On variants of Specht polynomials and Random Geometry

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28 March 2024 @IPAM: Statistical Mechanics and Discrete Geometry

Joint works with A. Lafay & J. Roussillon (and A. Karrila)

FINNISH CENTRE OF EXCELLENCE IN RANDOMNESS AND STRUCTURES 2022–2029

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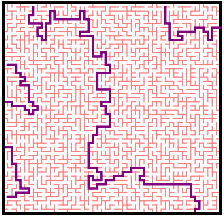
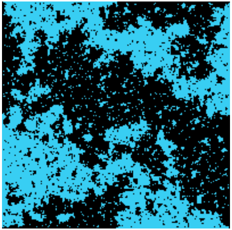
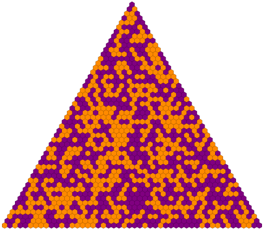
Critical lattice models in 2D statistical physics

- discrete models on (planar) graphs, e.g. $\mathbb{Z}^2$
- (continuous) phase transitions $\Rightarrow$ **critical phenomena**







random walks, percolation, Ising model, Potts model, dimer models, 6-vertex model, random cluster model, Gaussian free field, $O(n)$ spin and loop models, ...

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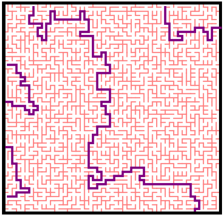
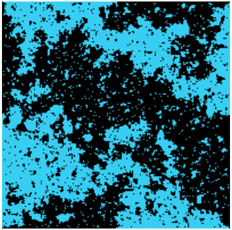
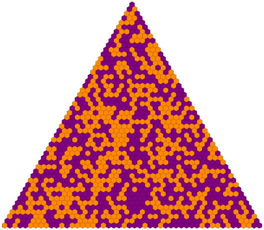
  - **self-similarity**: *fractal* behavior, scale invariance
  - **universality**: microscopic details irrelevant for *large-scale* properties







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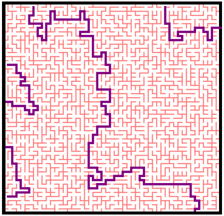
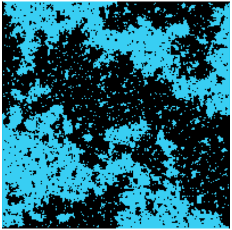
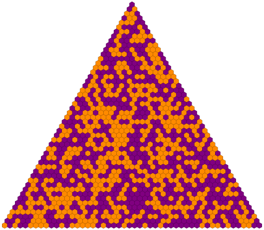
- **scaling limits**: **conformal field theories** (CFT) ? $\delta\mathbb{Z}^2$, $\delta \to 0$

[Belavin, Polyakov & Zamolodchikov '84; Cardy '84; Nienhuis '84]







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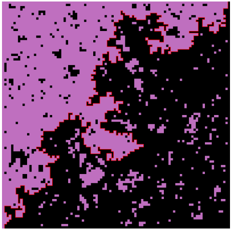
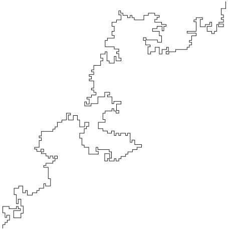
SCALING LIMITS OF CRITICAL INTERFACES — SLE(κ) CURVES

- $\kappa > 0$ labels *universality class* (e.g. $\kappa = 4$ double-dimer & GFF level lines)
- key feature: **conformal invariance** $\rightsquigarrow$ *central charge* $c(\kappa)$





(critical) interface $\xrightarrow{\delta \to 0}$ Schramm-Loewner Evolution, SLE(κ)

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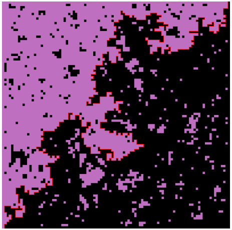

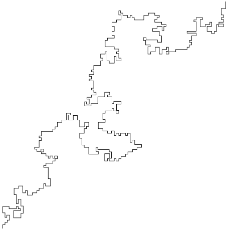

$$\text{(critical) interface} \xrightarrow{\delta \to 0} \text{Schramm-Loewner Evolution, SLE}(\kappa)$$

Usual proof strategy:

1. *tightness* (e.g. control via crossing estimates, RSW etc.)

   [Aizenman & Burchard '99, Kemppainen & Smirnov '17, ...]

2. *identification* of the limit (e.g. via discrete holomorphic observable)

   [Kenyon '00, Chelkak & Smirnov '01–'11, ...] □

General heuristics for SLE–CFT correspondence

- *universality class* labeled by $\kappa > 0$, central charge $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} \leq 1$
- discrete **crossing probabilities**

  $\overset{\delta\to 0}{\longrightarrow}$ probabilities of connectivities of $\mathrm{SLE}(\kappa)$ curves:

$$\lim_{\delta\to 0} \mathbb{P}^{\delta}_{\beta}[\text{ connectivity} = \alpha\,] = \mathcal{K}_{\alpha,\beta}\, \frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{\mathcal{U}_\beta(x_1,\ldots,x_{2N})}$$

- amplitudes for α encoded in *pure* partition functions $\mathcal{Z}_\alpha$
- boundary conditions β encoded in *b.c.c.* partition functions $\mathcal{U}_\beta$

[Feĭgin-Fuchs '80s; Dotsenko-Fateev '80s; Cardy '80s; Bauer-Bernard-Kytölä '05; Dubédat '07; Flores-Kleban-Simmons-Ziff '17; P. & Wu (et al.) '18–22'; Izyurov '22...]

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- **CFT prediction:** $\mathcal{Z}_\alpha, \mathcal{U}_\beta$ should be "$\langle\, \Phi_{1,2}(x_1) \cdots \Phi_{1,2}(x_{2N})\,\rangle$" singled out via *fusion rules* $\rightsquigarrow$ works for $\kappa \in (0,8]$

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  singled out via *fusion rules* $\rightsquigarrow$ works for $\kappa \in (0, 8]$
- *Coulomb gas* integral formulas

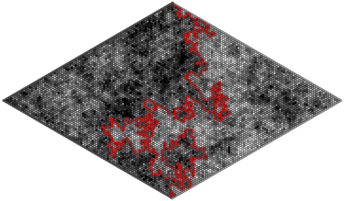
$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) = \prod_{i<j} (x_i - x_j)^{2/\kappa} \int_{\Gamma_\alpha} \prod_{r,j} (w_r - x_j)^{-4/\kappa} \prod_{r<s} (w_r - w_s)^{8/\kappa} \, \mathrm{d}w_1 \cdots \mathrm{d}w_N$$

[Feĭgin-Fuchs '80s; Dotsenko-Fateev '80s; Cardy '80s; Bauer-Bernard-Kytölä '05; Dubédat '07; Flores-Kleban-Simmons-Ziff '17; P. & Wu (et al.) '18–22'; Izyurov '22...]

Exact solvability

Some models carry additional combinatorial structure!

E.g. models building on the **Gaussian free field** (GFF):



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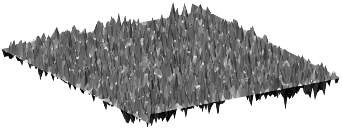


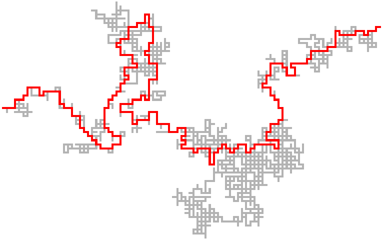
- double-dimer model: $\kappa = 4$, $c = 1$
- triple-dimer model: $c = 2$ (no κ)
- multi-dimer models: $c = 3, 4, \ldots$

$\Longrightarrow$ $\mathcal{Z}_\alpha$ and $\mathcal{U}_\beta$ given by **variants of Specht polynomials**

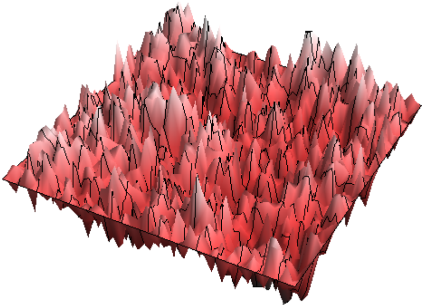
- uniform spanning tree: $\kappa = 8$, $c = -2$
- branches (LERW): $\kappa = 2$, $c = -2$

$\Longrightarrow$ $\mathcal{Z}_\alpha$ and $\mathcal{U}_\beta$ given by **determinants of Fomin type**

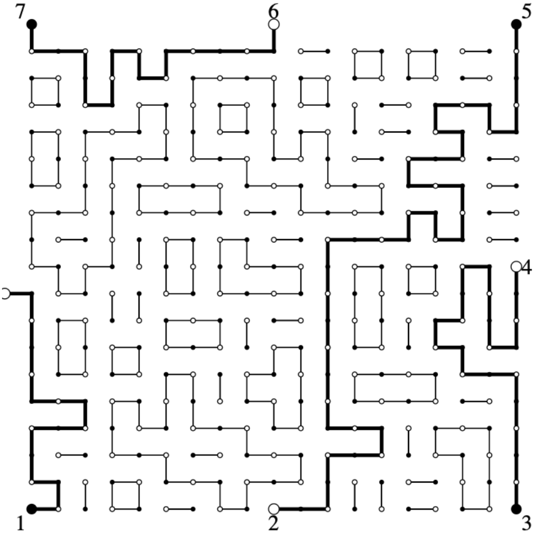
Loop-erased random walk ($c = -2$, $\kappa = 2$)



Gaussian free field ($c = 1$, $\kappa = 4$)



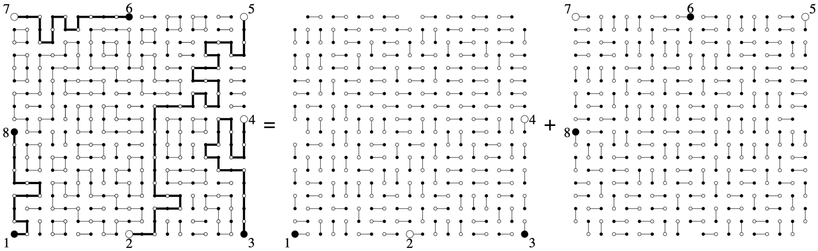
DOUBLE DIMERS ($c = 1$) AND VIRASORO CONFORMAL BLOCKS



Double-dimer interfaces

Conjecture. Interfaces converge to GFF level lines!

Double-dimer interfaces $\xrightarrow{\delta \to 0}$ Schramm-Loewner Evolution, SLE(4)



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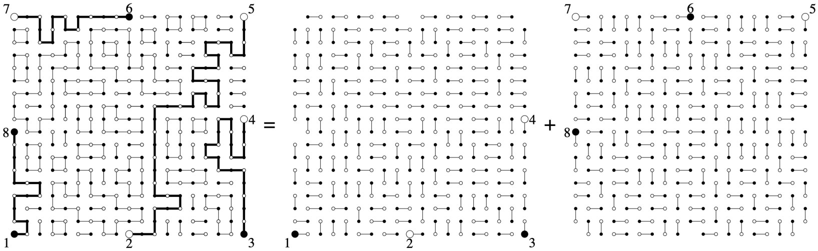
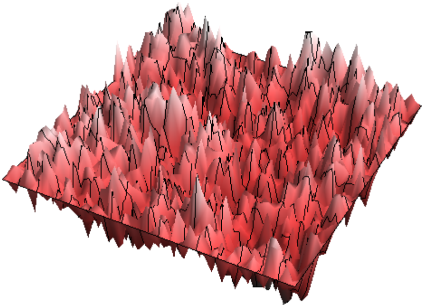


- convergence of crossing probabilities is known [Kenyon & Wilson '11]
- convergence of interfaces is **open**
- **very strong** evidence [Dubédat '19, Basok & Chelkak '21, Bai & Wan '22]

Gaussian free field (GFF)

- "Gaussian distribution" h
- centered with covariance given by the Green function $G_\Omega(x, y)$
- for test functions φ, the action of h gives (h, φ): a **Gaussian random vector**



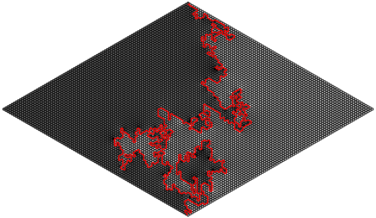
© Watson

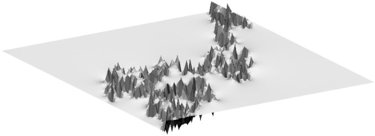
$$\mathrm{cov}((\mathrm{h}, \varphi_1), (\mathrm{h}, \varphi_2)) = \int\int \varphi_1(x)\, G_\Omega(x, y)\, \varphi_2(y) \mathrm{d}x\mathrm{d}y$$

- **GFF with boundary data:**
  - take function f on $\partial\Omega$
  - **take** h **plus the harmonic extension** of f into Ω

$\rightsquigarrow$ $\mathrm{h} + f$ is **GFF with boundary data** f

Gaussian Free Field Level Lines





- **Dobrushin boundary data**:
  $-\lambda < 0$ on the left,
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- "zero level set": **level line** γ

Thm. [Schramm & Sheffield '06]
γ equals in distribution to SLE(4)

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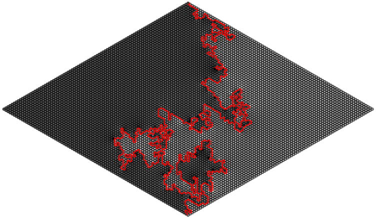

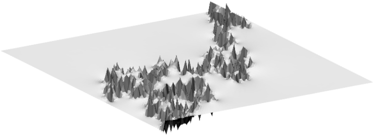

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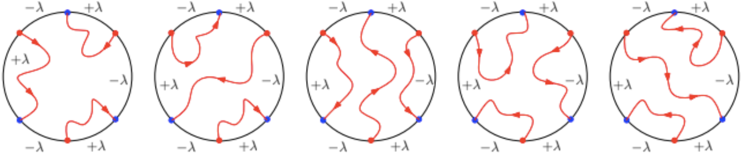
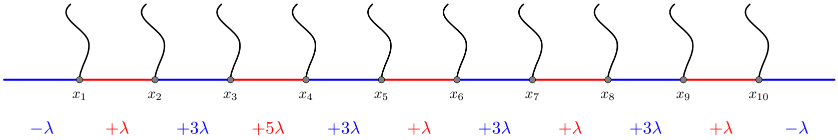
Alternating boundary data $\Longrightarrow$ multiple SLE(4) curves



Consider pw constant boundary conditions with jumps of sizes $\pm 2\lambda$.



- level lines form *random connectivity* ϑ_{GFF} (planar pairing)
- boundary condition encoded into *Dyck path* β

Thm. [Kenyon & Wilson '11; P. & Wu '17; Liu & Wu '21]

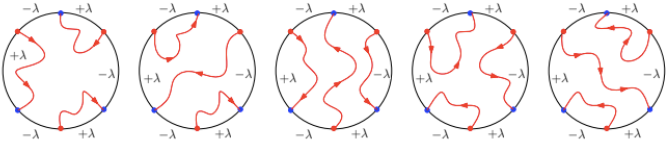
The **connection probabilities** are explicitly given by

$$\mathbb{P}_\beta[\vartheta_{\mathrm{GFF}} = \alpha] = \mathcal{K}_{\alpha,\beta} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{\mathcal{U}_\beta(x_1, \ldots, x_{2N})}$$

where $\mathcal{U}_\beta$ are explicit *conformal block functions* and

$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) := \sum_\beta \mathcal{K}^{-1}_{\alpha,\beta} \mathcal{U}_\beta(x_1, \ldots, x_{2N})$$

and $\mathcal{K}_{\alpha,\beta} \in \{0, 1\}$ and $\mathcal{K}^{-1}_{\alpha,\beta} \in \mathbb{Z}$ combinatorial matrix elements.



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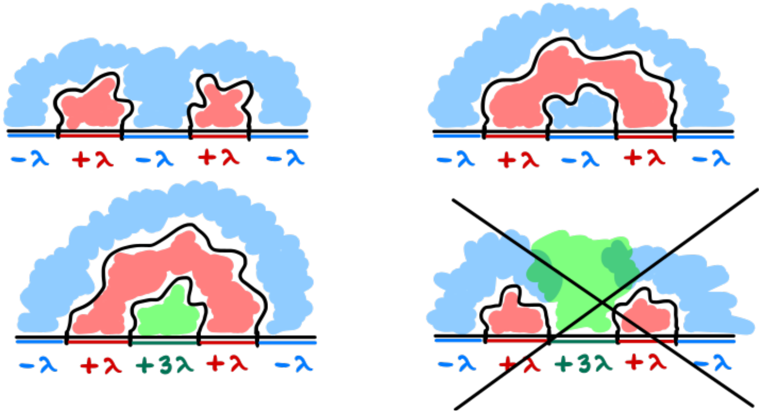
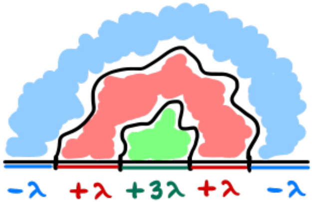
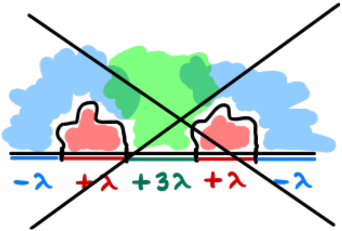
Multiple level lines of GFF



- **boundary condition** encoded into *Dyck path* β:
  height jumps across the boundary of sizes $\pm 2\lambda$
- **level lines** form *random connectivity* ϑ_{GFF} (planar pairing):
  pairwise connection of level line curves

Conformal blocks





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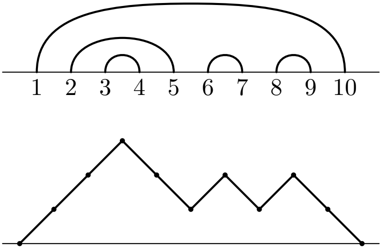
- $\mathcal{U}_\alpha$ is the **conformal block function**

$$\mathcal{U}_\alpha(x_1, \ldots, x_{2N}) := \left(\prod_{i<j} (x_j - x_i)^{-1/2} \right) \mathcal{P}_{\alpha^T}(x_1, \ldots, x_{2N})$$

- $\mathcal{P}_{\alpha^T}$ is the **transpose Specht polynomial**

$$\mathcal{P}_{\alpha^T}(x_1, \ldots, x_{2N}) := \prod_{\substack{\text{rows } R \\ \text{in } \alpha}} \prod_{\substack{i<j \\ \text{in row } R}} (x_j - x_i)$$

Planar Pairings $\longleftrightarrow$ Dyck paths $\longleftrightarrow$ standard Young tableaux of shape (N, N)



1	2	3	6	8
4	5	7	9	10

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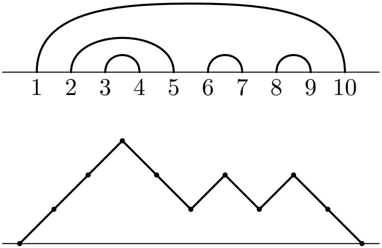

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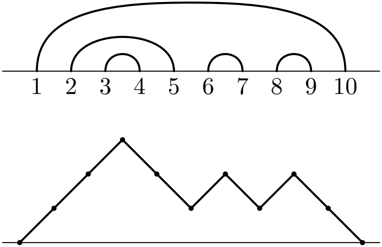
- any of these sets indexes basis for *simple module* $\mathrm{span}\{\mathcal{P}_\alpha : \alpha \in \mathrm{SYT}^{(N,N)}\}$ of *symmetric group* $\mathfrak{S}_{2N}$
- basis elements $\mathcal{P}_\alpha$ are Specht polynomials

$$\mathcal{P}_\alpha(x_1, \ldots, x_{2N}) = \prod_{\substack{\text{columns } C \\ \text{in } \alpha}} \prod_{\substack{i<j \\ \text{in column } C}} (x_j - x_i)$$

- $\mathfrak{S}_{2N}$ acts naturally by permutation of variables

Planar Pairings $\longleftrightarrow$ Dyck paths $\longleftrightarrow$ standard Young tableaux of shape (N, N)



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Example:

$$\mathcal{P}_\alpha(x_1, \ldots, x_{10}) = \prod_{\substack{\text{columns } C \\ \text{in } \alpha}} \prod_{\substack{i<j \\ \text{in column } C}} (x_j - x_i)$$

$$= (x_4 - x_1)(x_5 - x_2)(x_7 - x_3)(x_9 - x_6)(x_{10} - x_8)$$

Symmetric group action on conformal blocks

$$\mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = \Big(\prod_{i<j} (x_j - x_i)^{-1/2} \Big) \mathcal{P}_{\alpha^T}(x_1, \ldots, x_{2N})$$

Thm. [Lafay, P. & Roussillon '24+]
The space $\mathrm{span}\{\mathcal{U}_\alpha : \alpha \in \mathrm{SYT}^{(N,N)}\}$ forms a simple module of the symmetric group, with action of $\sigma \in \mathfrak{S}_{2N}$ given by

$$\sigma.\mathcal{U}_\alpha = \mathrm{sgn}(\sigma) \left(\prod_{i<j} (x_j - x_i)^{-1/2} \right) \sigma.\mathcal{P}_{\alpha^T}(x_1, \ldots, x_{2N}).$$

Example:

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$$\mathcal{P}_{\alpha^T}(x_1, \ldots, x_{10}) = \prod_{\substack{\text{rows } R \\ \text{in } \alpha}} \prod_{\substack{i<j \\ \text{in row } R}} (x_j - x_i)$$

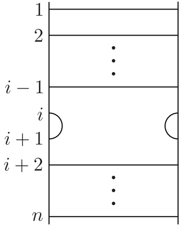
$$= (x_2-x_1)(x_3-x_1)(x_6-x_1)(x_8-x_1)(x_3-x_2)(x_6-x_2)(x_8-x_2)(x_6-x_3)(x_8-x_3)(x_8-x_6)$$
$$\cdot(x_5-x_4)(x_7-x_4)(x_9-x_4)(x_{10}-x_4)(x_7-x_5)(x_9-x_5)(x_{10}-x_5)(x_9-x_7)(x_{10}-x_7)(x_{10}-x_9)$$

Temperley-Lieb action on conformal blocks

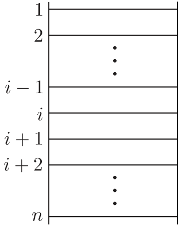
Cor. [Lafay, P. & Roussillon '24+]
The representation $\mathrm{span}\{\mathcal{U}_\alpha : \alpha \in \mathrm{SYT}^{(N,N)}\}$ of $\mathfrak{S}_{2N}$ descends to a representation of $\mathrm{TL}_{2N}(2)$ with fugacity parameter 2, isomorphic to the *standard (cell) module* with no defects.

Recall that Temperley-Lieb algebra $\mathrm{TL}_{2N}(2)$ is generated by diagrams $e_1, e_2, \ldots, e_{2N-1}$ and unit 1:

e_i :=



1 =



The product in $\mathrm{TL}_{2N}(2)$ is *concatenation of diagrams* with the rule that *loops* are resolved as multiplicative **fugacity factor** "2".

Conformal blocks — CFT properties

Thm. [P. & Wu '17; paraphrased] The functions $\mathcal{U}_\alpha$ satisfy

- **BPZ PDEs** of 2nd order $\forall j$ with $\kappa = 4$, $c = 1$,

$$\left\{ 2 \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left(\frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{1/2}{(x_i - x_j)^2} \right) \right\} \mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = 0$$

- **Möbius covariance**

$$\mathcal{U}_\alpha(f(x_1), \ldots, f(x_{2N})) = \Big(\prod_{1 \leq j \leq 2N} \left| f'(x_j) \right|^{-1/4} \Big) \, \mathcal{U}_\alpha(x_1, \ldots, x_{2N})$$

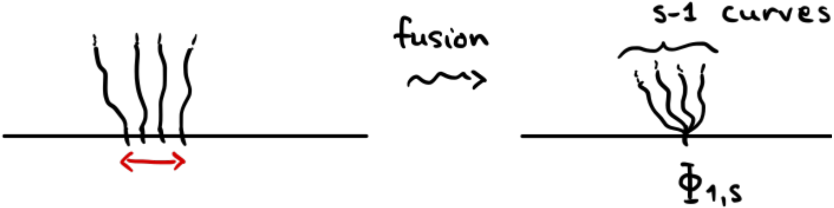
(for $f : \mathbb{H} \to \mathbb{H}$ s.t. $f(x_1) < \cdots < f(x_{2N})$)

- specific **asymptotics** related to fusion rules in $c = 1$ CFT

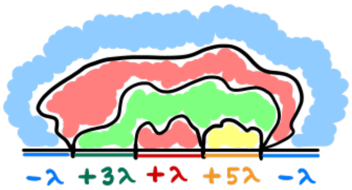
NB: Compared to physics lit. this is a *new* basis of blocks!

$$\mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = \Big(\prod_{i<j} (x_j - x_i)^{-1/2} \Big) \, \mathcal{P}_{\alpha^T}(x_1, \ldots, x_{2N})$$

Fusion

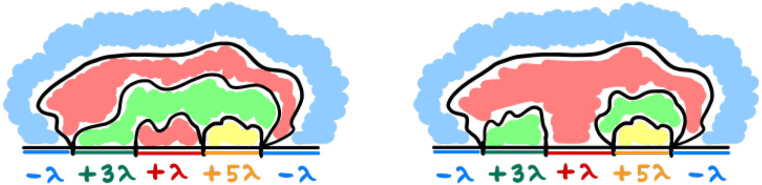


Multiple level lines of GFF: fusion



- **valences** $\varsigma = (s_1, \ldots, s_d)$: *composition* of $2N$, i.e., $s_1 + \cdots + s_d = 2N$
- **boundary condition** encoded into *walks* β:
  height jumps across the boundary of sizes $\in \pm 2\lambda\mathbb{Z}$

Multiple level lines of GFF: fusion



- **valences** $\varsigma = (s_1, \ldots, s_d)$: *composition* of $2N$, i.e., $s_1 + \cdots + s_d = 2N$
- **boundary condition** encoded into *walks* β:
  height jumps across the boundary of sizes $\in \pm 2\lambda\mathbb{Z}$
- **conformal block functions**

$$\mathcal{U}_\alpha(x_1, \ldots, x_d) = \Big(\prod_{i<j} (x_j - x_i)^{-s_i s_j / 2} \Big) \mathcal{P}_{\alpha^T}(x_1, \ldots, x_d)$$

- $\mathcal{P}_{\alpha^T}$ is (transpose) fused Specht polynomial

Fusion setup

- **valences** $\varsigma = (s_1, \ldots, s_d)$: *composition* of $2N$, i.e., $s_1 + \cdots + s_d = 2N$
- $\mathrm{Fill}_\varsigma^\lambda$: *fillings* of Young diagrams of shape $\lambda \vdash 2N$, that is, each k appears s_k times
- $\mathrm{RSYT}_\varsigma^\lambda \subset \mathrm{Fill}_\varsigma^\lambda$: **row-strict Young tableaux**, entries weakly increasing down columns and *strictly* increasing along rows
- Example: $\varsigma = (2, 1, 2, 1)$

1	2	3
1	3	4

Fusion setup

- **valences** $\varsigma = (s_1, \ldots, s_d)$: *composition* of $2N$, i.e., $s_1 + \cdots + s_d = 2N$
- $\mathrm{Fill}^{\lambda}_{\varsigma}$: *fillings* of Young diagrams of shape $\lambda \vdash 2N$, that is, each k appears s_k times
- $\mathrm{RSYT}^{\lambda}_{\varsigma} \subset \mathrm{Fill}^{\lambda}_{\varsigma}$: **row-strict Young tableaux**, entries weakly increasing down columns and *strictly* increasing along rows
- **Lemma.** $\alpha \in \mathrm{RSYT}^{\lambda}_{\varsigma} \implies \tilde{\alpha} \in \mathrm{SYT}^{\lambda}$

$$\alpha = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 3 & 4 \\ \hline \end{array} \quad \Longrightarrow \quad \tilde{\alpha} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}$$

- **Definition.** **fused Specht polynomial**

$$\mathcal{P}_{\alpha}(y_1, \ldots, y_d) = \left[\frac{p_{\varsigma} . \mathcal{P}_{\tilde{\alpha}}(x_1, \cdots, x_{2N})}{\prod_{k=1}^{d} \prod_{q_k \leq i < j < q_{k+1}} (x_j - x_i)} \right]_{x_{q_k}, x_{q_k+1}, \ldots, x_{q_{k+1}-1} = y_k \; \forall\, k}$$

Fused Specht polynomials

$$\mathcal{P}_\alpha(y_1,\ldots,y_d) = \left[\frac{p_\varsigma.\,\mathcal{P}_{\tilde{\alpha}}(x_1,\cdots,x_{2N})}{\prod_{k=1}^{d}\prod_{q_k\le i<j<q_{k+1}}(x_j-x_i)}\right]_{x_{q_k},x_{q_k+1},\ldots,x_{q_{k+1}-1}=y_k\ \forall k}$$

- $p_\varsigma = \frac{1}{s_1!\cdots s_d!}\prod_{k=1}^{d}\sum_{\sigma\in\mathfrak{S}_{s_k}}\mathrm{sgn}(\sigma)\,\sigma$ is the *antisymmetrizer*
- $q_k = 1+\sum_{j=0}^{k-1} s_j$ are sizes of the "fused groups"

Prop. [Lafay, P. & Roussillon '24+]
$\mathcal{P}_\alpha$ has an explicit alternative formula as a linear combination involving monomials and Schur polynomials

Conformal blocks — CFT properties

Thm. [Lafay, P. & Roussillon '24+] The functions $\mathcal{U}_\alpha$ satisfy

- d **BPZ PDEs** of orders $(s_1 + 1, \ldots, s_d + 1)$ with $\kappa = 4$, $c = 1$,
- **Möbius covariance**

$$\mathcal{U}_\alpha(f(x_1), \ldots, f(x_d)) = \Big(\prod_{1 \le j \le d} |f'(x_j)|^{-s_j^2/4} \Big) \mathcal{U}_\alpha(x_1, \ldots, x_d)$$

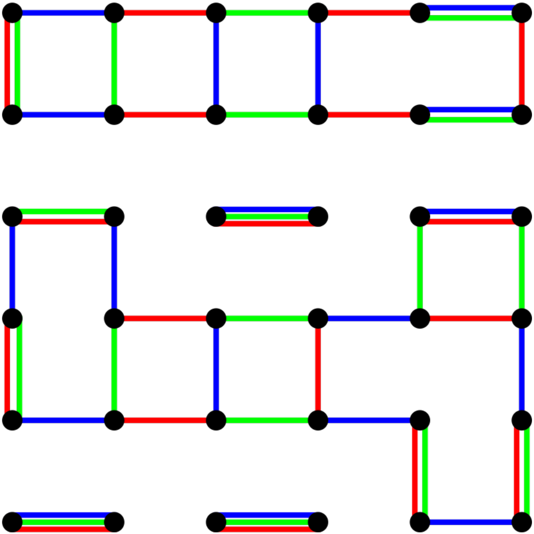
(for $f \colon \mathbb{H} \to \mathbb{H}$ s.t. $f(x_1) < \cdots < f(x_{2N})$)

- specific (incomplete) **asymptotics** related to fusion rules in CFT

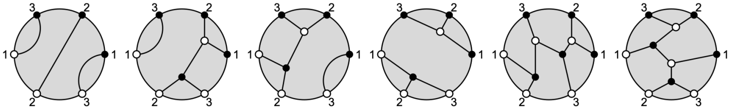
Proof ingredients:

- bootstrap arguments from lower order to higher order
- generalizing [Dubédat '15] ($c \notin \mathbb{Q}$) to $c = 1$ (and $c = -2$ for UST/LERW)
- structure of Verma modules over the Virasoro algebra
- framework of **Virasoro Uniformization** à la [Kontsevich '87]
- (in spirit, closely related to *Segal's sewing formalism* in CFT)

Triple dimers ($c = 2$) and W_3-Conformal blocks



W_3-Conformal blocks and a loose conjecture



Conj. [Lafay & Roussillon '24: arXiv:2402.12013]
Verified in special cases using formulas of [Kenyon & Shi '24].

$$\mathbb{P}^{\delta}_{\beta}[\text{triple dimer web} = \alpha] \xrightarrow{\delta \to 0} \mathcal{K}_{\alpha,\beta} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_d)}{\mathcal{U}_\beta(x_1, \ldots, x_d)}$$

- where $\mathcal{U}_\beta$ is the **conformal block function**

$$\mathcal{U}_\beta(x_1, \ldots, x_d) = \Big(\prod_{i<j} (x_j - x_i)^{-s_i s_j/3} \Big) \mathcal{P}_{\beta^T}(x_1, \ldots, x_d)$$

- $\mathcal{Z}_\alpha := \sum_\beta \mathcal{K}^{-1}_{\alpha,\beta} \mathcal{U}_\beta$
- $\mathcal{P}_{\beta^T}$ is a (transpose) *fused Specht polynomial*
- $\mathcal{K}_{\alpha,\beta} \in \mathbb{Z}_{\geq 0}$ and $\mathcal{K}^{-1}_{\alpha,\beta} \in \mathbb{Z}$ are *combinatorial* matrix elements

Setup (no fusion here)

- $\varsigma = (s_1, \ldots, s_d)$: *composition* of $3N$, i.e., $s_1 + \cdots + s_d = 3N$
- current work [Lafay & Roussillon '24] assumes $s_i \in \{1, 2\}$ for all i
- $\mathrm{Fill}_\varsigma^\lambda$: *fillings* of Young diagrams of shape $\lambda \vdash 3N$, that is, each k appears s_k times
- $\mathrm{RSYT}_\varsigma^\lambda \subset \mathrm{Fill}_\varsigma^\lambda$: **row-strict Young tableaux**, entries weakly increasing down columns and *strictly* increasing along rows
- Example: $\varsigma = (2, 1, 3, 1, 1, 1)$

1	2	3
1	3	4
3	5	6

Setup (no fusion here)

- $\varsigma = (s_1, \ldots, s_d)$: *composition* of $3N$, i.e., $s_1 + \cdots + s_d = 3N$
- $\mathrm{Fill}_\varsigma^\lambda$: *fillings* of Young diagrams of shape $\lambda \vdash 3N$, that is, each k appears s_k times
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- **Lemma.** $\alpha \in \mathrm{RSYT}_\varsigma^\lambda \implies \tilde{\alpha} \in \mathrm{SYT}^\lambda$

$$\alpha = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 3 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \implies \tilde{\alpha} = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array}$$

- **Definition.** transpose **Specht polynomial**

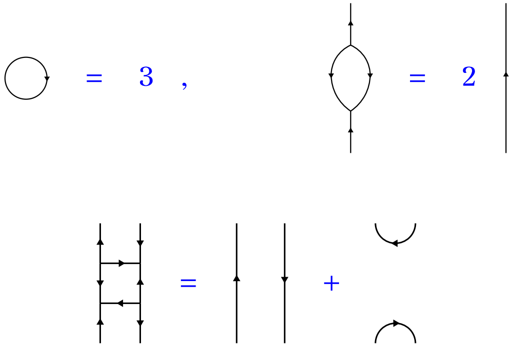
$$\mathcal{P}_\alpha(y_1, \ldots, y_d) = \left[\mathcal{P}_{\tilde{\alpha}^T}(x_1, \ldots, x_{3N})\right]_{x_{q_k}, x_{q_k+1}, \ldots, x_{q_{k+1}-1} = y_k \, \forall k}$$

E.g.
$(y_3 - y_1)(y_2 - y_1)(y_3 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_6 - y_5)(y_6 - y_3)(y_5 - y_3)$

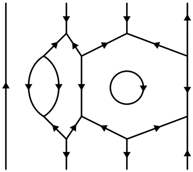
Kuperberg action on W_3-Conformal blocks

Thm. [Lafay & Roussillon '24]
The space $\mathrm{span}\{\mathcal{U}_\alpha : \alpha \in \mathrm{RSYT}_\varsigma^{(N,N,N)}\}$ forms a simple web module of the $\mathfrak{sl}_3$ Kuperberg algebra $\mathrm{K}_\varsigma(3)$ with fugacity parameter 3.

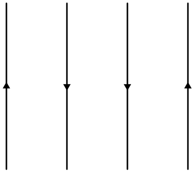
Kuperberg algebra $\mathrm{K}_\varsigma(3)$ is a planar diagram algebra w. relations



Example of web reduction



= 6



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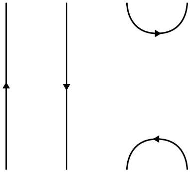

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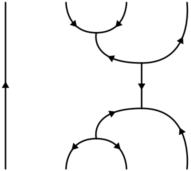

W_3-Conformal blocks — CFT properties

Thm. [Lafay & Roussillon '24+] $s_i \in \{1, 2\}$. The functions $\mathcal{U}_\alpha$ satisfy

- d 3rd order **BPZ W_3-null-state PDEs**
  with central charge $c = 2$ (NB: no corresponding κ exists – im-Toda?)
- 8 PDEs which are W_3 **global Ward identities**
  (five 2nd order and three 1st order), yielding Möbius covariance
- specific (incomplete) **asymptotics** related to fusion rules in CFT

Upshot: Triple-dimer connection probabilities should be given by **specific CFT correlation functions with W_3 algebra symmetry** !

We believe that analogous results hold also for *higher rank* webs,
Young tableaux of more columns or rows... with W algebras