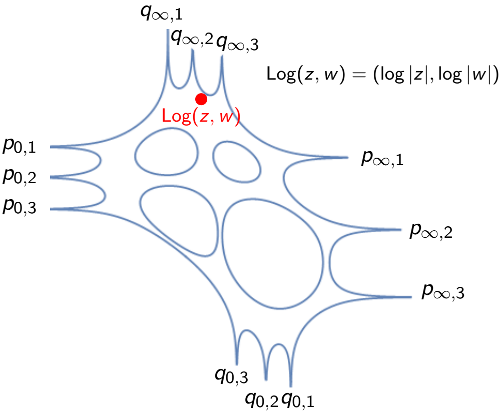
# The action function

Let $(\xi, \eta) \in (-1, 1)^2$ be global coordinates. The action function $F$ is defined for $q = (z, w) \in \mathcal{R}$ by

$$F(q; \xi, \eta) = \frac{k}{2}(1 - \xi) \log w - \frac{\ell}{2}(1 - \eta) \log z - \log \frac{\prod_{i=1}^{\ell} E(q_{0,i}, q)^k}{\prod_{j=1}^{k} E(p_{0,j}, q)^\ell},$$

where $E$ is a prime form (locally meromorphic with $E(p, q) = 0$ iff $q = p$).



# The action function

$$F(q;\xi,\eta) = \frac{k}{2}(1-\xi)\log w - \frac{\ell}{2}(1-\eta)\log z - \log \frac{\prod_{i=1}^{\ell} E(q_{0,i},q)^k}{\prod_{j=1}^{k} E(p_{0,j},q)^\ell}$$

Let $K_{\mathrm{Az}}$ be the Kasteleyn matrix for the Aztec diamond of size $k\ell N$. Then

$$K_{\mathrm{Az}}^{-1} = \frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_1}\int_{\gamma_2} \mathrm{e}^{N(F(q_1;\xi,\eta)-F(q_2;\xi,\eta))} G(q_1,q_2)\frac{z_1^{\zeta'}}{w_1^{\kappa'}}\frac{w_2^{\kappa}}{z_2^{\zeta}}\frac{\mathrm{d}z_2\,\mathrm{d}z_1}{z_2(z_2-z_1)}$$

where $(\zeta,\kappa),(\zeta',\kappa')\in\mathbb{Z}^2$ are the local coordinates, and $\gamma_1$ and $\gamma_2$ are curves in $\mathcal{R}$.

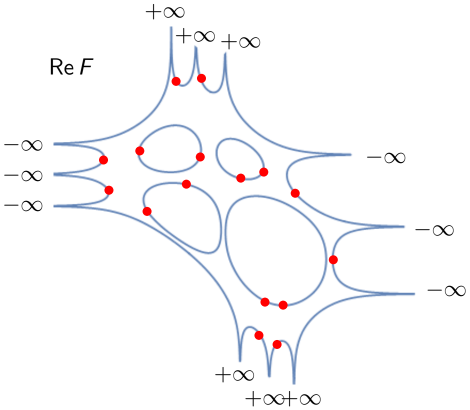
The proof goes via non-intersecting paths, a Wiener–Hopf factorization using a result from B–Duits '19, and a linear flow on the Jacobian of the spectral curve.

# The action function

Let $(\xi, \eta) \in (-1,1)^2$ be global coordinates. The action function $F$ is defined for $q = (z, w) \in \mathcal{R}$ by

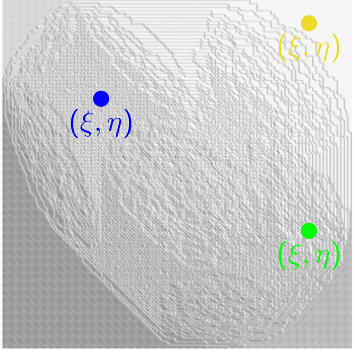
$$F(q; \xi, \eta) = \frac{k}{2}(1-\xi)\log w - \frac{\ell}{2}(1-\eta)\log z - \log \frac{\prod_{i=1}^{\ell} E(q_{0,i}, q)^k}{\prod_{j=1}^{k} E(p_{0,j}, q)^\ell}.$$

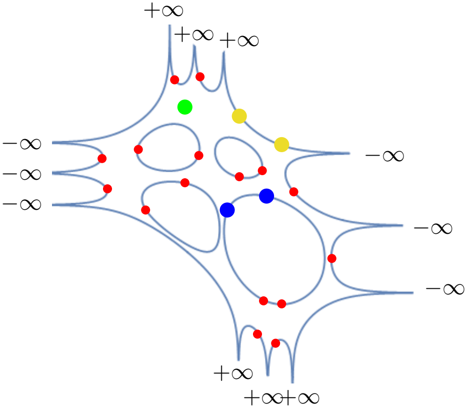
All but two of the critical points of $F$:



# Definition of the frozen, rough and smooth regions

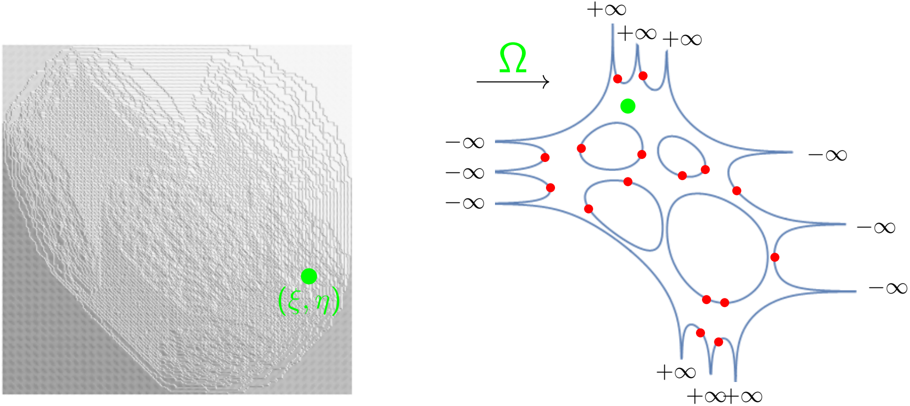
The final two critical points of the $F$ are both real or comes as conjugate pairs. The location of the these critical points determines the phases: $(\xi, \eta)$ is in the frozen region, rough disordered region and smooth disordered region.



# Definition of the map $\Omega$

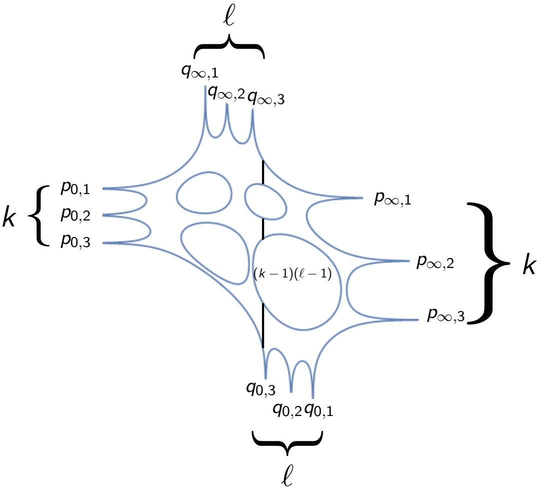
We define the map $\Omega$ from the rough region to the interior of the amoeba.



# Main results

# Assumptions

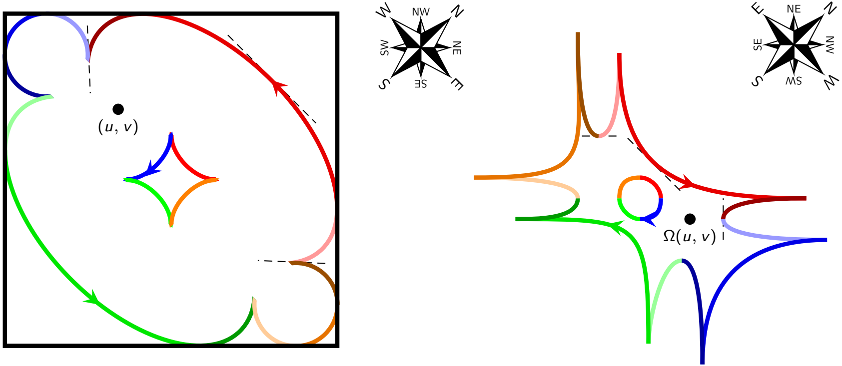
- $\beta_i^v < 1 < \alpha_i^v/\gamma_i^v$ for $i = 1, \ldots, \ell$.
- There are $2k$ horizontal tentacles, $2\ell$ vertical tentacles and $g = (k-1)(\ell-1)$ compact ovals (holds generically.)



# The arctic curve

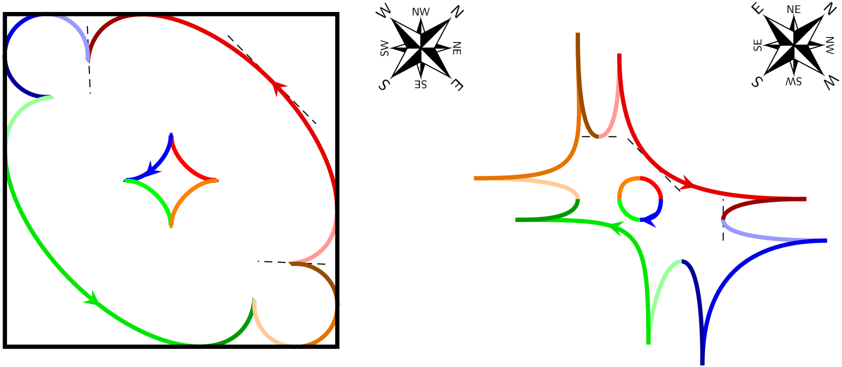
## Theorem (Berggren–Borodin '23)

*The critical point map $\Omega$ is a homeomorphism from the closure of the rough region to the amoeba. Moreover, the induced map between the boundaries, in a correct coordinate system, preserves the slope of the tangent lines of the respective curves.*




The coordinates $(\xi, \eta)$ are chosen so that the scaled Aztec diamond is the square $(-1,1)^2$. The coordinate system in which the homeomorphism of the theorem preserves the slope is given by $u = -\frac{\xi+1}{2\ell}$ and $v = -\frac{\eta+1}{2k}$.

# The arctic curve




- The number of smooth (frozen) regions is equal to the number of bounded (unbounded) components of the complement of the amoeba.
- The rough region is locally convex at all smooth points of the arctic curve.

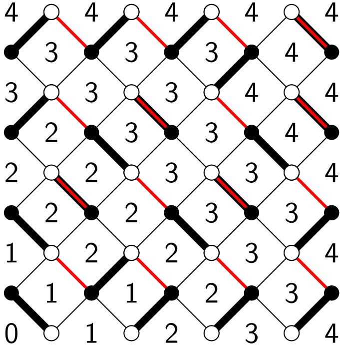
  Known fact by the work of Astala–Duse–Prause–Zhong '20
- The arctic curve has four cusps in each smooth region, and one cusp in each frozen region, except the north, east, south and west frozen regions.

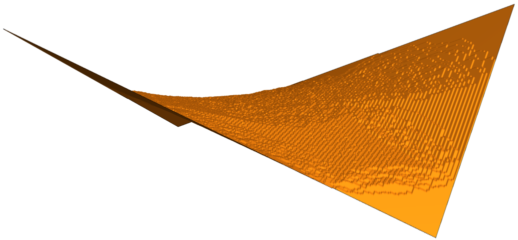
# Height function

If $\mathrm{f}$ and $\mathrm{f}'$ are two faces in $G_{\mathsf{Az}}$ (the Aztec diamond graph), we define the height function $h$ for a dimer covering $M$ so that

$$h(\mathrm{f}') - h(\mathrm{f}) = \sum_{e=\mathrm{wb}} (\pm)\left(1_{e\in M} - 1_{e\in \mathrm{N}}\right),$$

where the sum runs over the edges intersecting the edges of a dual path of $G_{\mathsf{Az}}$ going from $f$ to $f'$, the sign is $+$ if the path intersects the edge $e$ with the white vertex on the right, and $-$ if it is on the left, and $\mathrm{N}$ is the set of north edges.



3D figures

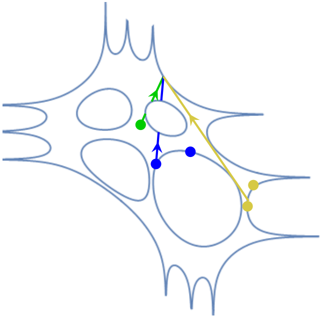
# Limit shape

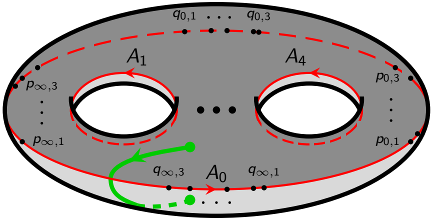
## Theorem (BB23)

*The limit of the normalized height function $\bar{h}$ and its gradient $\nabla \bar{h}$ are given by*

$$\bar{h}(\xi,\eta) = \frac{1}{k\ell}\frac{1}{2\pi \mathrm{i}}\int_{\gamma_{\xi,\eta}} \mathrm{d}F + 1 \quad \textit{and} \quad \nabla \bar{h}(u,v) = \left(\frac{1}{2\pi \mathrm{i}}\int_{\gamma_{\xi,\eta}} \frac{\mathrm{d}w}{w}, -\frac{1}{2\pi \mathrm{i}}\int_{\gamma_{\xi,\eta}} \frac{\mathrm{d}z}{z}\right),$$

*where the curve $\gamma_{\xi,\eta}$ is as indicated in the figure.*



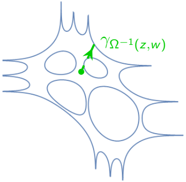


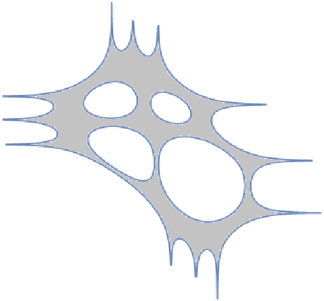

Frozen region, Rough region, Smooth Region.

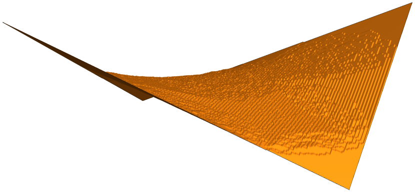
# Limit shape in the rough region

In the rough region, we obtain an explicit parametrization of the limit shape:

$$\mathcal{R}_0 \ni (z, w) \mapsto \left( \Omega^{-1}(z, w), \frac{1}{k\ell} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_{\Omega^{-1}(z,w)}} \mathrm{d}F + 1 \right) \in \mathbb{R}^3.$$



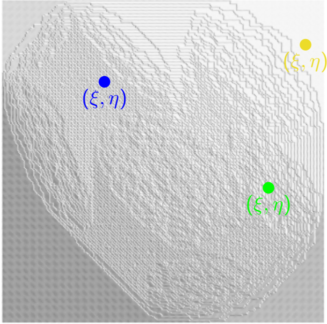


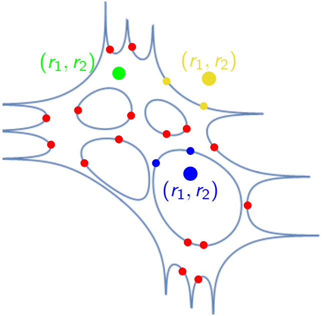
The inverse $\Omega^{-1}(z, w)$ is explicitly given.

# Local fluctuations

## Theorem (BB23)

*The local statistics of the dimer model (away from the arctic curve) converge to those of the ergodic translation-invariant Gibbs measure with slope given by* $\nabla \bar{h}(u, v)$.

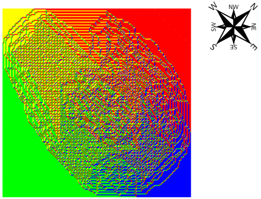


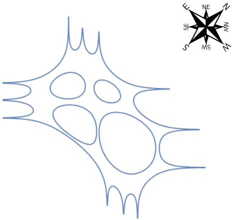

Let $K_{\mathrm{Az}}$ be the Kasteleyn matrix for the Aztec diamond of size $k\ell N$, then

$$\lim_{N\to\infty} K_{\mathrm{Az}}^{-1} = \frac{1}{(2\pi \mathrm{i})^2} \int_{|z|=\mathrm{e}^{r_1}} \int_{|w|=\mathrm{e}^{r_2}} \left(K_{G_1}(z,w)^{-1}\right)_{\mathrm{b}_{i,j}\mathrm{w}_{i',j'}} \frac{z^{\zeta'-\zeta}}{w^{\kappa'-\kappa}} \frac{\mathrm{d}w}{w} \frac{\mathrm{d}z}{z}.$$

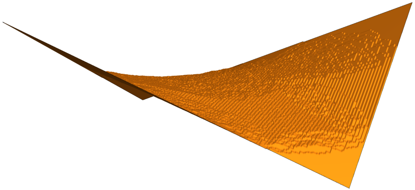
# Recap

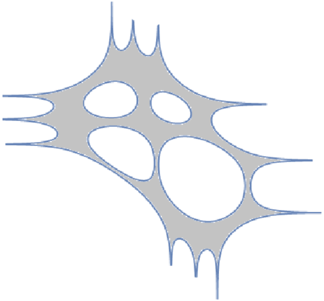


Random domino tilings of the Aztec diamond with periodic edge weights.



The amoeba of the spectral curve.





Thank you for your attention!