Minimal dimers and Maximal surfaces

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GSI Workshop 1
Minimal dimers and Maximal surfaces
... and Laplacians

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Motivation
Ergodic Gibbs measures for periodic planar bipartite dimers

Kenyon Okounkov Sheffield 2006

- periodic graph + weights
  \[ P(z, w) = \det K(z, w) = 5 - z - \frac{1}{z} - w - \frac{1}{w} \]
- spectral curve \( C = \{ P(z, w) = 0 \} \)
- amoeba \( A \): image of \( C \) by \( (z, w) \mapsto (\log |z|, \log |w|) \).
- \( A \): phase diagram of the dimer model on \( G \)
- \( N \): Newton polygon of \( P \)
Theorem (Kenyon-Okounkov-Sheffield)

Ergodic Gibbs measures: determinantal processes on edges:

\[ \mathbb{P}((w_1, b_1), \ldots, (w_k, b_k) \text{ dimers}) = \left( \prod_{i=1}^{k} K_{w_i, b_i} \right) \det_{1 \leq i, j \leq k} \left( A_{b_i, w_j}^{(B_x, B_y)} \right) \]

- \( A^{(B_x, B_y)} \): inverse of \( K \) from Fourier analysis,

\[ A_{b_i, w_j}^{(B_x, B_y)} = \int \int_{|z|=e^{B_x}} |w|=e^{B_y} \frac{z^{-y} w^{x} Q(z, w)_{i,j}}{P(z, w)} \frac{dz}{2i\pi z} \frac{dw}{2i\pi w} \]

- 3 phases:
  - inside \( A \): liquid/rough
  - bounded component of \( A^c \): smooth/gas
  - unbounded component of \( A^c \): frozen
# The dimer spectral theorem

Kenyon-Okounkov, Goncharov-Kenyon

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- In one direction:
  - from weights, get \( P(z, w) \), then \( C \)
  - divisor: points where the column \( \text{Com} K(z, w)_o \cdot = 0 \)
The dimer spectral theorem

Kenyon-Okounkov, Goncharov-Kenyon

**Theorem**

\[ \{ \text{dimer models on } G \} / \text{gauge eq.} \]

\[ \leftrightarrow \]

\[ \{ \text{Harnack curves on } N + \text{standard divisor} \} \]

- In one direction:
  - from weights, get \( P(z, w) \), then \( C \)
  - divisor: points where the column \( \text{Com} K(z, w)_o \cdot = 0 \)

- In the other direction?
Fock’s inverse construction

- In an algebraic setting (no positivity)
- input: algebraic curve + a standard divisor
- output: periodic graph and a Kasteleyn matrix which would give these spectral data
- entries of $K$ are given by theta functions and prime forms on the curve
- square move $\leftrightarrow$ Fay’s trisecant identity
What about not periodic graphs?
Kenyon’s trigonometric weights on isoradial graphs

Kenyon 2002

• $K_{\circ,\bullet} = e^{i\beta} - e^{i\alpha}$ satisfies Kasteleyn’s rule
• explicit inverse:

$$K_{\circ,\bullet}^{-1} = \frac{1}{4i\pi^2} \oint \prod_{j} (\lambda - e^{i\alpha_j})^{\pm} \log \lambda d\lambda$$

• locality: depends only on the geometry of a path from $\circ$ to $\bullet$
Kenyon-Schlenker

- can always draw the diamond graph of a planar graph $G$
- $G$ has an isoradial embedding if and only if

Thurston, Postnikov

- more general notion: *minimal* or *reduced*
Fock’s Kasteleyn operator on minimal graphs
our goal

- work with all minimal bipartite graphs simultaneously
- replace $\hat{C}$ by higher genus surface $\Sigma$
  - “maximal curve”
  - plays the role of spectral curve, given a priori
- extra data (real point on the Jacobian)
- look at Fock’s Kasteleyn operator
  - family of inverses
  - probabilistic quantities read on $\Sigma$
- alternative hierarchy of complexity
  (genus of $\Sigma$ vs. size of fund. domain)
Maximal surface $\Sigma$

- compact Riemann surface of genus $g \geq 1$
- antiholomorphic involution $\sigma$
- fixed points of $\sigma$: $g + 1$ circles $A_0, \ldots, A_g$
- $A_1, \ldots, A_g, B_1, \ldots, B_g$ homology basis
- $(u_j)$ holomorphic forms adapted to $(A_i)$
Maximal surface $\Sigma$

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- $(u_j)$ holomorphic forms adapted to $(A_i)$
- $\Omega = (\Omega_{ij}) = (\int_{B_i} u_j)$
- $\theta(z) = \sum_{n \in \mathbb{Z}^g} e^{i\pi(n \cdot \Omega + 2n \cdot z)}$, $z \in \mathbb{C}^g$
- prime form $E(u, v)$
- Fay's trisecant identity

\[
\frac{\theta(s + u - \alpha - \beta)E(\alpha, \beta)}{E(\alpha, u)E(\beta, u)} + \frac{\theta(s + u - \beta - \gamma)E(\beta, \gamma)}{E(\beta, u)E(\gamma, u)} + \frac{\theta(s + u - \gamma - \alpha)E(\gamma, \alpha)}{E(\alpha, u)E(\gamma, u)} = 0
\]
- Minimality $\Rightarrow$ partial cyclic order on the strands
- Each strand $T$ gets a parameter $\alpha_T \in A_0$
Parameters on minimal graphs

- Minimality $\Rightarrow$ partial cyclic order on the strands
- Each strand $T$ gets a parameter $\alpha_T \in A_0$

- Discrete Abel map

\[ d(b) = d(f') + \alpha = d(f) + \beta = d(w) + \alpha + \beta \]
Fock’s Kasteleyn operator

\[ K_{w,b} = \frac{E(\alpha, \beta)}{\theta(t + d(f))\theta(t + d(f'))} \]

Lemma

if

- \( \Sigma \) is maximal
- \( T \mapsto \alpha_T \) is monotone
- \( t \in \mathbb{R}^g \)

then \( K \) satisfies Kasteleyn’s rule
Functions in the kernel of $K$

\[ g_{b,f}(u) = \frac{\theta(-t + u - d(b))}{E(\beta, u)} = g_{f,b}(u)^{-1} \]

\[ g_{f,w}(u) = \frac{\theta(t + u + d(w))}{E(\alpha, u)} = g_{w,f}(u)^{-1} \]

\[ b = x_0, x_1, \ldots, x_n = w \]

\[ g_{b,w}(u) = \prod_{j=0}^{n-1} g_{x_j, x_{j+1}}(u) \]

Proposition

- $u \mapsto g_{b,w}(u)$ meromorphic 1-form on $\Sigma$
- $\sum_w g_{b,w}(u)K_w, b' = 0$, $\sum_b K_w, b g_{b,w}(u) = 0$
Proposition

- \( u \mapsto g_{b,w}(u) \) meromorphic 1-form on \( \Sigma \)
- \( \sum_w g_{b,w}(u)K_{w,b'} = 0, \quad \sum_b K_{w,b}g_{b,w'}(u) = 0 \)

Proof.

Fay’s trisecant identity \( \Rightarrow K_{w,b}g_{b,w} \) is telescopic \( \square \)
Inverse(s) of $K$

**Theorem**

Let $u_0 \in \Sigma^+$

$$A_{b,w}^{u_0} = \frac{1}{2i\pi} \int_{C_{b,w}^{u_0}} g_{b,w}(u)$$ is an inverse of $K$

- locality property
- under condition (⋆)

$$\left( \prod_{j=1}^{k} K_{w_j,b_j} \right) \det \left( A_{b_i,w_j}^{u_0} \right)_{1 \leq i,j \leq k}$$

defines a probability measure on dimers

- if $u_0 \in A_0$: frozen
- if $u_0 \in A_j$: smooth/gas
- otherwise: rough/liquid
Aztec Diamond with Fock’s weights

[B.-de Tilière]
There is a double integral formula on $\Sigma$ for the inverse of the Kasteleyn matrix of the Aztec diamond constructed from the blocks $g_{bw}$, and $E$, which generalizes the formula of Berggren and Borodin.
Laplacians on isoradial graphs
Spectral theorem for Laplacians

\[ \Delta f(x) = \sum_{y \sim x} c_{xy} (f(x) - f(y)) \]

Theorem (T. George)

\{ biperiodic planar electrical networks \}/elem. transf.

\[ \sim \]

\{ Harnack curves of even genus with \((z, w) \mapsto \left( \frac{1}{z}, \frac{1}{w} \right)\) involution

\[ + \text{ divisor in the Prym variety} \} \]

- conductances in terms of Prym theta functions
- \(Y - \Delta\) transformation \(\leftrightarrow\) Fay’s quadrisection identity
- Tempting to reproduce the construction in this context
Geometric setting for Laplacians on infinite planar graphs

- $\Sigma$ maximal Riemann surface of genus 2$g$
- $\sigma$ antiholomorphic involution, $A_0, A_1, \ldots, A_{2g}$
- $\tau$ holomorphic involution with two fixed points $a, b \notin A_j$
- $\tau$ and $\sigma$ commute
- $\Sigma/\tau$ is also a maximal Riemann surface
- $e \in \mathbb{R}^g$
Relation between $K$ and $\Delta$

- Temperley’s bijection between spanning trees on $G$ and dimers on the medial graph $G_D$
- $G_D$ minimal $\Rightarrow$ $G$ (and thus $G_D$) Kenyon-Schlenker
Relation between \( K \) and \( \Delta \)

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On \( \mathbb{Z}^2 \), \( K^* \cdot K = \Delta_G \oplus \Delta_{G^*} \)

On \( G \), with (slightly modified) George’s conductances, there are local gauge \( D_W, D_B \) such that \( L = D_W K D_B \) and

\[
L^* L = \Delta_G \oplus \Delta_{G^*}
\]
Green function

• Define special harmonic functions $h_{xy}(u)$ from $g_{xy}(u)$ and gauges [related to algebro-geometric solution of $B$-quadrilaterals/discrete BKP by Doliwa]

• For every vertex $y$, define a 1-form $\omega_y$ with good properties

• Define $G$ the Green function

$$G_{x,y} = \frac{1}{2i\pi} \int_{C_{xy}} h_{xy}(u)\omega_y$$

• $G_{x,y} = \frac{1}{2\pi} \log |\Psi(y) - \Psi(x)| + O(1)$

• locality
Green function

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- $G_{x,y} = \frac{1}{2\pi} \log |\psi(y) - \psi(x)| + O(1)$
- locality
- measure for spanning trees
- free energy, etc.
Work in progress and open questions

• Similar setting for odd maximal surfaces with involution (massive Laplacians)
• Ising model(?)
Work in progress and open questions

- Similar setting for odd maximal surfaces with involution (massive Laplacians)
- Ising model (?)
- How to define/select inverses with a probabilistic meaning?
Work in progress and open questions

- Similar setting for odd maximal surfaces with involution (massive Laplacians)
- Ising model(?)
- How to define/select inverses with a probabilistic meaning?
- Drawing of $G$ (or $G^*$) from asymptotics of inverses of $K$ or $\Delta$ related to $t$-embeddings?