Geometric objects associated to planar bipartite graphs

Pavel Galashin (UCLA)

IPAM Workshop “Statistical Mechanics and Discrete Geometry”
March 26, 2024

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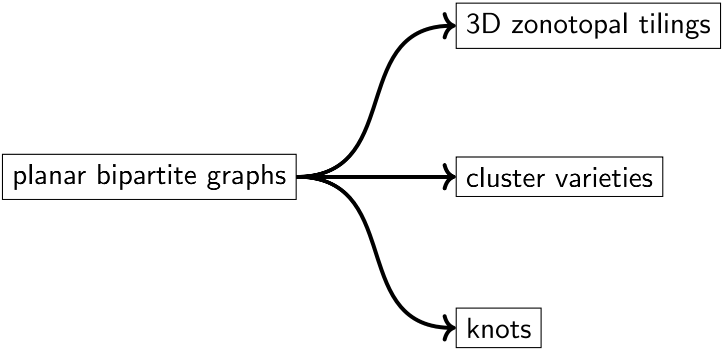

planar bipartite graphs
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cluster varieties
knots

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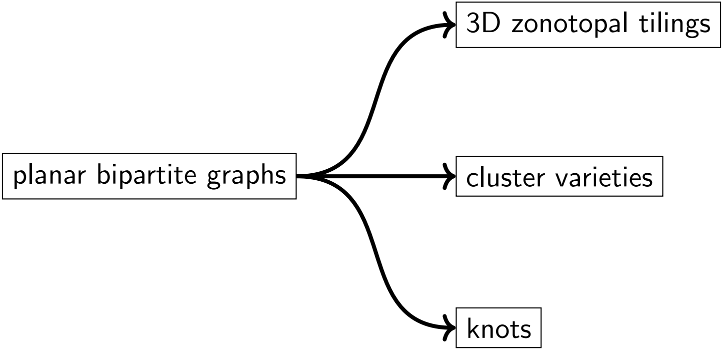

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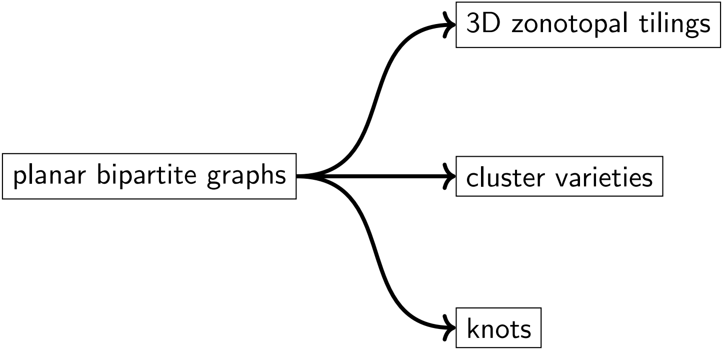
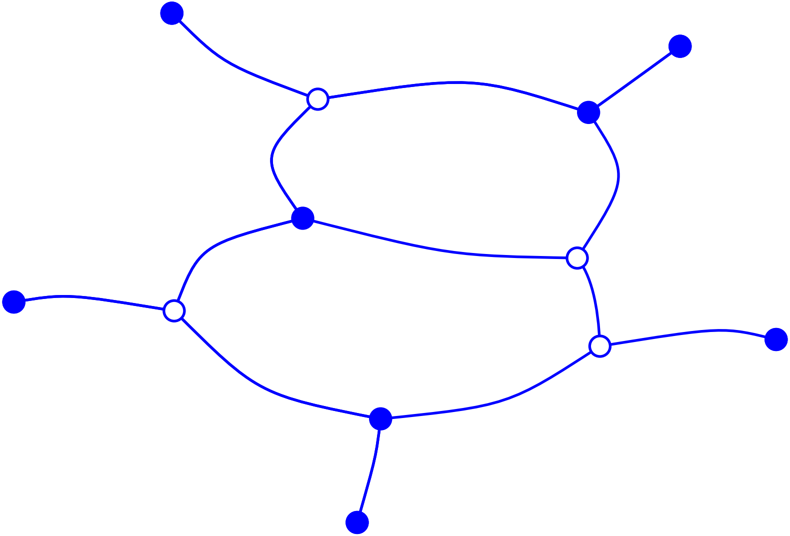


- Mostly will focus on reduced bipartite graphs embedded in a disk.
- Other interesting classes to keep in mind are non-reduced graphs and graphs on a torus and other surfaces.

Part 1: Plabic graphs



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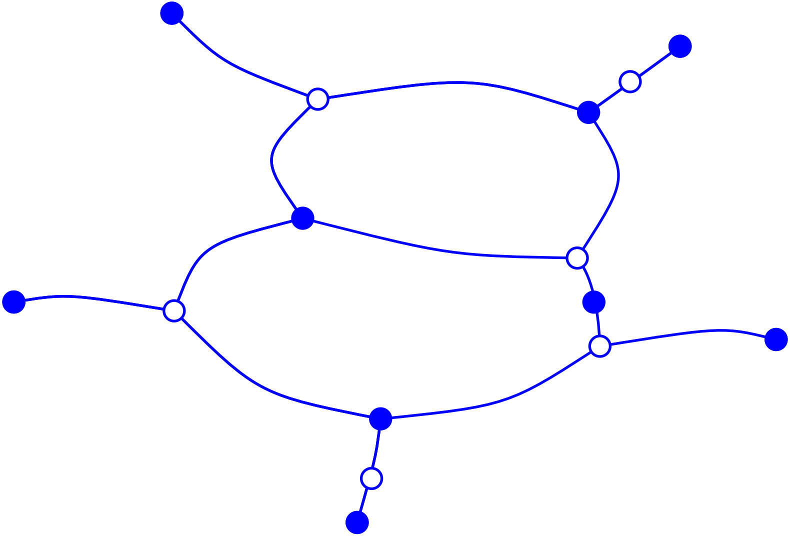

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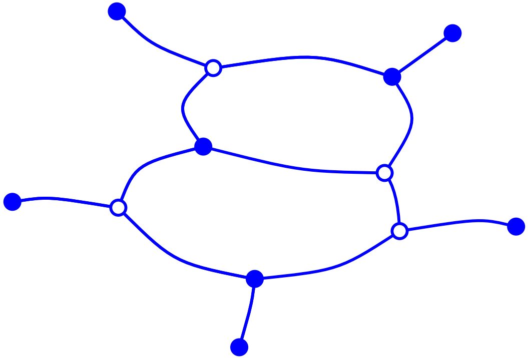
A *plabic graph* is a planar graph embedded in a disk, with n boundary vertices of degree 1, and the remaining vertices colored black and white.



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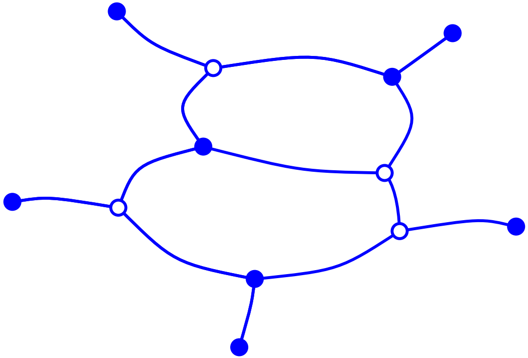

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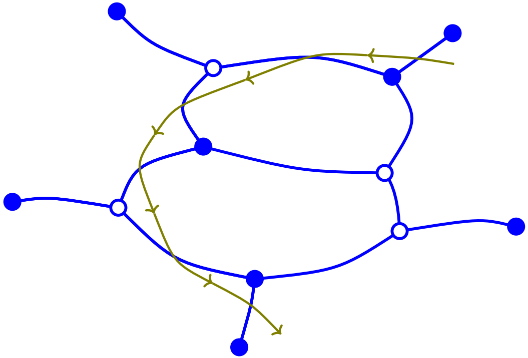

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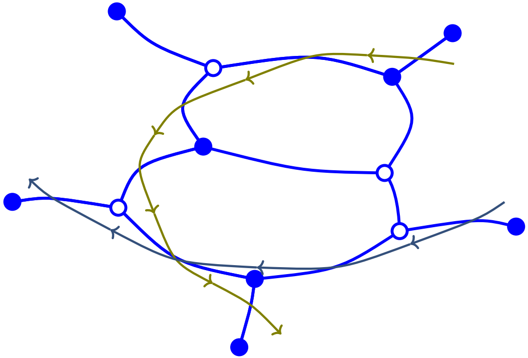

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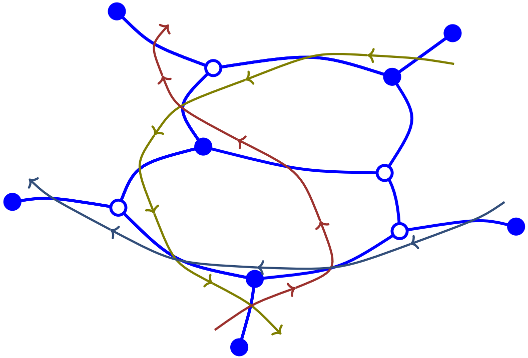

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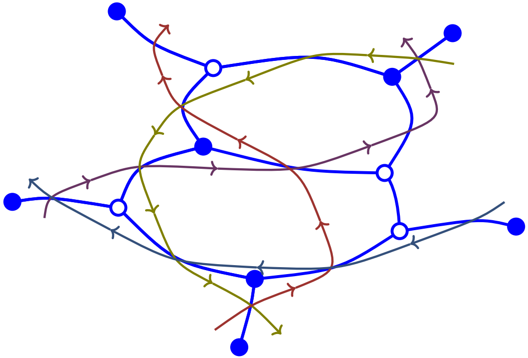

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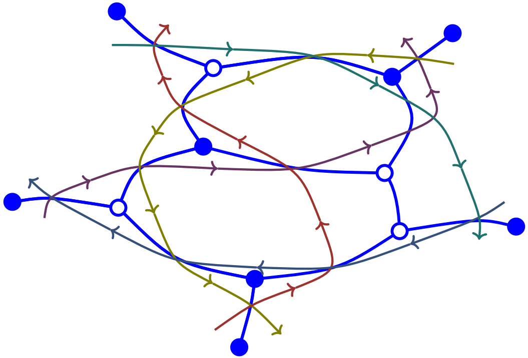

(k, n)-plabic graphs

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A plabic graph is *reduced* if it satisfies:

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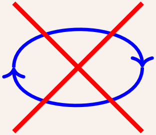

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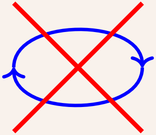

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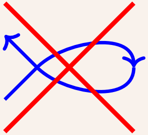

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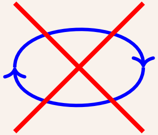

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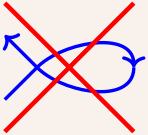

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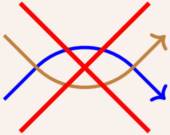

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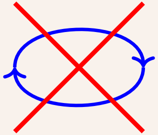

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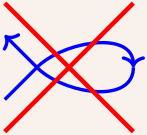

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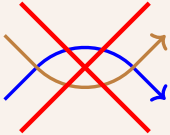

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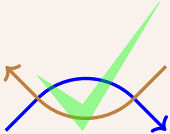

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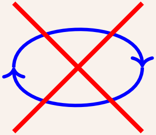

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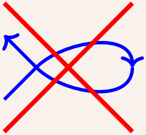

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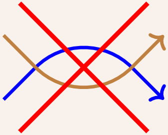

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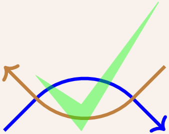

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A *(k, n)-plabic graph* is a reduced plabic graph such that:

- the strand that starts at i ends at $i + k$ modulo n for all i.

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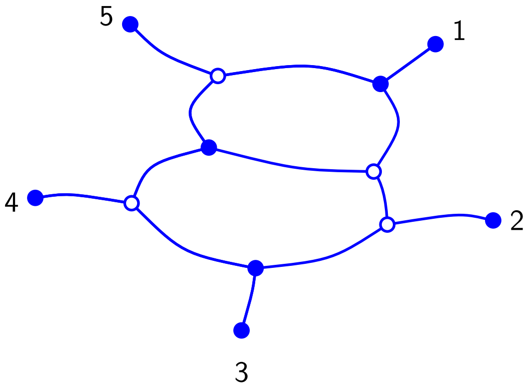

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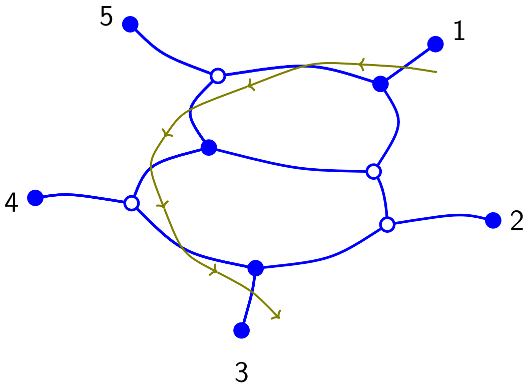

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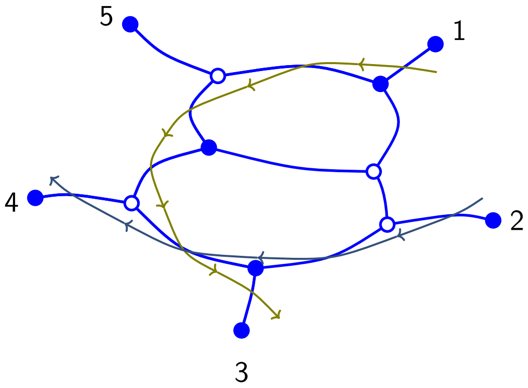

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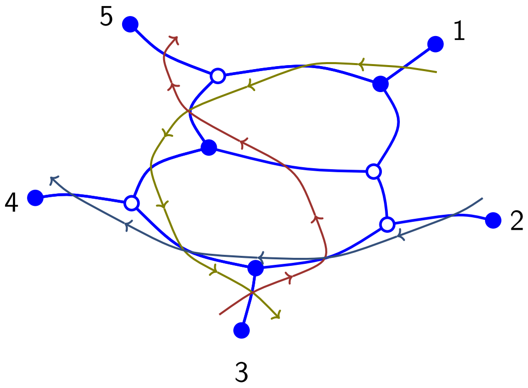

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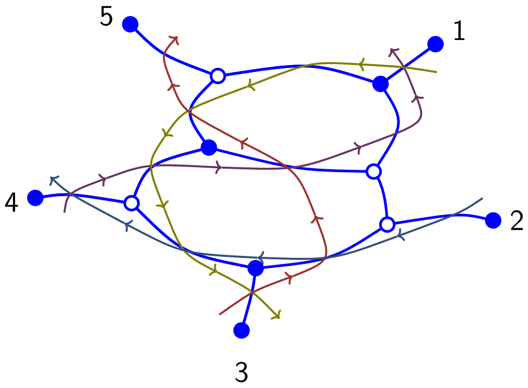

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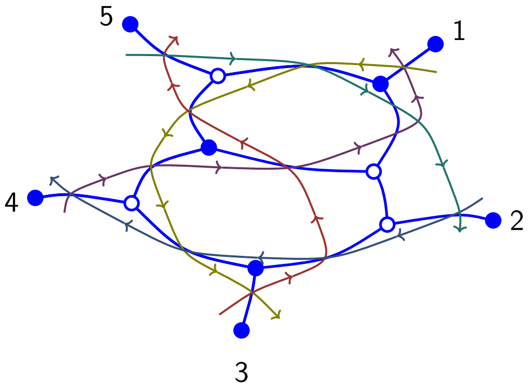
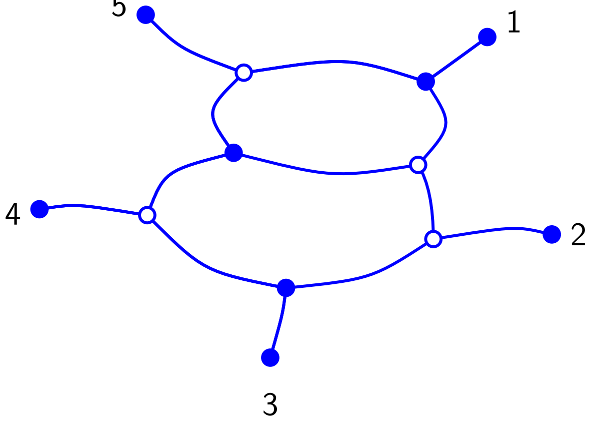
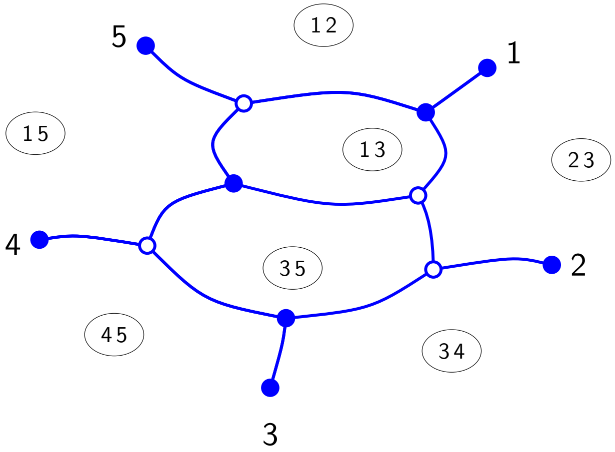


Face labels



Face labels

Label each face of a (k, n)-plabic graph by a k-element set:



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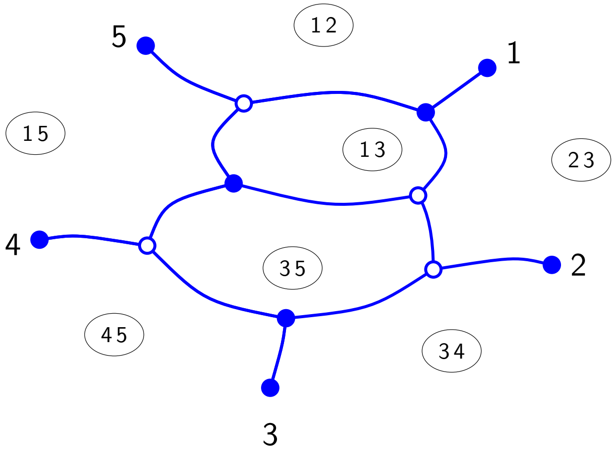

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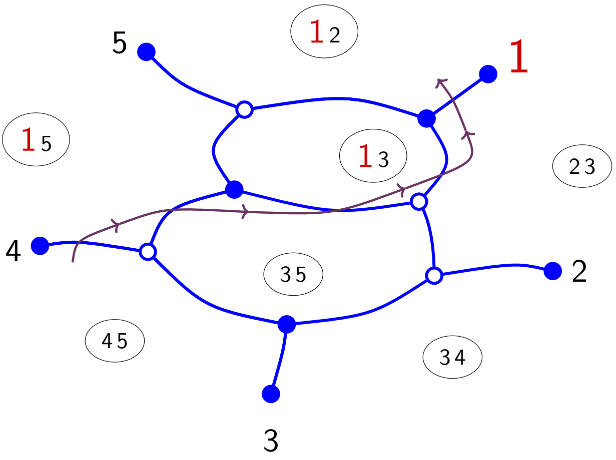

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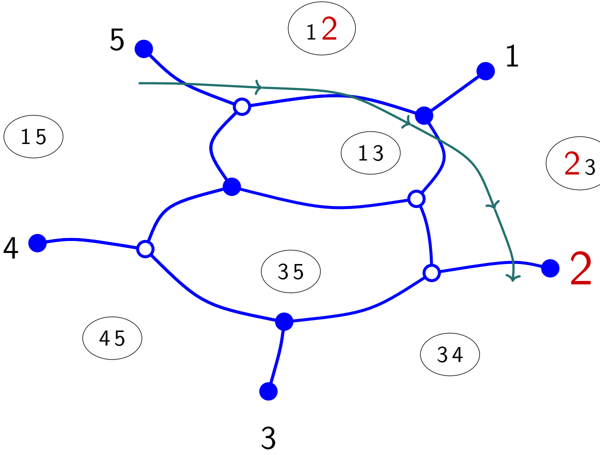

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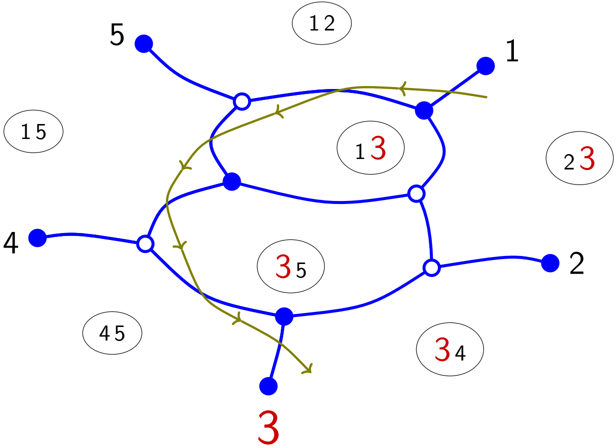

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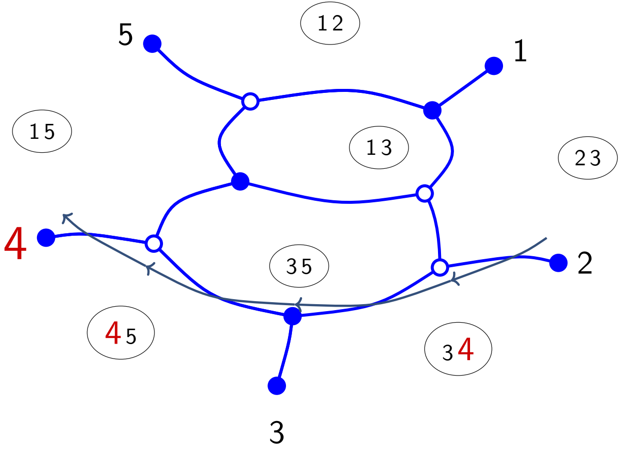

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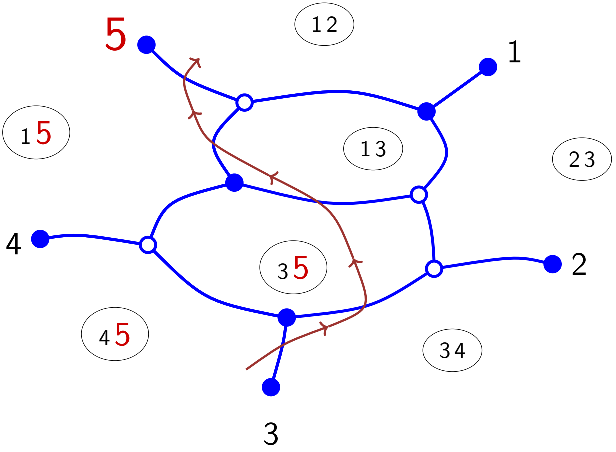

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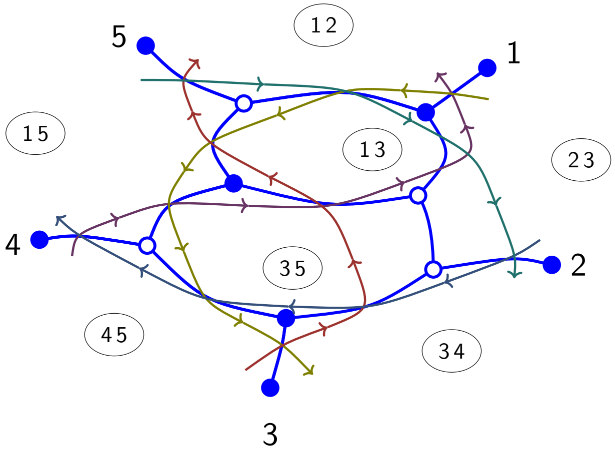

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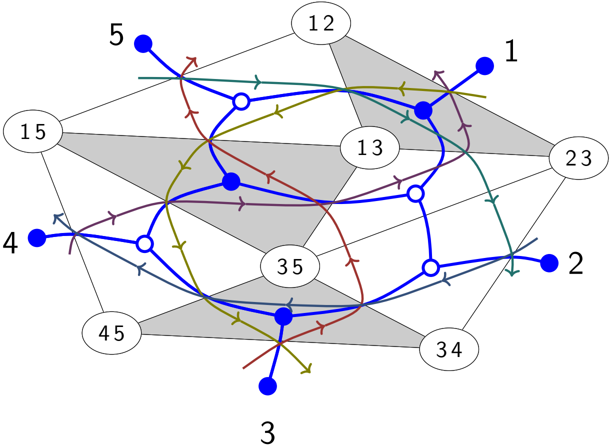
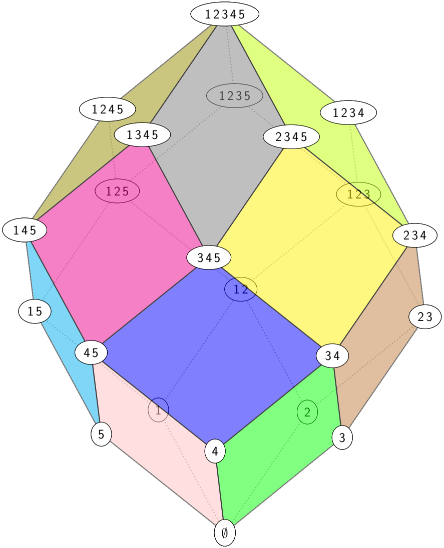


Related work: [Oh–Postnikov–Speyer], [Danilov–Karzanov–Koshevoy], [Leclerc–Zelevinsky].

Part 1: Zonotopal tilings



Zonotopes

Definition (Minkowski sum)

$$A, B \subseteq \mathbb{R}^d, \quad A + B := \{a + b \mid a \in A, b \in B\}.$$

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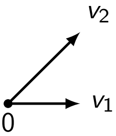
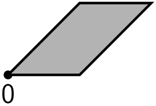
Zonotope:

$$\mathcal{Z}_{\mathbf{V}} := [0, v_1] + [0, v_2] + \cdots + [0, v_n] \subseteq \mathbb{R}^d.$$

Two-dimensional zonotopes

$\mathbf{V} =$ 

$\mapsto$

$\mathcal{Z}_{\mathbf{V}} =$ 

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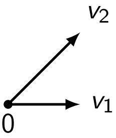
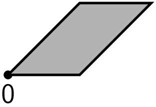
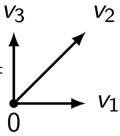
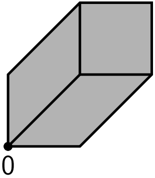
$\mathbf{V} =$ 

$\mapsto$ $\mathcal{Z}_{\mathbf{V}} =$ 

$\mathbf{V} =$ 

$\mapsto$ $\mathcal{Z}_{\mathbf{V}} =$ 

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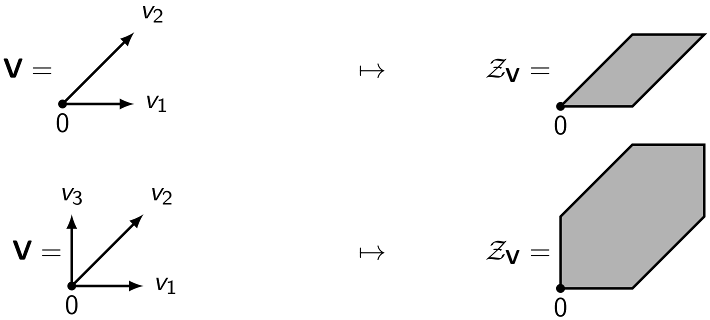

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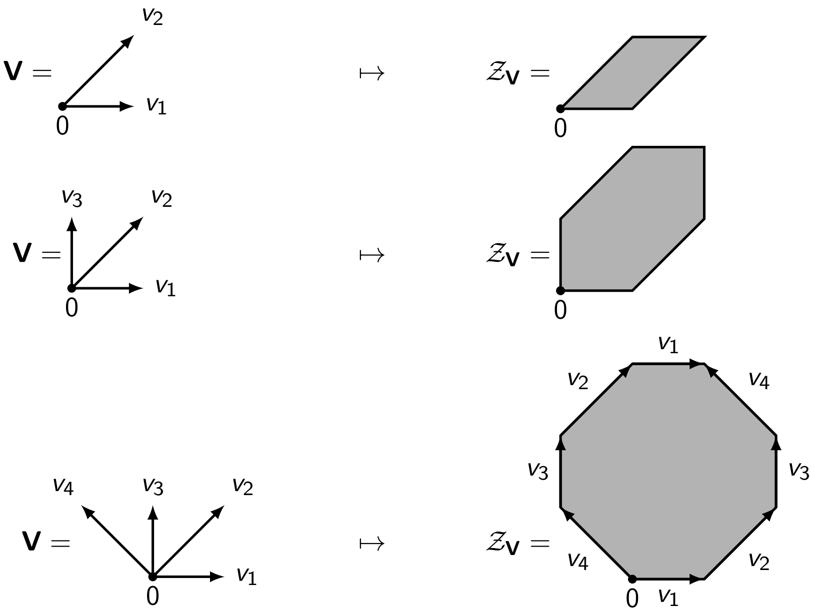
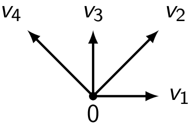


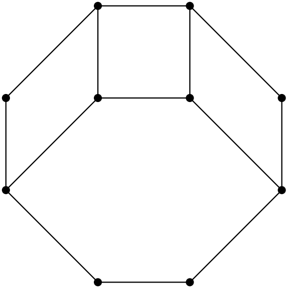
Zonotopal tilings

Definition

A *fine zonotopal tiling* of $\mathcal{Z}_{\mathbf{V}}$ is a polyhedral subdivision Δ of $\mathcal{Z}_{\mathbf{V}}$ into zonotopes of the form $\sum_{i \in B}[0, v_i]$, where $\{v_i \mid i \in B\}$ form a basis of $\mathbb{R}^d$.



$\mathbf{V}$



Δ (not fine)

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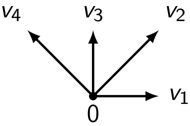
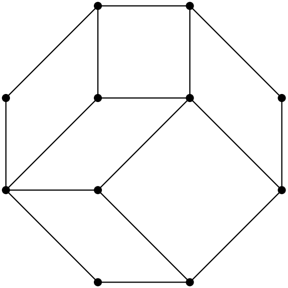


$\mathbf{V}$



Δ (fine)

Zonotopal tilings

Definition

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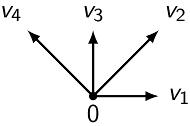
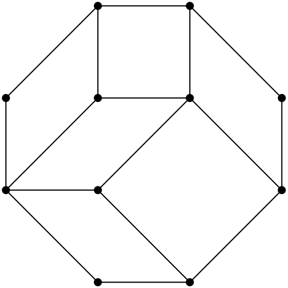
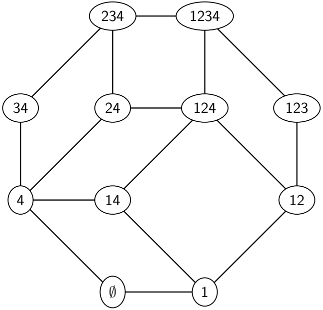


$\mathbf{V}$



Δ (fine)



Vert(Δ)$\subseteq 2^{[n]}$

Zonotopal tilings

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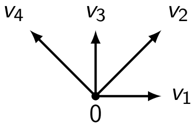
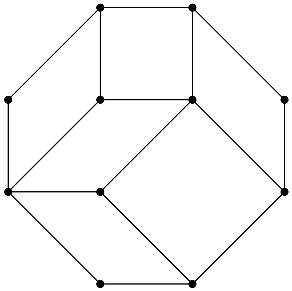
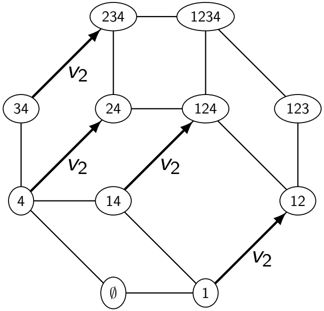


$\mathbf{V}$

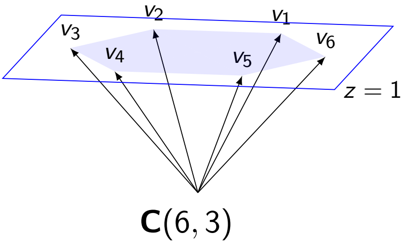


Δ (fine)



$\mathrm{Vert}(\Delta) \subseteq 2^{[n]}$

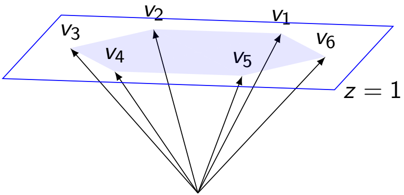
3D zonotopes



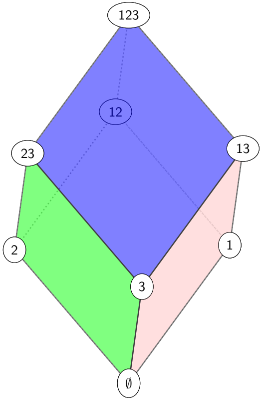
Definition

- $\mathbf{C}(n, 3)$: endpoints of $v_1, v_2, \ldots, v_n$ form a convex n-gon in the $z = 1$ plane.

3D zonotopes



$\mathbf{C}(6,3)$

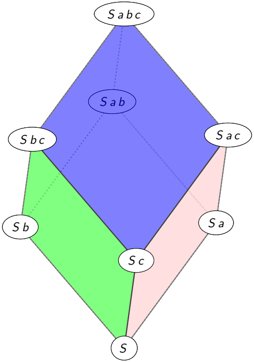


$\mathcal{Z}(3,3)$

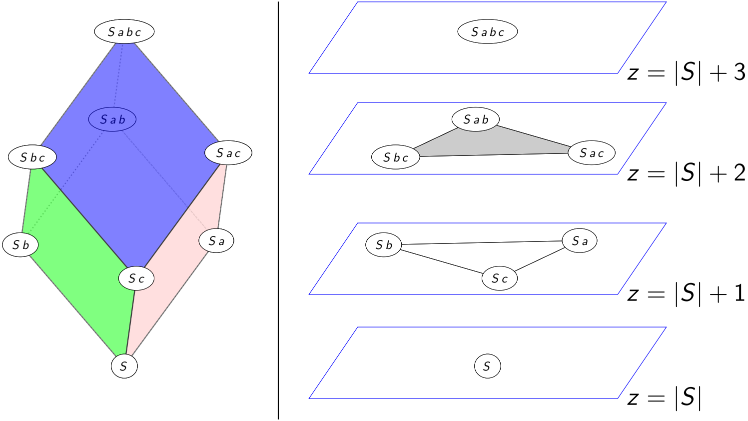
Definition

- $\mathbf{C}(n,3)$: endpoints of $v_1, v_2, \ldots, v_n$ form a convex n-gon in the $z = 1$ plane.
- $\mathcal{Z}(n,3) := \mathcal{Z}_{\mathbf{C}(n,3)}$.

Sections of tiles



Sections of tiles



Plabic graphs vs zonotopal tilings

Theorem (G.)

trivalent (k, n)*-plabic graphs* $\xleftrightarrow[\textit{dual}]{\textit{planar}}$ *horizontal sections at level* k *of fine zonotopal tilings of* $\mathcal{Z}(n, 3)$

Plabic graphs vs zonotopal tilings

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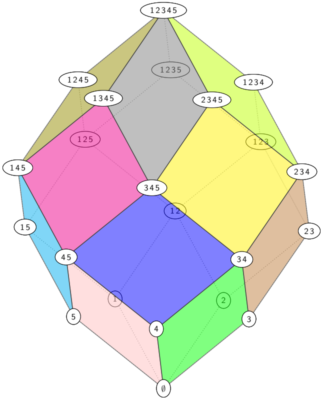

$\mathcal{Z}(n, 3)$ for $n = 5$

Plabic graphs vs zonotopal tilings

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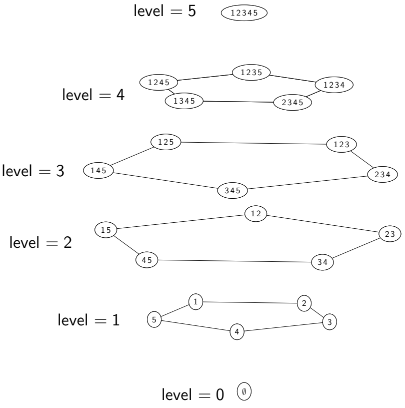
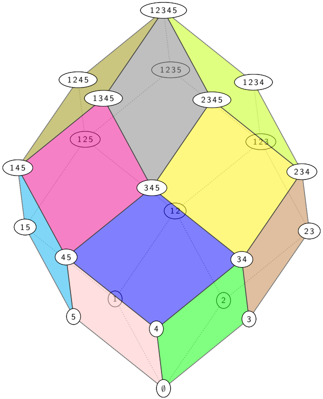
trivalent (k, n)*-plabic graphs* $\overset{\textit{planar}}{\underset{\textit{dual}}{\longleftrightarrow}}$ *horizontal sections at level k of fine zonotopal tilings of* $\mathcal{Z}(n, 3)$





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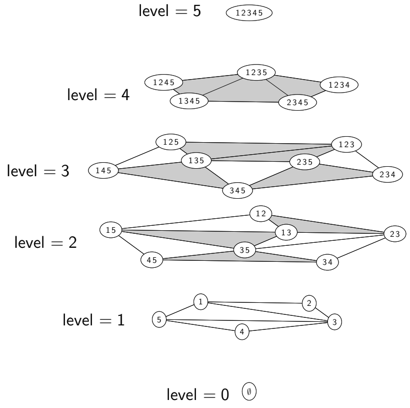

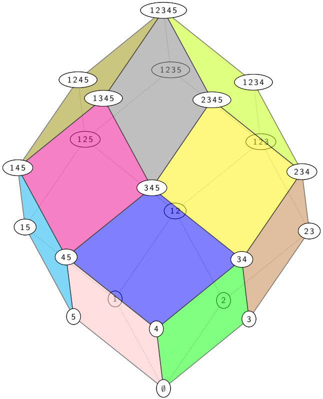

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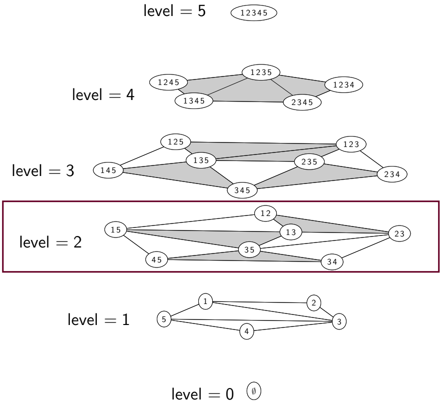

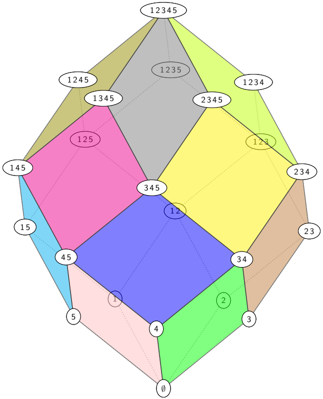

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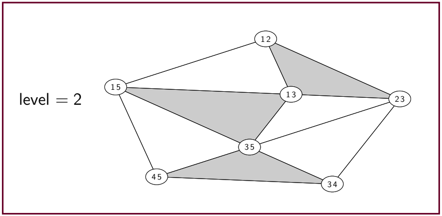

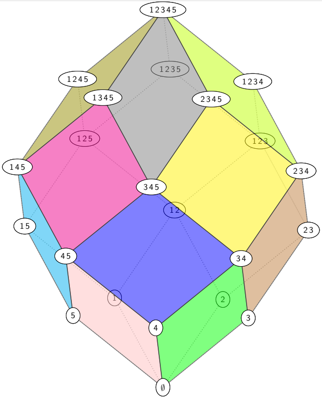

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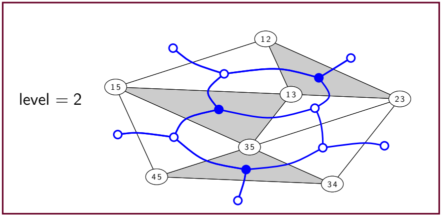

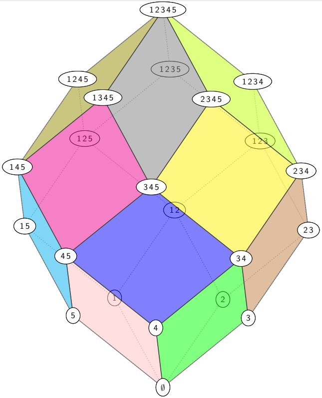

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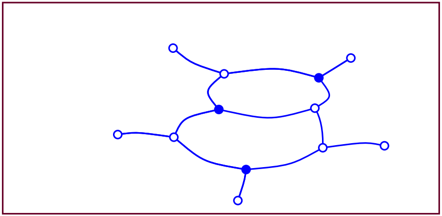
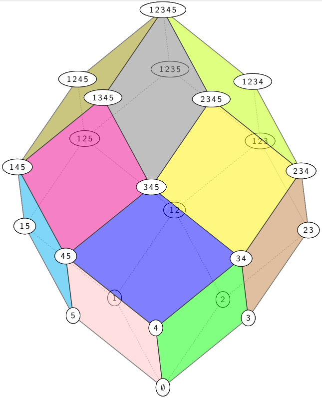


a trivalent $(2, 5)$-plabic graph

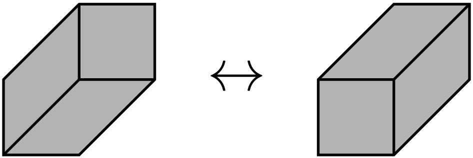


$\mathcal{Z}(n, 3)$ for $n = 5$

Flips of zonotopal tilings

Lemma

$\mathcal{Z}(d+1, d)$ *admits exactly* *two* *fine zonotopal tilings.*



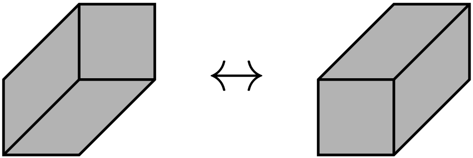
Flips of zonotopal tilings

Lemma

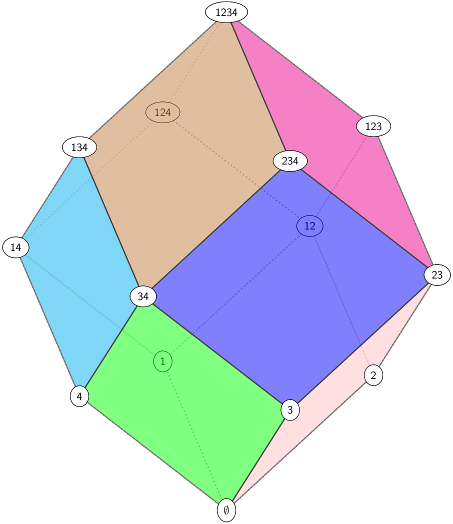
$\mathcal{Z}(d+1, d)$ admits exactly two fine zonotopal tilings.

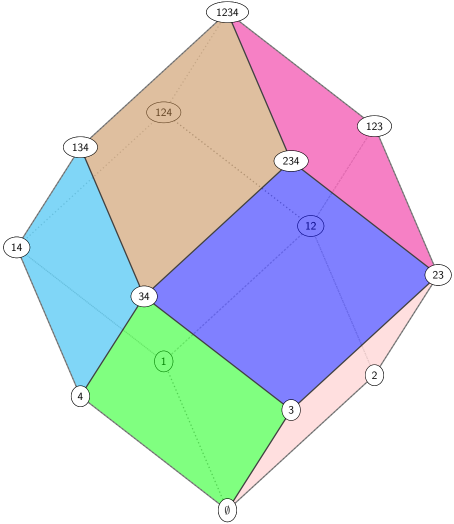
Definition

The local transformation interchanging them is called a flip.



Fine zonotopal tilings of $\mathcal{Z}(4,3)$





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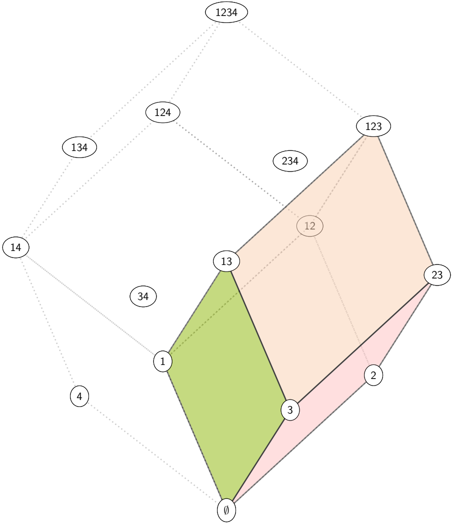

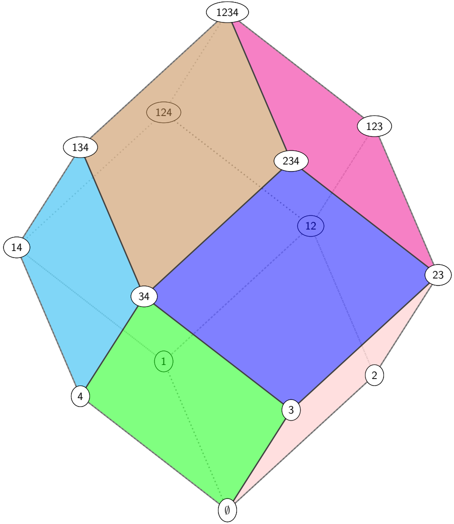

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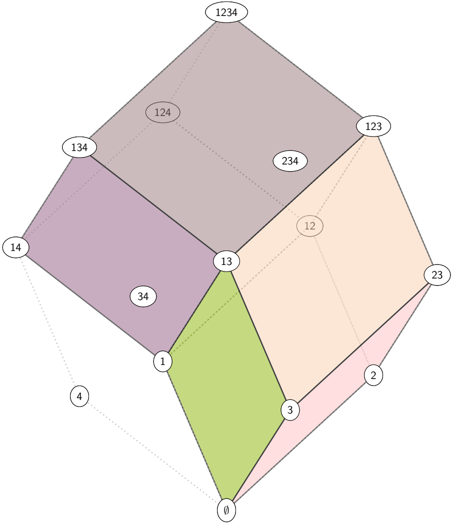

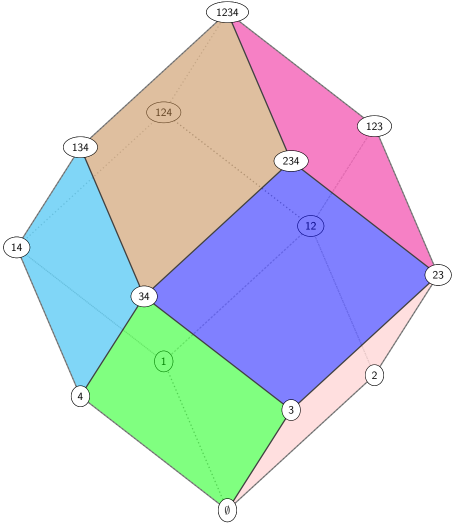

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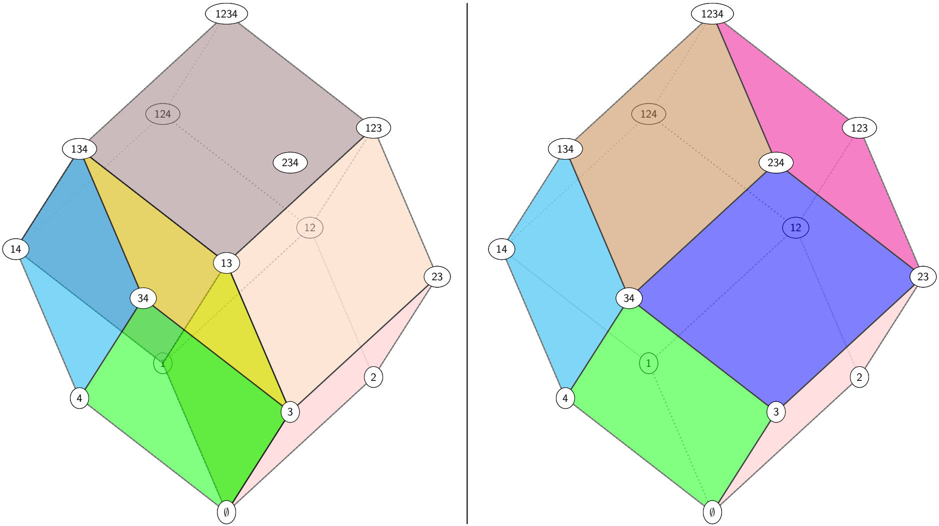

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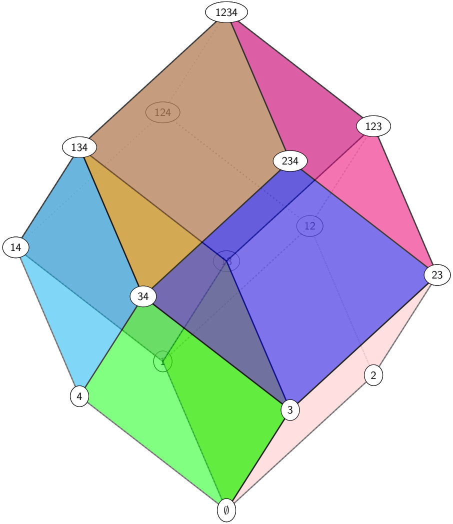

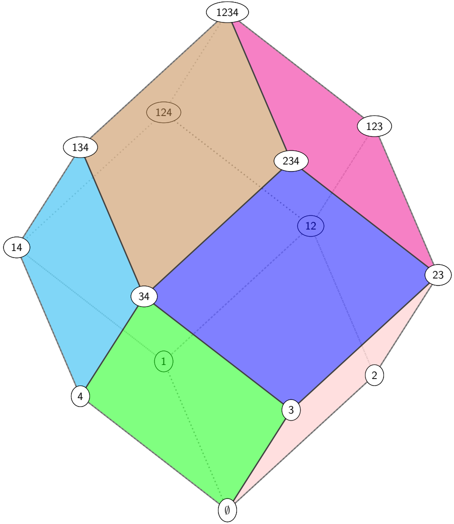

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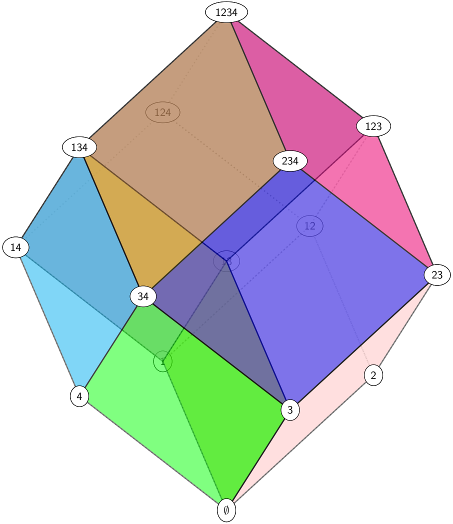

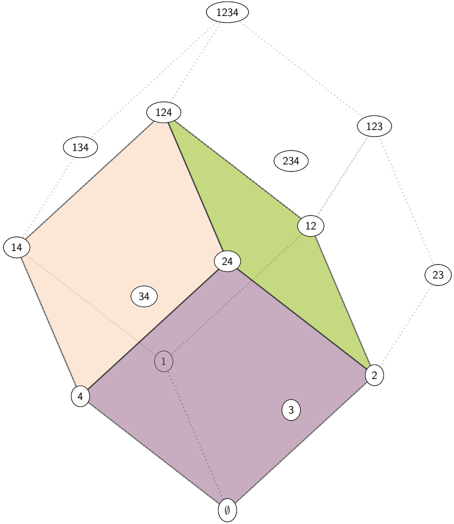

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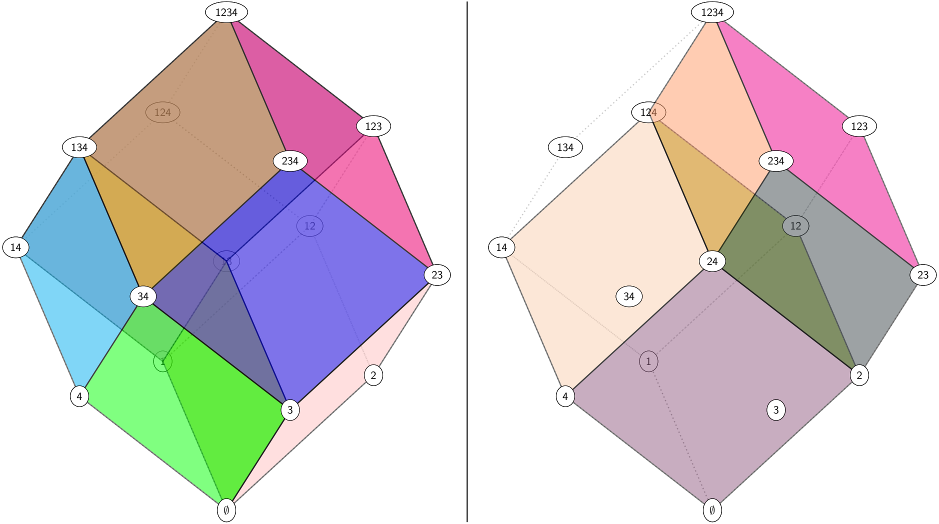

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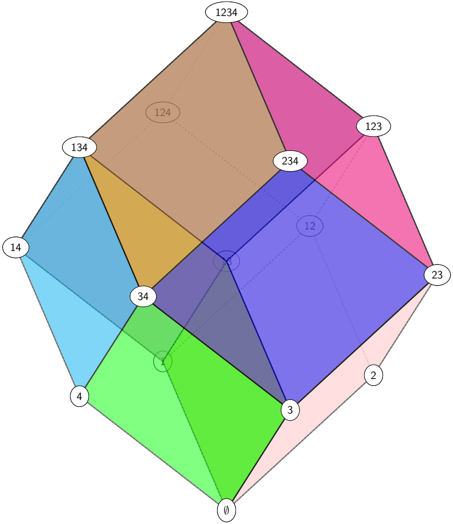

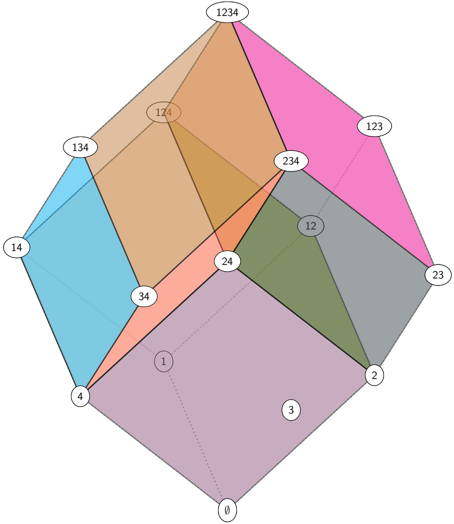

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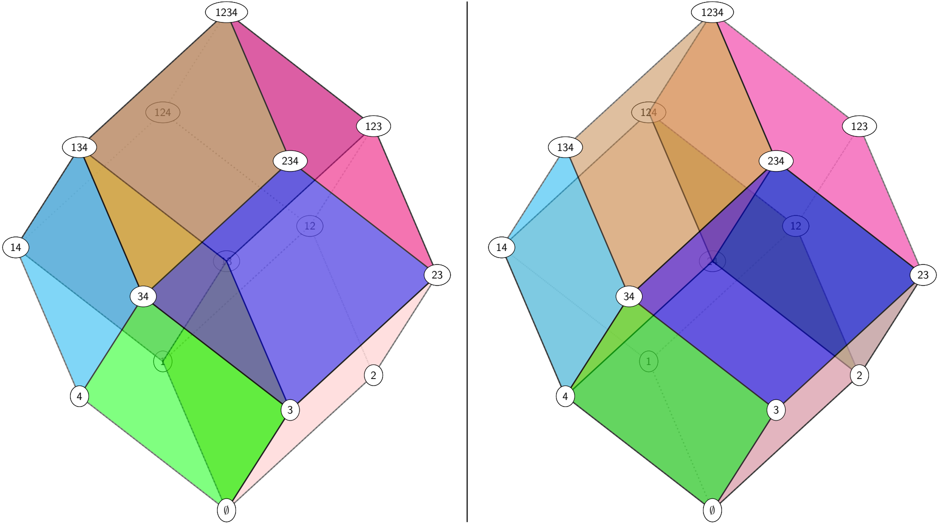
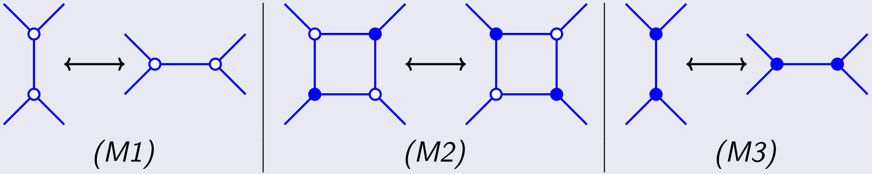


Moves and flips

Theorem (Postnikov)

Any two trivalent (k, n)-plabic graphs are connected by a sequence of moves:

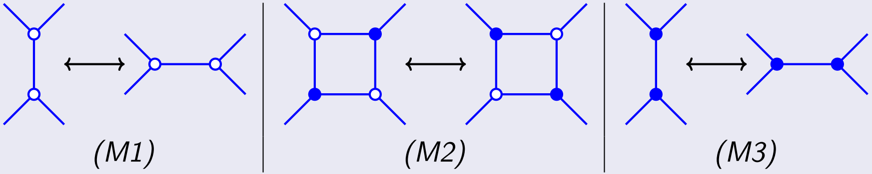


(M1) (M2) (M3)

Moves and flips

Theorem (Postnikov)

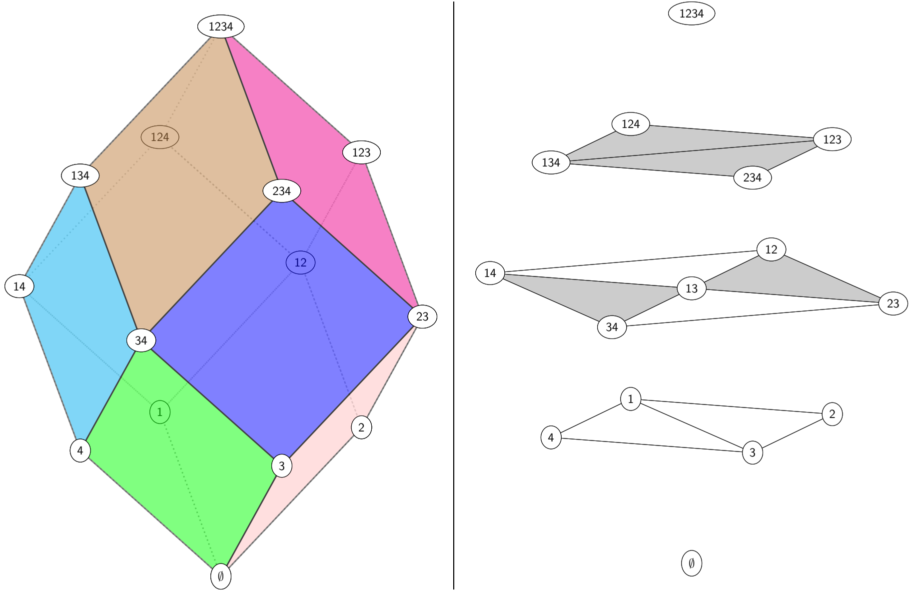
Any two trivalent (k, n)-plabic graphs are connected by a sequence of moves:



Theorem (Ziegler)

Any two fine zonotopal tilings of $\mathcal{Z}(n, 3)$ are connected by a sequence of flips.

Example: $n = 4$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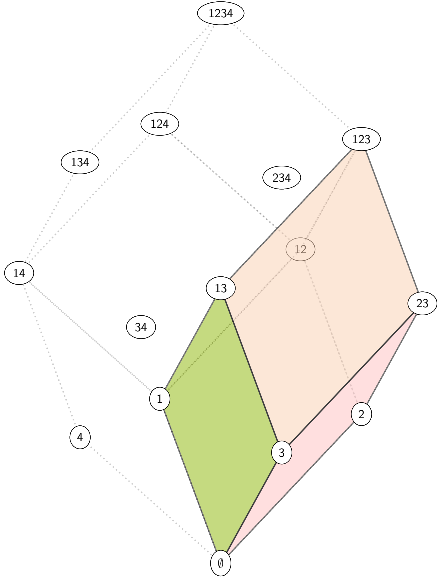

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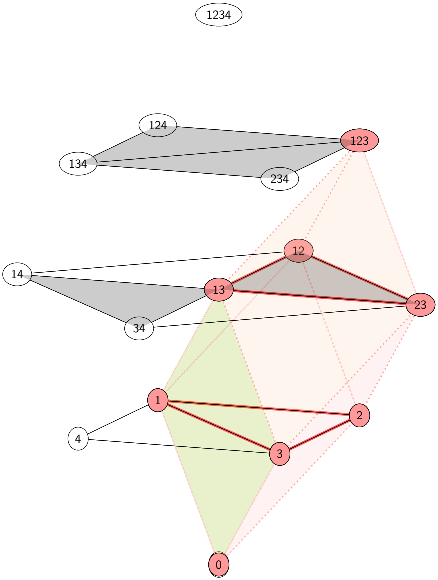

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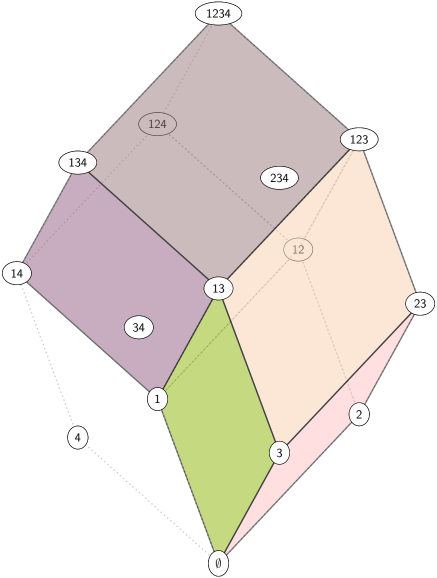
Example: $n = 4$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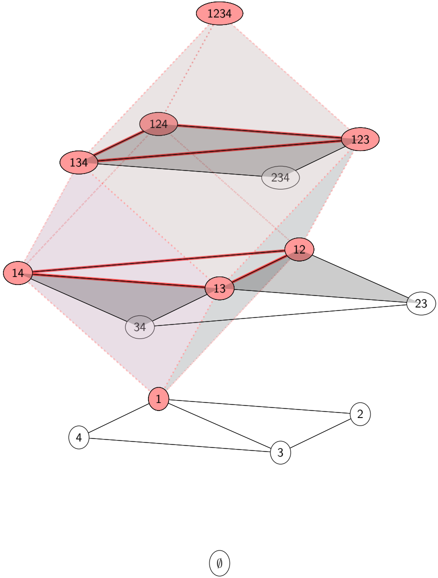
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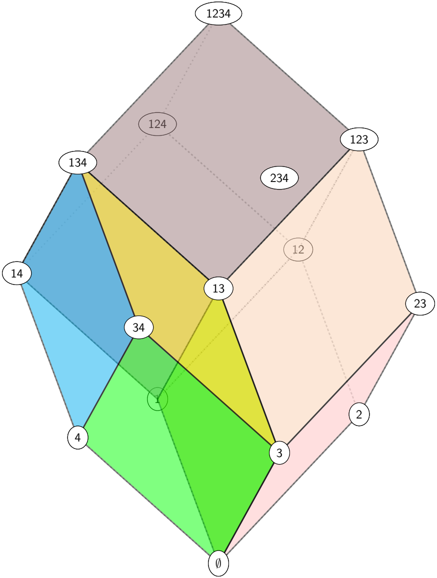

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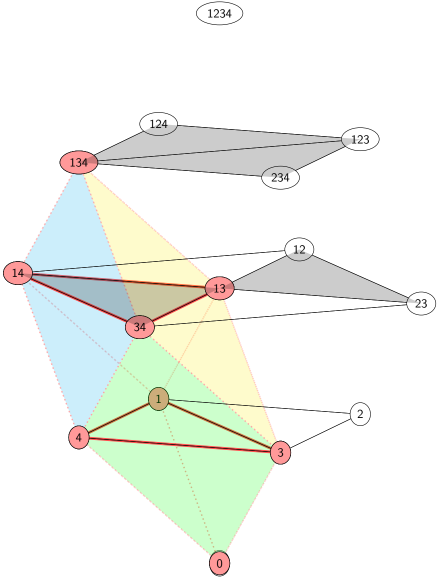

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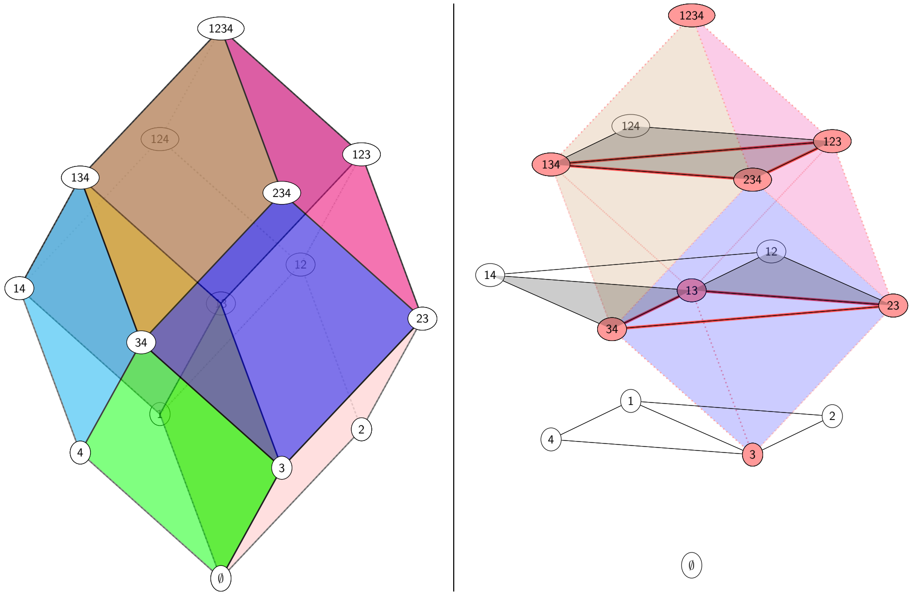

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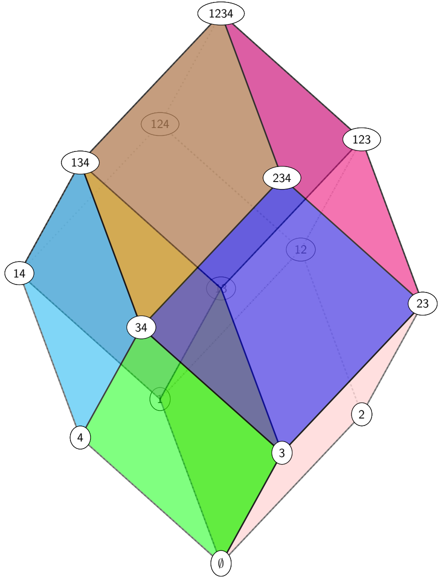

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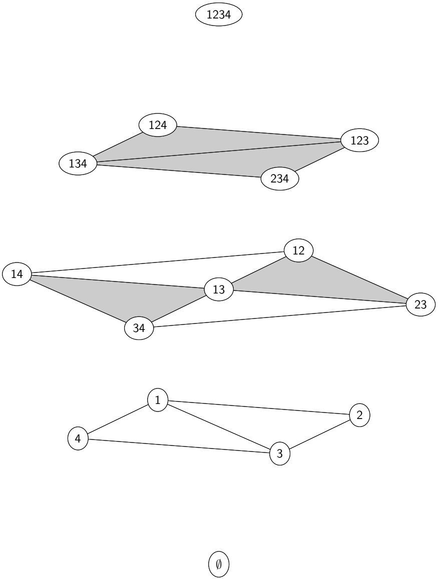

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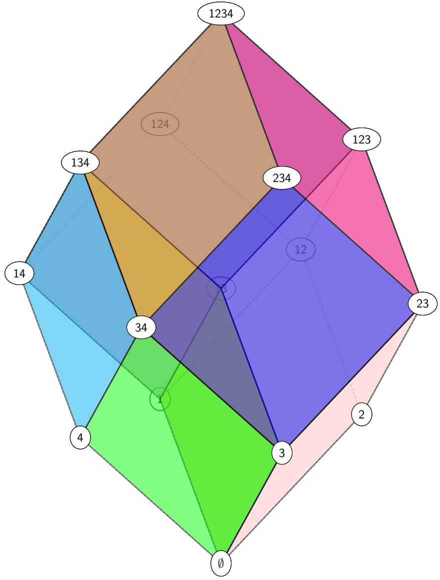
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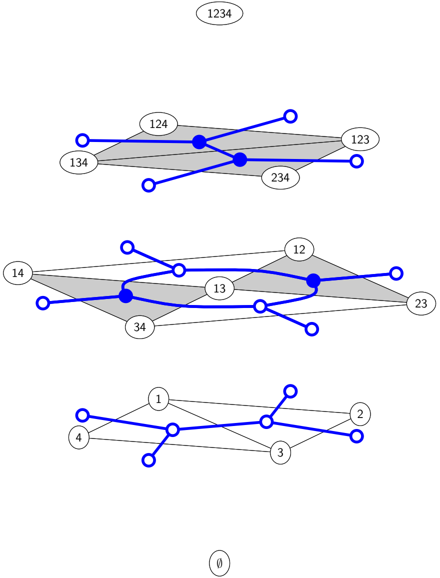
1234
124
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34
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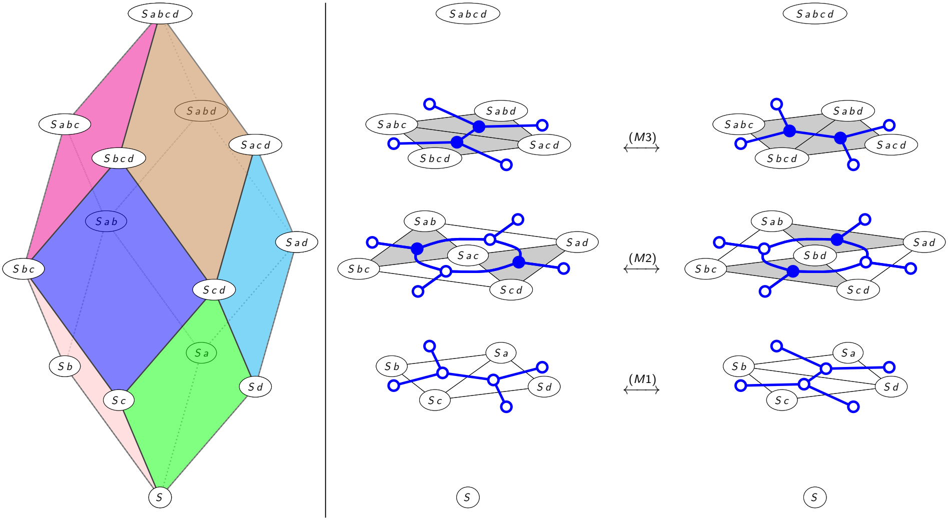
Moves = sections of flips



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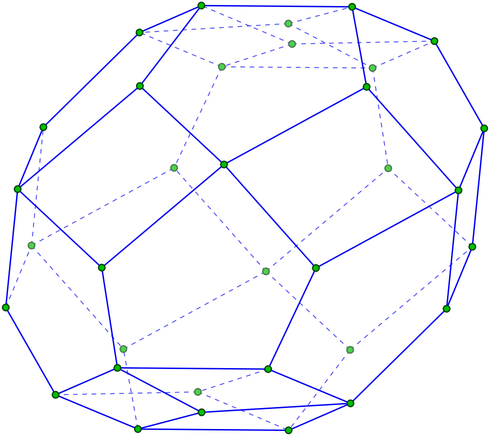

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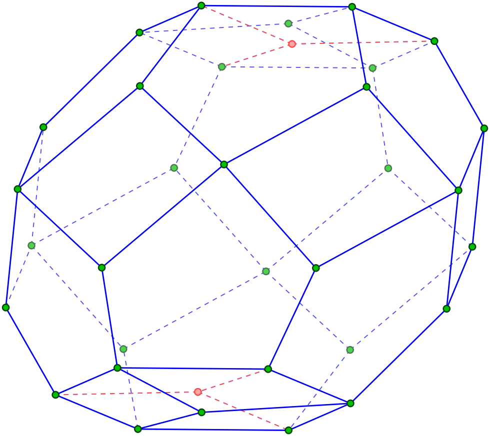

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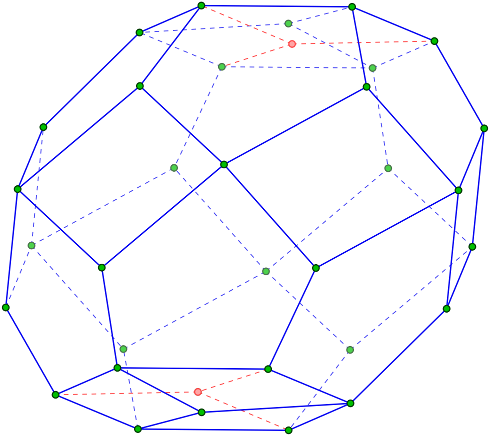

The 32 "regular" plabic graphs form a $(3, 6)$-Higher Associahedron.

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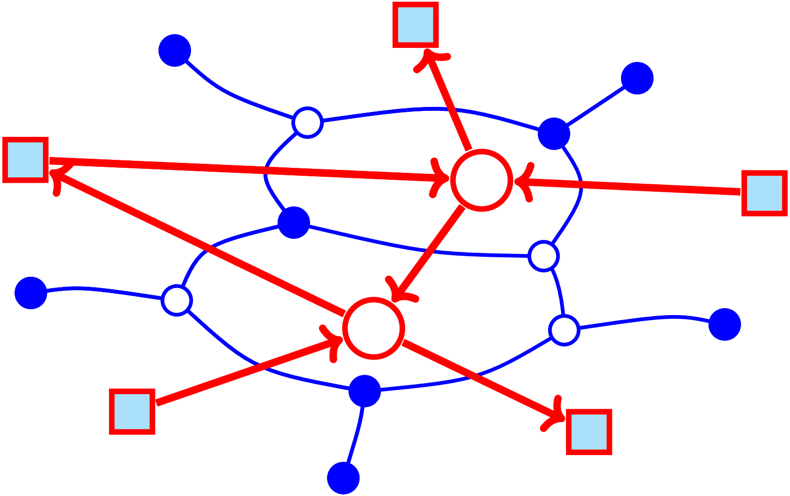
- For minimal planar bipartite graphs on a torus [Goncharov–Kenyon], get horizontal sections of periodic fine zonotopal tilings of $\mathbb{R}^3$ [G.–George].

Part 2: Cluster varieties



Quivers and cluster varieties

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- Still, in a lot of "nice" cases, it produces a tractable geometric object: $\mathcal{A}(Q)$ is isomorphic to the ring of polynomial functions on some interesting algebraic variety.
- For any Q, the cluster variety $\mathcal{X}(Q)$ is defined as $\operatorname{Spec} \mathcal{A}(Q)$.

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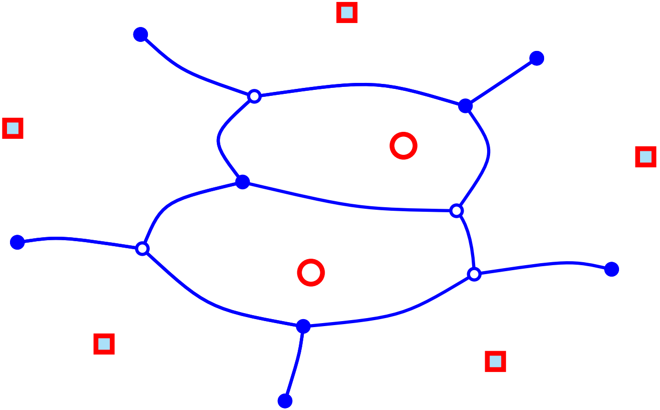
(Ignore unicolored edges. Orient each arrow so that white vertex is to the left.)



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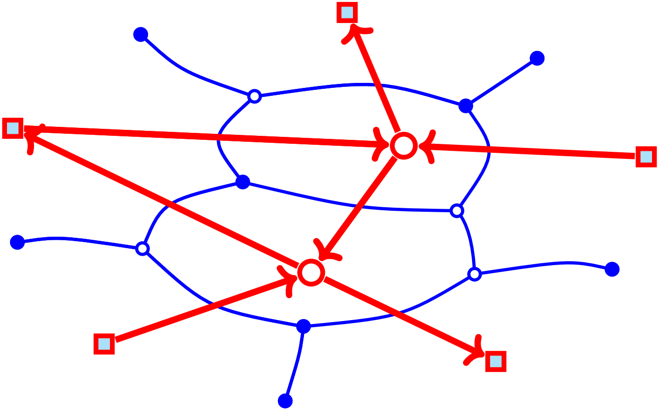

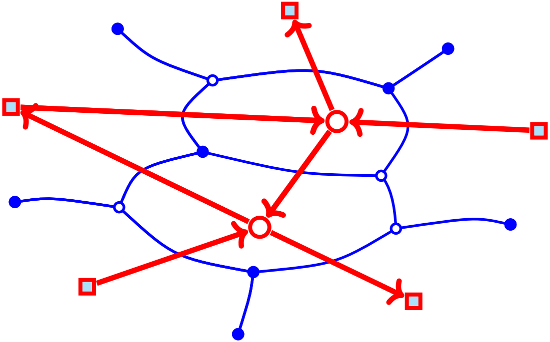

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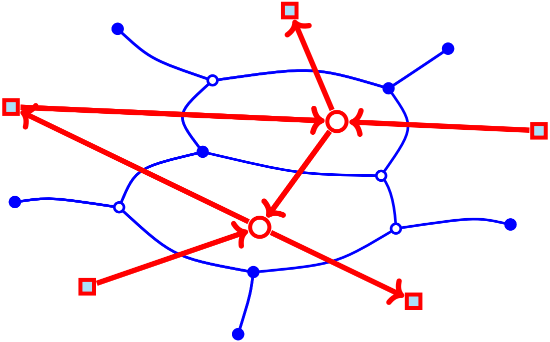

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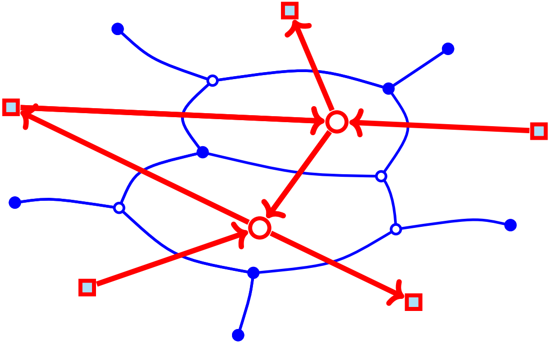

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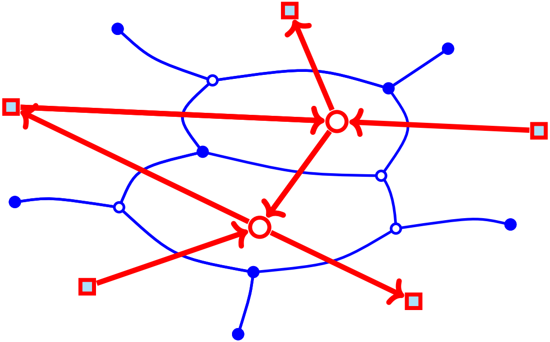

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- When G is not reduced on a disk, or minimal on a torus, not much is known about these varieties.

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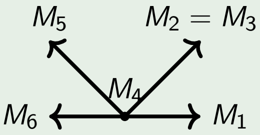

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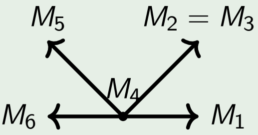

$$f_M = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

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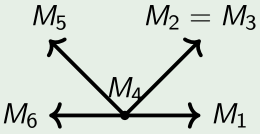

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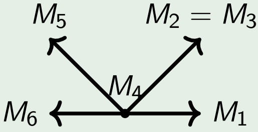

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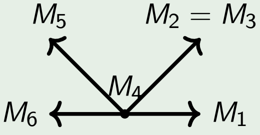

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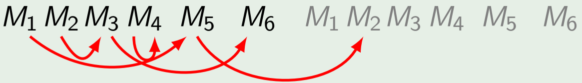

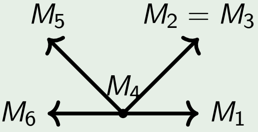

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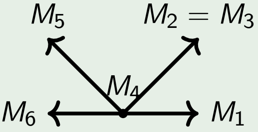

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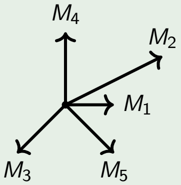

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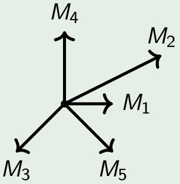

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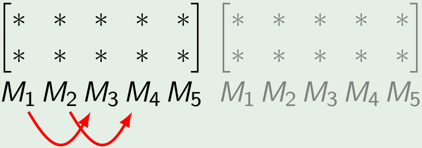

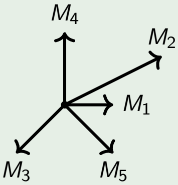

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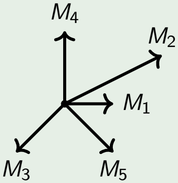

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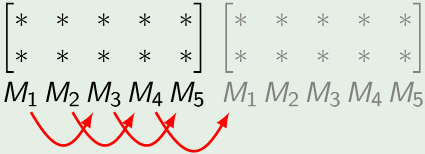

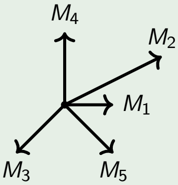

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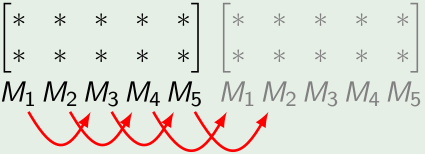

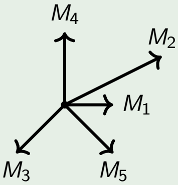

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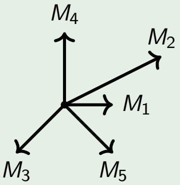
$M_1 \; M_2 \; M_3 \; M_4 \; M_5 \quad M_1 \; M_2 \; M_3 \; M_4 \; M_5$



Let $f_{k,n}(i) \equiv i + k \pmod{n}$ for all $i = 1, 2, \ldots, n$. Then

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where $\Delta_{i_1 \cdots i_k}(M) = \det(M_{i_1} M_{i_2} \cdots M_{i_k})$.

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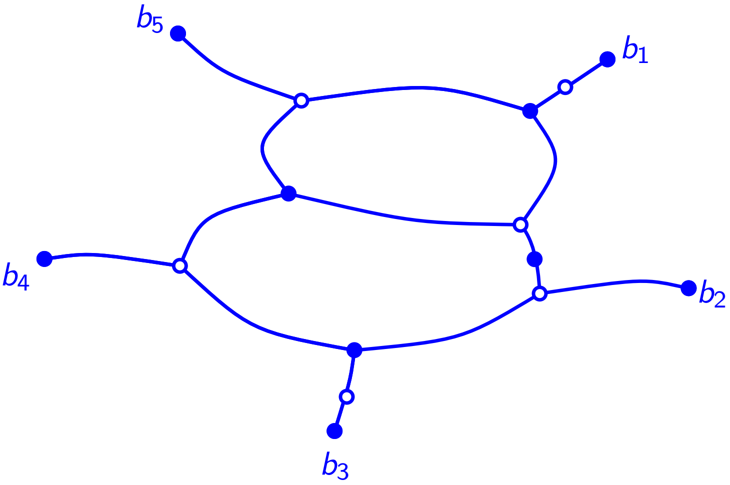
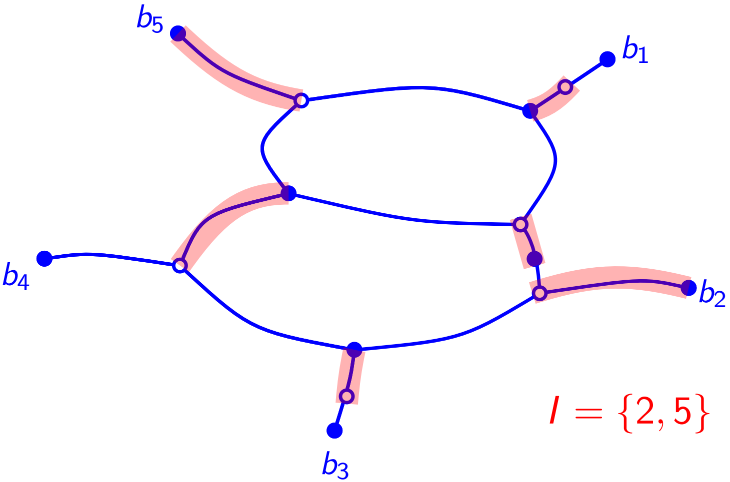
Theorem (G.–Lam)

If G is a reduced plabic graph with strand permutation f then $\mathcal{X}(Q_G) \cong \Pi_f^\circ$.

Partial progress: [Serhiyenko–Sherman-Bennett–Williams], [Leclerc], [Muller–Speyer], [Scott].



b_5
b_1
b_4
b_2
b_3



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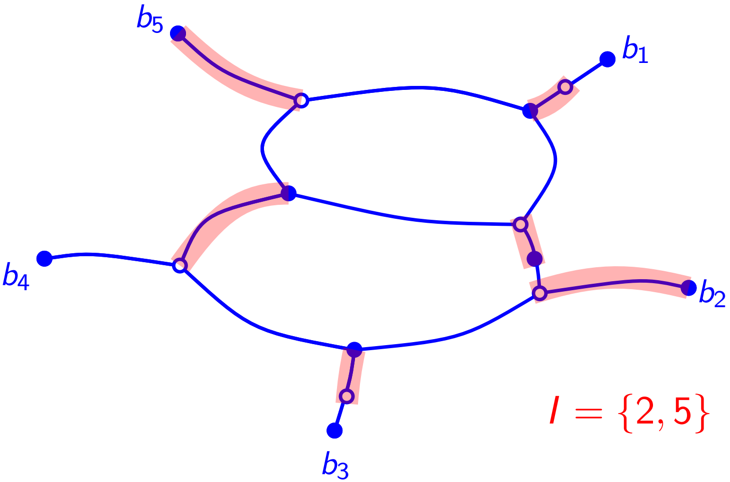
$$\Delta_I(M) = \text{dimer partition function of } G \setminus \{b_j\}_{j \notin I}.$$



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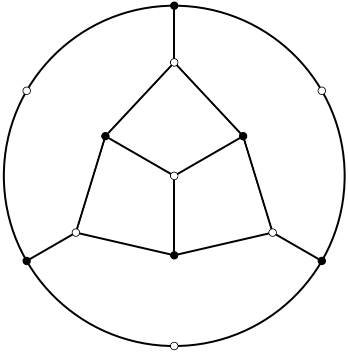
- A plabic graph G on a surface $\mathbb{S}$ gives rise to a link L_G in $\mathbb{S} \times S^1$ [Shende–Treumann–Williams–Zaslow], [Fomin–Pylyavskyy–Shustin–Thurston].



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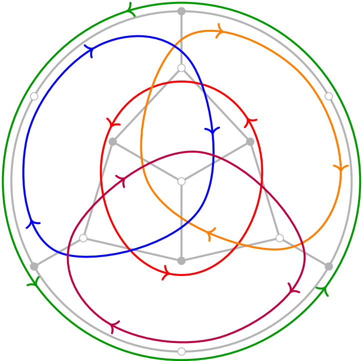

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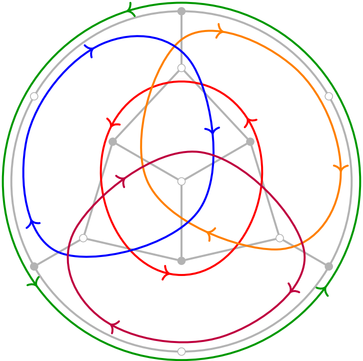

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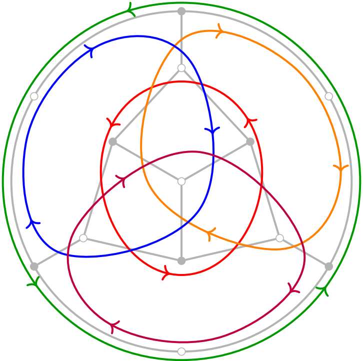

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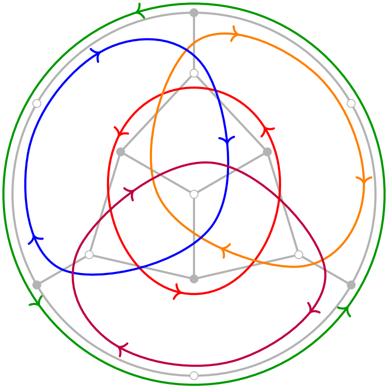

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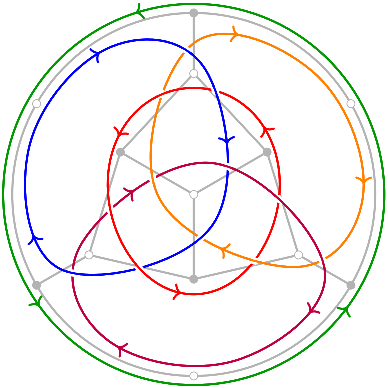

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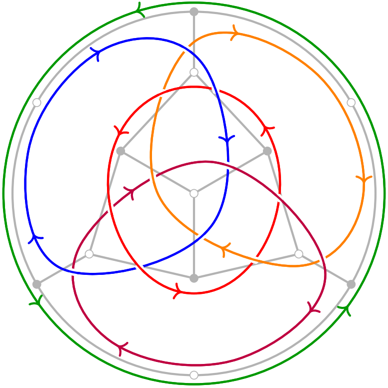

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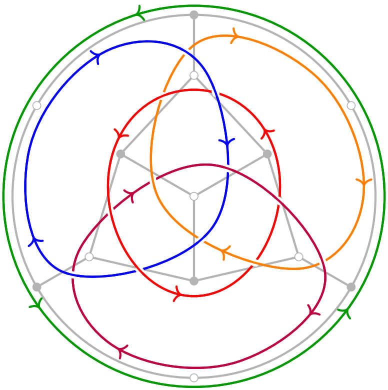

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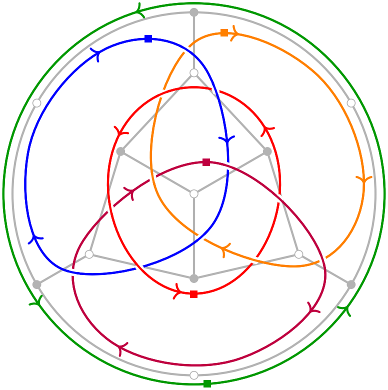

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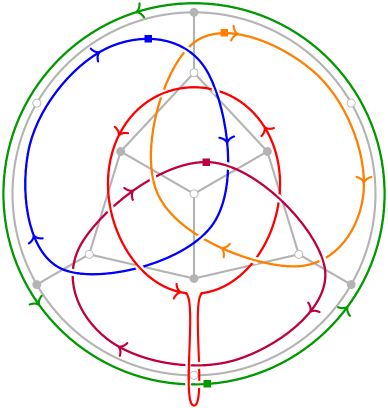

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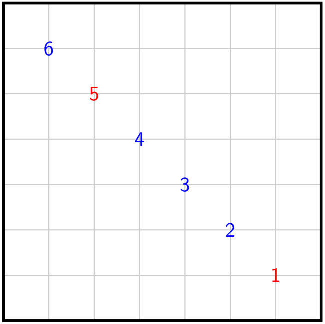

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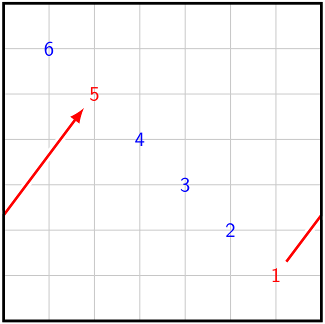

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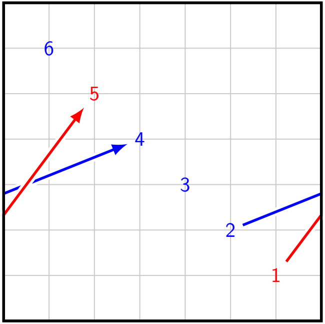

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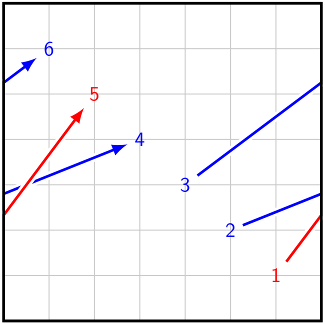

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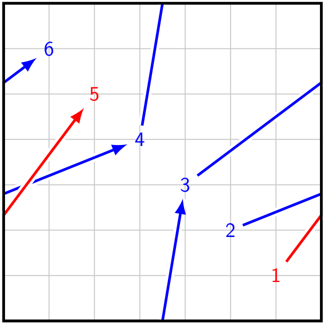

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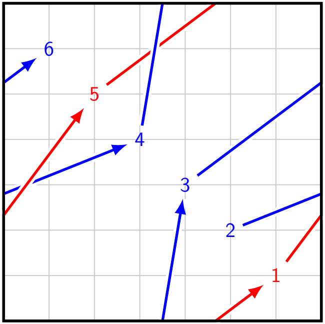

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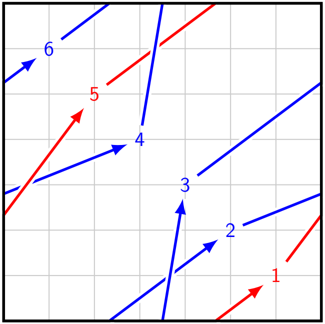

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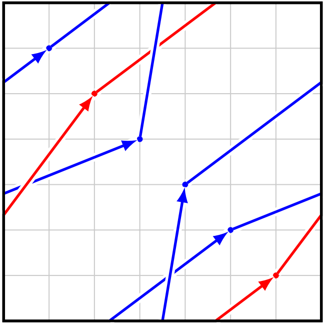

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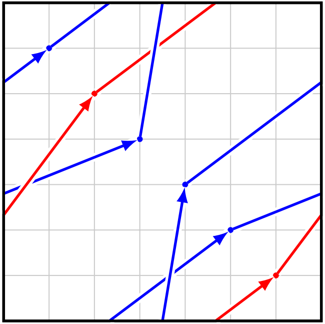

This construction: [G.–Lam]. Related constructions: [Shende–Treumann–Williams–Zaslow], [Fomin–Pylyavskyy–Shustin–Thurston], [Casals–Gorsky–Gorsky–Simental]

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Question

What happens for other classes of plabic graphs?

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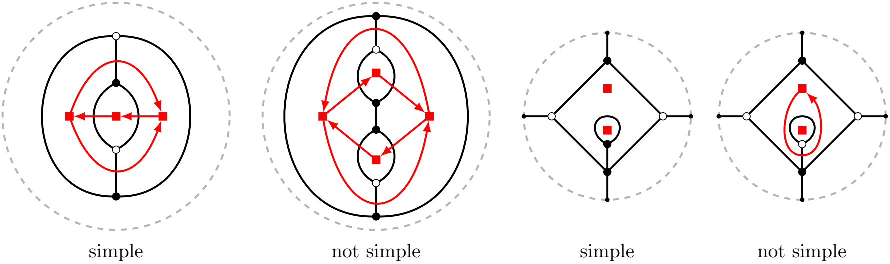

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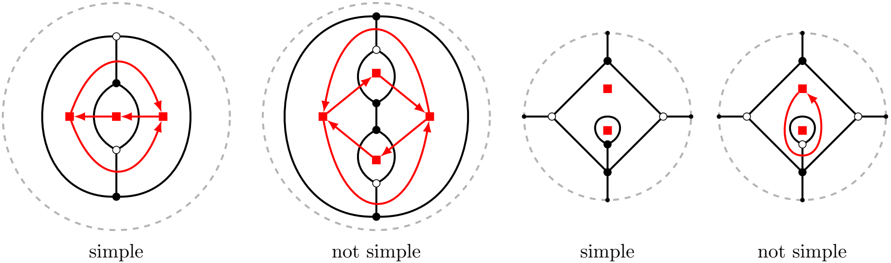

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When G is simple, the $\mathbb{F}_q$-point count of $\mathcal{X}(Q_G)$ is a specialization of the HOMFLY polynomial of L_G.

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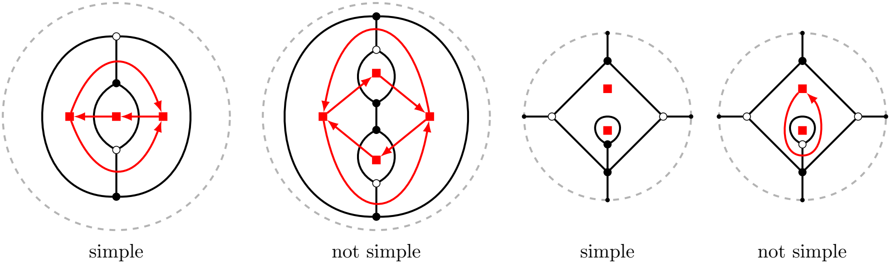

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