Minimal dimers and Maximal surfaces

Cédric Boutillier, David Cimasoni, Béatrice de Tilière

GSI Workshop 1

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… and Laplacians

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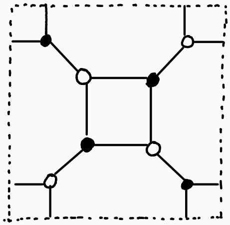
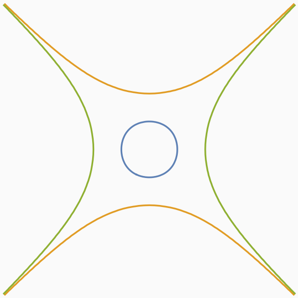
Motivation

Ergodic Gibbs measures for periodic planar bipartite dimers

Kenyon Okounkov Sheffield 2006



- periodic graph + weights
- $P(z,w) = \det K(z,w) = 5 - z - \frac{1}{z} - w - \frac{1}{w}$
- spectral curve $C = \{P(z,w) = 0\}$
- amoeba A: image of C by $(z,w) \mapsto (\log|z|, \log|w|)$.
- A: phase diagram of the dimer model on G
- N: Newton polygon of P



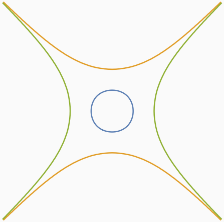
Theorem (Kenyon-Okounkov-Sheffield)

Ergodic Gibbs measures: determinantal processes on edges:

$$\mathbb{P}((w_1,b_1),\dots,(w_k,b_k)\ dimers) = \left(\prod_{i=1}^{k} K_{w_i,b_i}\right) \det_{1\le i,j\le k}(A^{(B_x,B_y)}_{b_i,w_j})$$

- $A^{(B_x,B_y)}$: inverse of K from Fourier analysis,

$$A^{(B_x,B_y)}_{b_i,w_j} = \iint_{\substack{|z|=e^{B_x}\\ |w|=e^{B_y}}} z^{-y} w^{x} \frac{Q(z,w)_{i,j}}{P(z,w)} \frac{dz}{2i\pi z} \frac{dw}{2i\pi w}$$



- 3 phases:
  - inside A: liquid/rough
  - bounded component of A^c: smooth/gas
  - unbounded component of A^c: frozen

The dimer spectral theorem

Kenyon-Okounkov, Goncharov-Kenyon

Theorem

$$\{\textit{dimer models on } G\}\ /\textit{gauge eq.} \overset{\sim}{\leftrightarrow} \{\textit{Harnack curves on } N\textit{+standard divisor}\}$$

- In one direction:
  - from weights, get $P(z, w)$, then C
  - divisor: points where the column $\mathrm{Com}\, K(z, w)_{\circ,\cdot} = 0$

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- In the other direction?

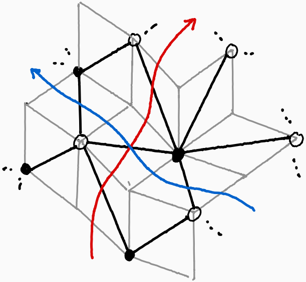
Fock's inverse construction

- In an algebraic setting (no positivity)
- input: algebraic curve + a standard divisor
- output: periodic graph and a Kasteleyn matrix which would give these spectral data
- entries of K are given by theta functions and prime forms on the curve
- square move $\leftrightarrow$ Fay's trisecant identity

What about not periodic graphs?

Kenyon's trigonometric weights on isoradial graphs

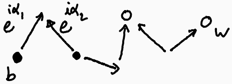
Kenyon 2002



- $K_{\circ,\bullet} = e^{i\beta} - e^{i\alpha}$ satisfies Kasteleyn's rule
- explicit inverse:

$$K^{-1}_{\bullet,\circ} = \frac{1}{4i\pi^2} \oint \prod_j (\lambda - e^{i\alpha_j})^{\pm} \log \lambda d\lambda$$

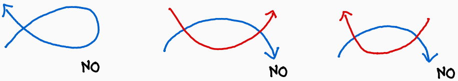
- **locality**: depends only on the geometry of a path from $\circ$ to $\bullet$



Isoradial versus minimal graphs

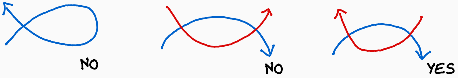
Kenyon-Schlenker

- can always draw the diamond graph of a planar graph G
- G has an isoradial embedding if and only if



Thurston, Postnikov

- more general notion: *minimal* or *reduced*



Fock’s Kasteleyn operator on minimal graphs

our goal

- work with all minimal bipartite graphs simultaneously
- replace $\hat{\mathbb{C}}$ by higher genus surface Σ
  - "maximal curve"
  - plays the role of spectral curve, given a priori
- extra data (real point on the Jacobian)
- look at Fock's Kasteleyn operator
  - family of inverses
  - probabilistic quantities read on Σ
- alternative hierarchy of complexity
  (genus of Σ vs. size of fund. domain)

Maximal surface Σ

- compact Riemann surface of genus $g \geq 1$
- antiholomorphic involution σ
- fixed points of σ: $g+1$ circles $A_0, \dots, A_g$
- $A_1, \dots, A_g, B_1, \dots, B_g$ homology basis
- (u_j) holomorphic forms adapted to (A_i)

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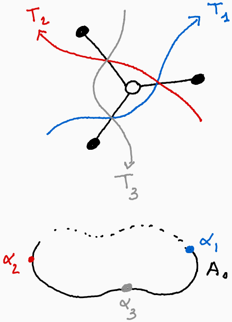
- (u_j) holomorphic forms adapted to (A_i)
- $\Omega = (\Omega_{ij}) = (\int_{B_i} u_j)$
- $\theta(z) = \sum_{n \in \mathbb{Z}^g} e^{i\pi(n \cdot \Omega n + 2n \cdot z)}$, $z \in \mathbb{C}^g$
- prime form $E(u, v)$
- Fay's trisecant identity

$$\frac{\theta(s+u-\alpha-\beta)E(\alpha,\beta)}{E(\alpha,u)E(\beta,u)\theta(s-\alpha)\theta(s-\beta)} + \frac{\theta(s+u-\beta-\gamma)E(\beta,\gamma)}{E(\beta,u)E(\gamma,u)\theta(s-\beta)\theta(s-\gamma)} + \frac{\theta(s+u-\gamma-\alpha)E(\gamma,\alpha)}{E(\alpha,u)E(\gamma,u)\theta(s-\alpha)\theta(s-\gamma)} = 0$$

Parameters on minimal graphs

- Minimality $\Rightarrow$ partial cyclic order on the strands
- Each strand T gets a parameter $\alpha_T \in A_0$



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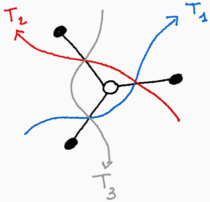

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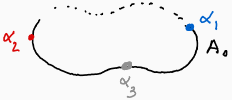
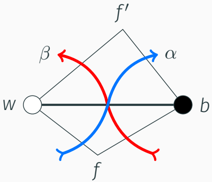


- discrete Abel map

$$d(b) = d(f') + \alpha = d(f) + \beta = d(w) + \alpha + \beta$$



Fock's Kasteleyn operator

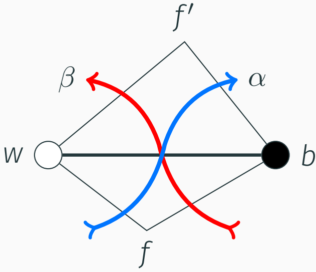
$$K_{w,b} = \frac{E(\alpha, \beta)}{\theta(t + d(f))\theta(t + d(f'))}$$

Lemma

if

- Σ *is maximal*
- $T \mapsto \alpha_T$ *is monotone*
- $t \in \mathbb{R}^g$

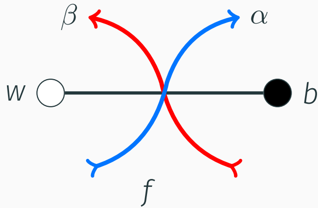
then K satisfies Kasteleyn's rule



Functions in the kernel of K

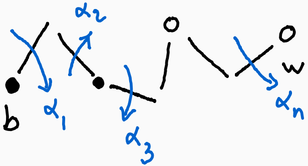
$$g_{b,f}(u) = \frac{\theta(-t+u-d(b))}{E(\beta,u)} = g_{f,b}(u)^{-1}$$

$$g_{f,w}(u) = \frac{\theta(t+u+d(w))}{E(\alpha,u)} = g_{w,f}(u)^{-1}$$



$b = x_0, x_1, \ldots, x_n = w$

$$g_{b,w}(u) = \prod_{j=0}^{n-1} g_{x_j,x_{j+1}}(u)$$



Proposition

- *$u \mapsto g_{b,w}(u)$ meromorphic 1-form on Σ*
- $\sum_w g_{b,w}(u) K_{w,b'} = 0, \qquad \sum_b K_{w,b} g_{b,w'}(u) = 0$

Functions in the kernel of K

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- *$u \mapsto g_{b,w}(u)$ meromorphic 1-form on Σ*
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Proof.

Fay's trisecant identity $\Rightarrow K_{w,b} g_{b,w}$ is telescopic $\square$

Inverse(s) of K

Theorem

Let $u_0 \in \Sigma^+$

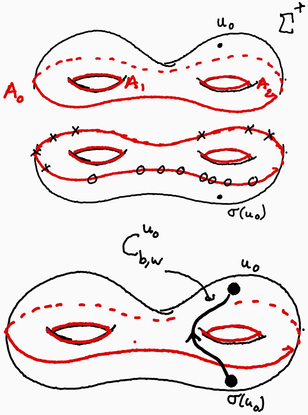
$$A^{u_0}_{b,w} = \frac{1}{2i\pi} \int_{C^{u_0}_{b,w}} g_{b,w}(u) \text{ is an inverse of } K$$

- *locality property*
- *under condition* $(\star)$

$$\left(\prod_{j=1}^{k} K_{w_j,b_j} \right) \det_{1 \le i,j \le k} (A^{u_0}_{b_i,w_j})$$

defines a probability measure on dimers

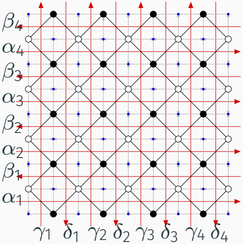
- *if* $u_0 \in A_0$*: frozen*
- *if* $u_0 \in A_j$*: smooth/gas*
- *otherwise: rough/liquid*

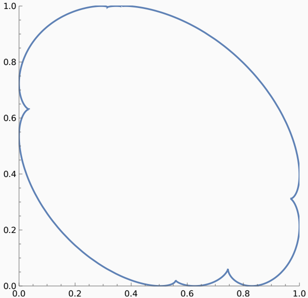


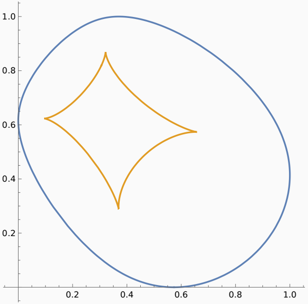
Aztec Diamond with Fock's weights

[B.-de Tilière]

There is a double integral formula on Σ for the inverse of the Kasteleyn matrix of the Aztec diamond constructed from the blocks g_{bw}, and E, which generalizes the formula of Berggren and Borodin.







Laplacians on isoradial graphs

Spectral theorem for Laplacians

$$\Delta f(x) = \sum_{y \sim x} c_{xy}(f(x) - f(y))$$

Theorem (T. George)

$$\{\textit{biperiodic planar electrical networks}\}/\textit{elem. transf.}$$

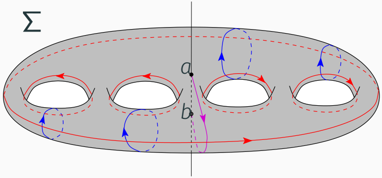
$$\overset{\sim}{\longleftrightarrow}$$

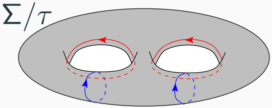
$$\{\textit{Harnack curves of even genus with } (z,w) \mapsto (\frac{1}{z}, \frac{1}{w}) \textit{ involution}$$
$$+ \textit{ divisor in the Prym variety}\}$$

- conductances in terms of Prym theta functions
- $Y - \Delta$ transformation $\leftrightarrow$ Fay's quadrisectant identity
- Tempting to reproduce the construction in this context

Geometric setting for Laplacians on infinite planar graphs

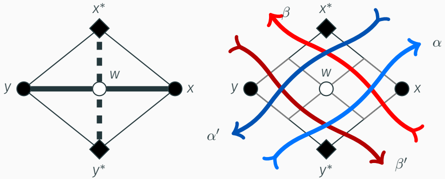
- Σ maximal Riemann surface of genus $2g$
- σ antiholomorphic involution, $A_0, A_1, \ldots, A_{2g}$
- τ holomorphic involution with two fixed points $a, b \notin A_j$
- τ and σ commute
- Σ/τ is also a maximal Riemann surface
- $e \in \mathbb{R}^g$





Relation between K and Δ

- Temperley's bijection between spanning trees on G and dimers on the medial graph G_D
- G_D minimal $\Rightarrow$ G (and thus G_D) Kenyon-Schlenker



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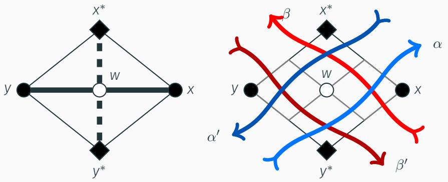
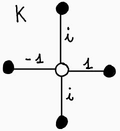




- On $\mathbb{Z}^2$, $K^* \cdot K = \Delta_G \oplus \Delta_{G^*}$
- On G, with (slightly modified) George's conductances, there are **local** gauge D_W, D_B such that $L = D_W K D_B$ and

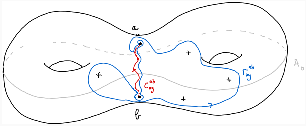
$$L^* L = \Delta_G \oplus \Delta_{G^*}$$

Green function

- Define special harmonic functions $h_{xy}(u)$ from $g_{xy}(u)$ and gauges [related to algebro-geometric solution of B-quadrilaterals/discrete BKP by Doliwa]
- For every vertex y, define a 1-form ω_y with good properties
- Define $\mathcal{G}$ the Green function

$$\mathcal{G}_{x,y} = \frac{1}{2i\pi} \int_{C_{xy}} h_{xy}(u)\omega_y$$

- $\mathcal{G}_{x,y} = \frac{1}{2\pi} \log|\Psi(y) - \Psi(x)| + O(1)$
- locality



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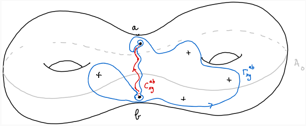
- locality
- measure for spanning trees
- free energy, etc.



Work in progress and open questions

- Similar setting for odd maximal surfaces with involution (massive Laplacians)
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- Ising model(?)
- How to define/select inverses with a probabilistic meaning?
- Drawing of G (or G^*) from asymptotics of inverses of K or Δ related to t-embeddings?