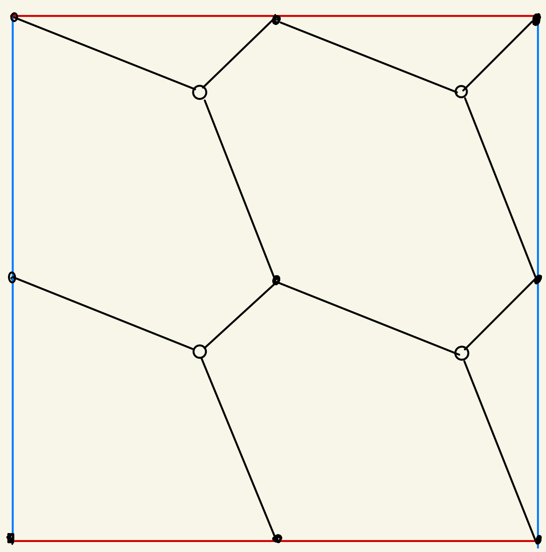
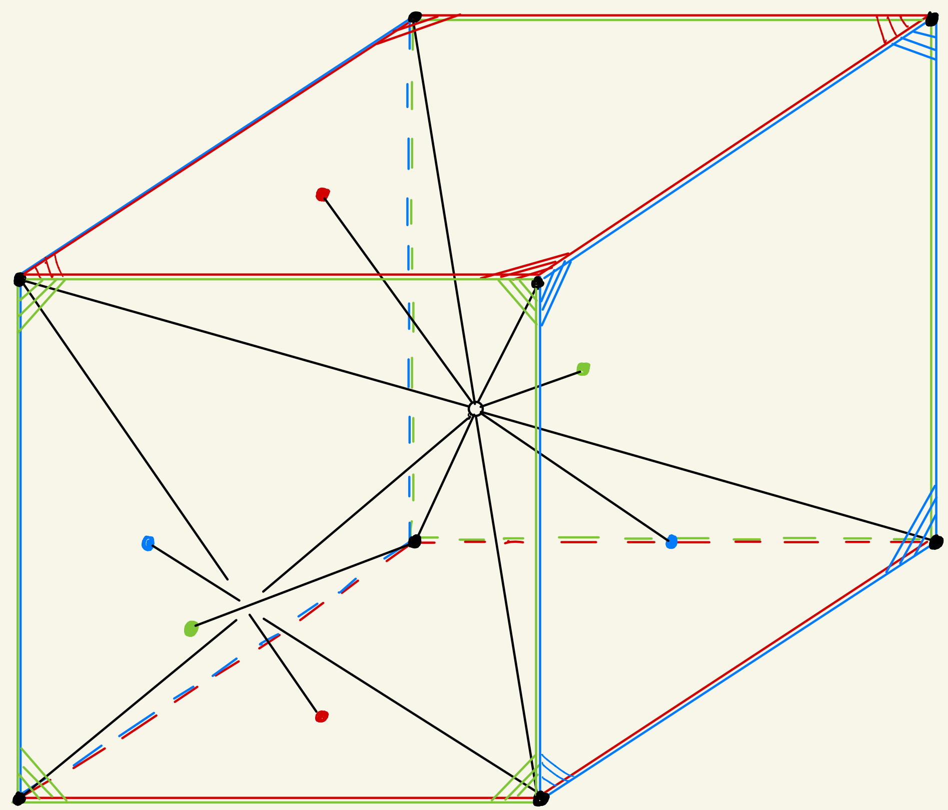
Bipartite Graphs and Mirror Coamoebae

Harold Williams (USC), w/ Chris Kuo (USC)

Workshop: Statistical Mechanics and Discrete Geometry



$$F(x,y) = a + bx + cy + dx^2 + exy + fy^2$$



$$F(x,y,z) = a + bx + cy + dz + ex^2 + fxy + gxz + hy^2 + iyz + jz^2$$

Mirrors, amoebae, and coamoebae

- A mirror of a variety X is a symplectic manifold $\check{X}$ (i.e. a manifold w/ nondegenerate 2-form $\omega \in \Omega^2(\check{X})$) whose symplectic geometry reflects the algebraic geometry of X.
- Rudimentary form: one-to-many correspondence between subvarieties $Z \subset X$ and Lagrangians $L \subset \check{X}$ equipped w/ a local system E (i.e. $\dim L = \frac{1}{2} \dim \check{X}$, $\omega|_L \equiv 0$, $E : \pi_1(L) \to GL_n\mathbb{C}$).

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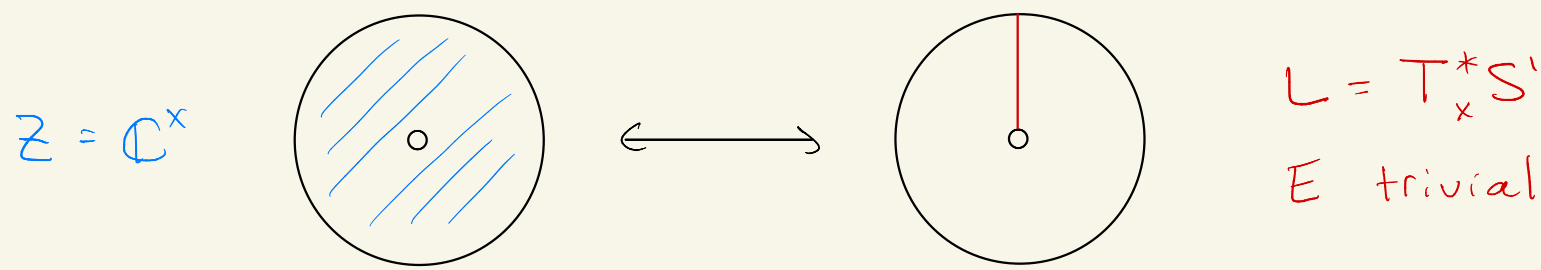
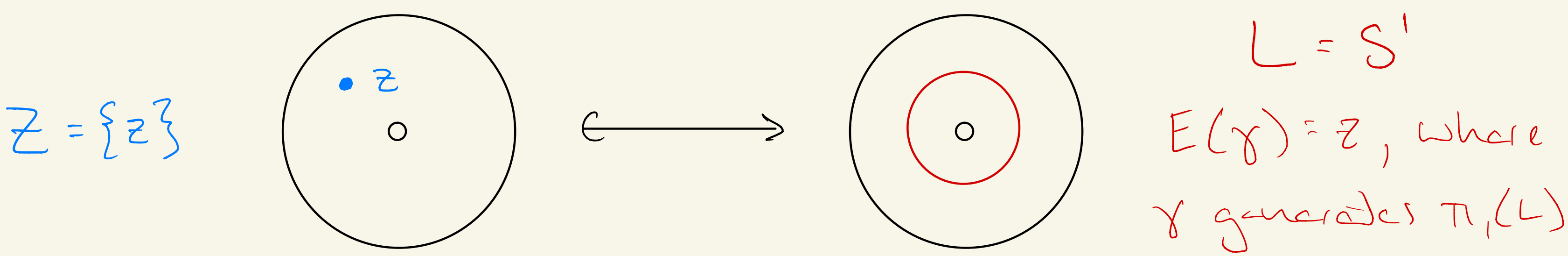
Ex $X = (\mathbb{C}^\times)^n$ is self-mirror (symplectic via $(\mathbb{C}^\times)^n \simeq T^*T^n$).





Mirrors, amoebae, and coamoebae

- To $Z \subset (\mathbb{C}^\times)^n$ one associates the following:
  - amoeba $A(Z) \subset \mathbb{R}^n$, its image under $(z_i) \mapsto (\log|z_i|)$
  - coamoeba $C(Z) \subset T^n$, its image under $(z_i) \mapsto (\operatorname{Arg} z_i)$
- Theme: interesting combinatorial approximations of $A(Z)$ and $C(Z)$.

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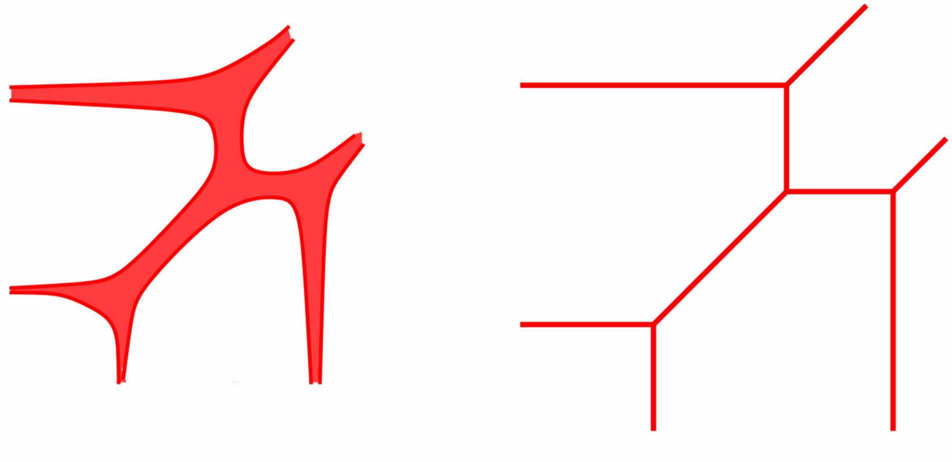
- Theme: interesting combinatorial approximations of $A(Z)$ and $C(Z)$.
- For Z a hypersurface, a suitable scaling limit takes $A(Z)$ to a tropical variety A_{trop}, an $(n-1)$-dim'l polyhedral encoding coarse features of $A(Z)$ (Mikhalkin '02)
- We may choose a mirror (L, E) of Z so that $A(L)$ is ε-close to A_{trop} for any $\varepsilon > 0$ (Hicks '19).



$\mathrm{Log}_t(\mathcal{C}_t)$ $\qquad$ $\lim_{t\to+\infty} \mathrm{Log}_t(\mathcal{C}_t)$

FIGURE 7. $\mathcal{C}_t : 1 - z - w + t^{-2}z^2 - t^{-1}zw + t^{-2}w^2 = 0$

(Brugallé - Itenberg - Mikhalkin - Shaw)

Mirrors, amoebae, and coamoebae

- Feng-He-Kennaway-Vafa '05: for $Z \subset (\mathbb{C}^\times)^2$ a curve, $C(Z)$ retracts in good cases onto a bipartite graph Γ for which Z arises as a spectral curve.

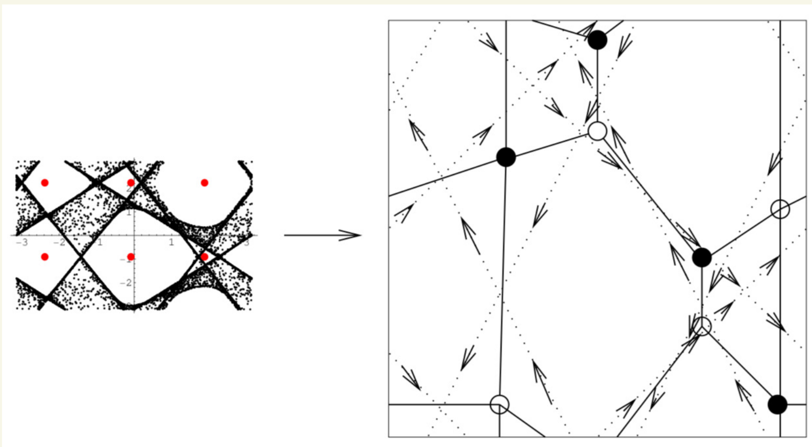


Figure 25: The alga for $P(z,w) = 1 + z + \frac{1+\imath}{z} + \frac{3-2\imath}{z^2} + w + \frac{-1-4\imath}{w}$.

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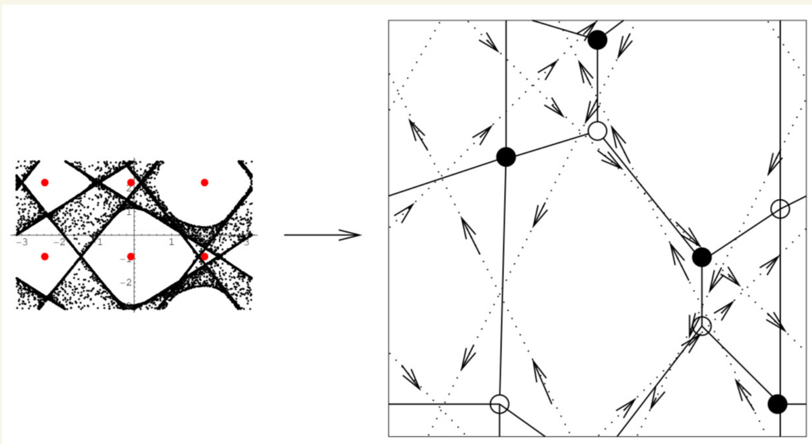
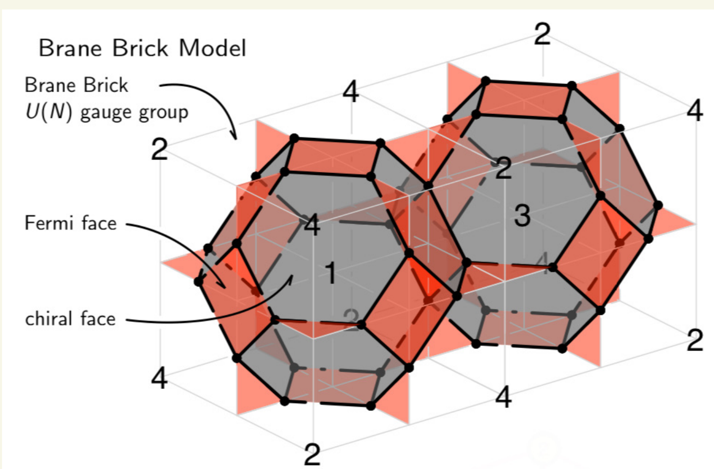
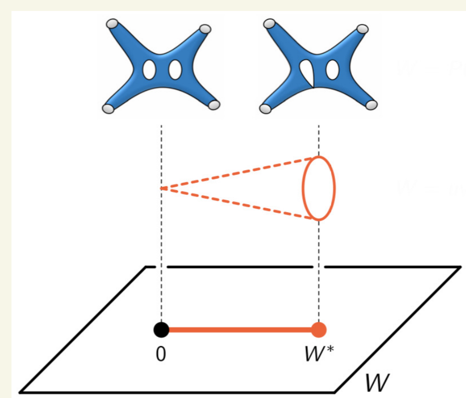


Figure 25: The alga for $P(z,w) = 1 + z + \frac{1+i}{z} + \frac{3-2i}{z^2} + w + \frac{-1-4i}{w}$.

- Futaki-Ueda '10, Franco-Lee-Seong-Vafa '16: for $Z = V(f) \subset (\mathbb{C}^\times)^n$ a hypersurface, $C(Z)$ retracts in good cases onto an $(n-1)$-dim'l polyhedral complex whose faces are the vanishing thimbles of f.



$$f(x,y,z) = 1 + x + y + z + \frac{1}{xyz}$$



Mirror coamoebae for $n=2$

- Homological MS: a Lagrangian $L \subset \check{X}$ w/ local system defines a coherent sheaf $\mathcal{F}$ on X ($Z \subset X \leftrightarrow \mathcal{O}_D \in Coh(X)$).
- For $Z = V(f) \subset (\mathbb{C}^\times)^2$ a curve, let $\Gamma \subset T^2$ be any minimal bipartite graph w/ the same Newton polygon as f (i.e. primitive edges of $N_f \leftrightarrow$ classes of zig-zags in $H_1(T^2) \cong \mathbb{Z}^2$).

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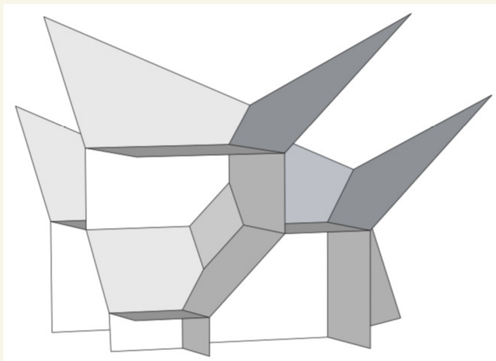
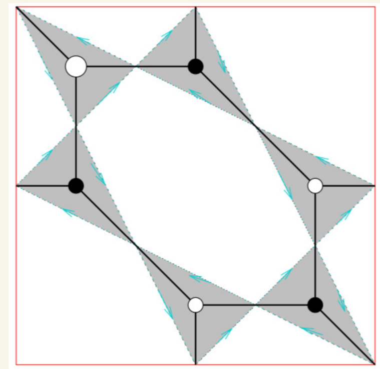
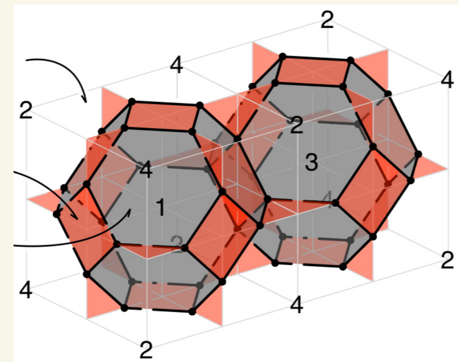
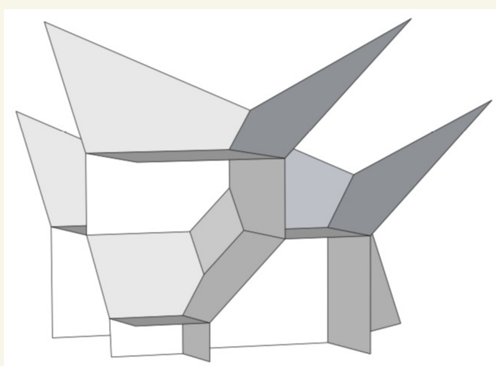
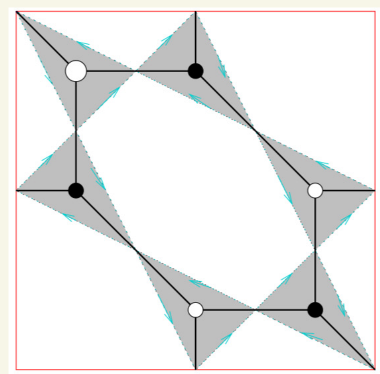
- Treumann-W.-Zaslow '18: for generic Z, there is a mirror (L, E) to Z s.t. $C(L)$ is ε-close to Γ.

Topologically L is the Goncharov-Kenyon conjugate surface of Γ, hence edge-weightings of Γ define local systems on L. The mirror sheaf of L w/ such a local system is the spectral transform of the Kasteleyn matrix $K(z,w)$ of the edge-weighting (i.e. $\operatorname{cok}\left(\mathbb{C}[z^{\pm 1}, w^{\pm 1}]^{\oplus \Gamma_b} \xrightarrow{K(z,w)} \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^{\oplus \Gamma_w}\right)$.

Summary: combinatorics of amoebae and coamoebae

	amoebae	coamoebae: $n=2$	coamoebae: $n>2$
hyper-surface $Z\subset(\mathbb{C}^*)^n$	 tropical varieties	 bipartite graphs	 brane brick models
mirror			
Lagrangian $L\subset(\mathbb{C}^\times)^n$	 tropical varieties	 bipartite graphs	

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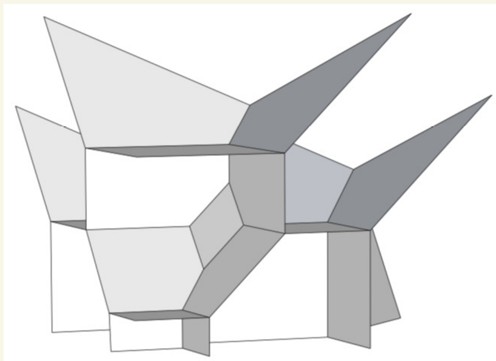
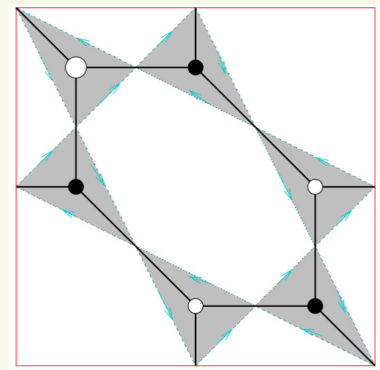
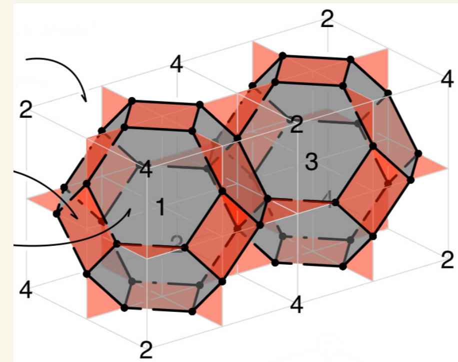
hyper-surface $Z\subset(\mathbb{C}^*)^n$	 tropical varieties	 bipartite graphs	 brane brick models
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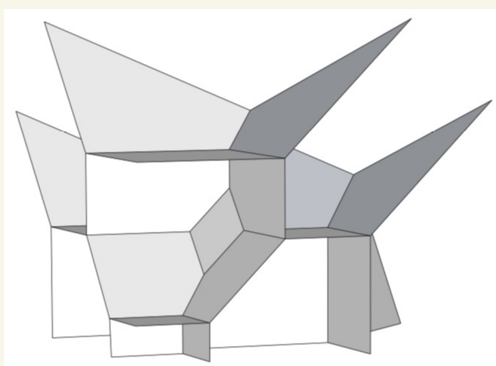
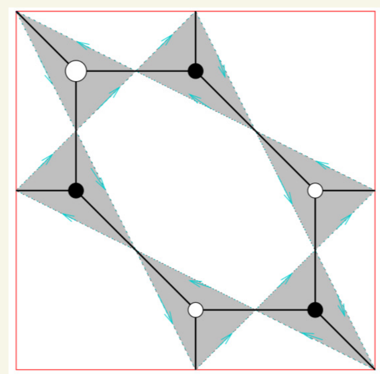
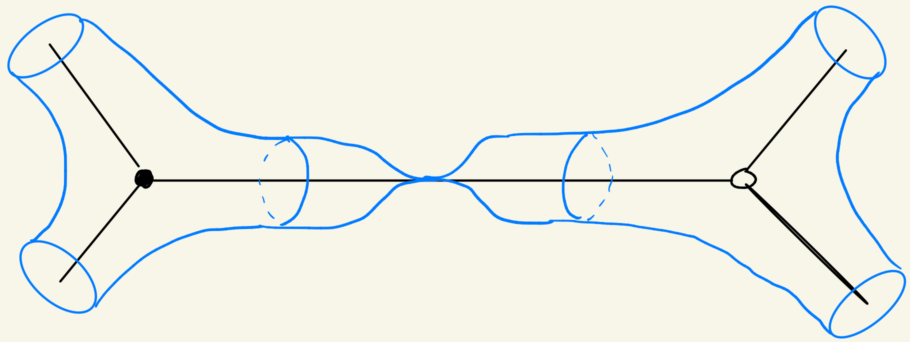
Lagrangian $L\subset(\mathbb{C}^\times)^n$	 tropical varieties	 bipartite graphs	?

Mirror coamoebae for $n>2$

- Let $\Gamma \subset M$ be a bipartite graph in an n-manifold, and define $\pi(\Lambda) \subset M$ by (1) smoothing an ε-nbhd of Γ, (2) pinching to a point in the middle of each edge.



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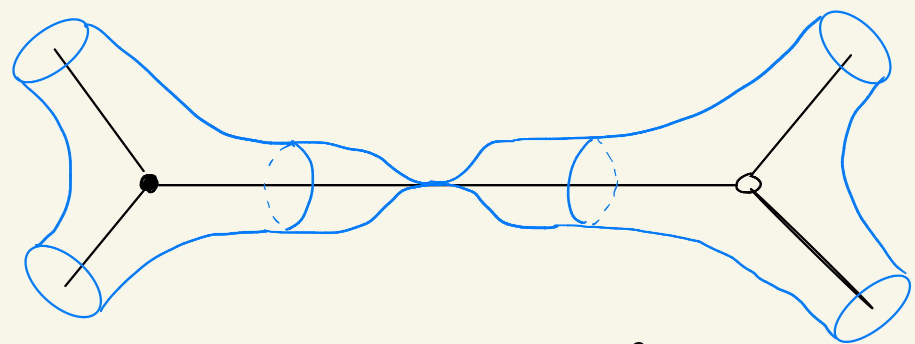
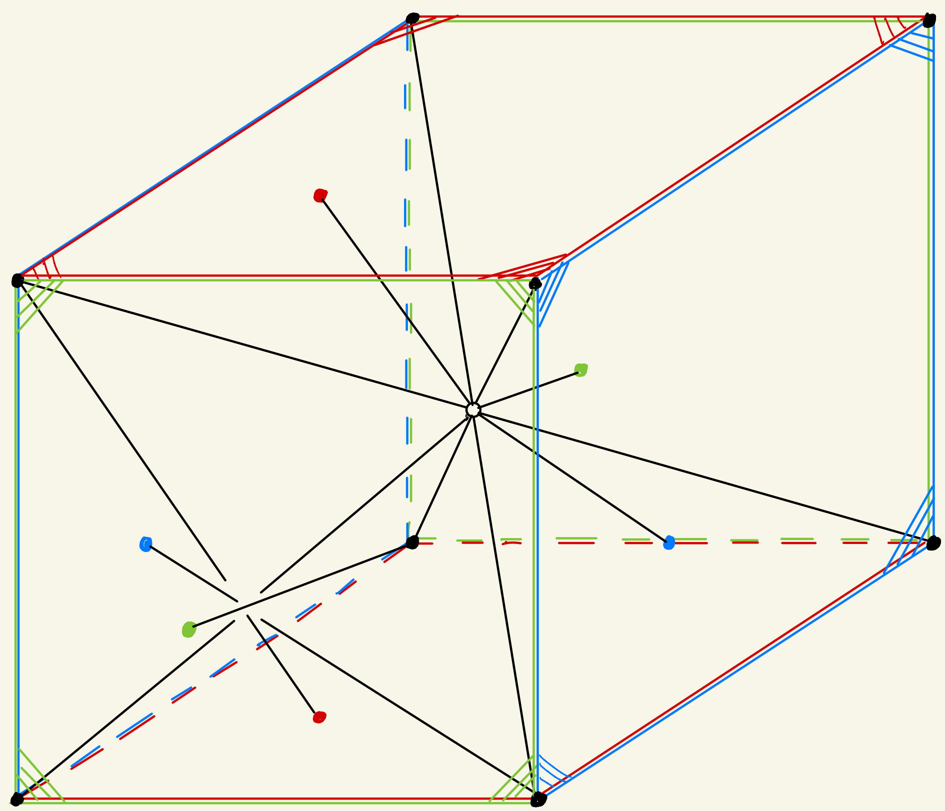


- Let $W, B \subset M$ be the union of the white and black regions, and take $f \in C^\infty(M)$ positive on $W \cup B$, zero elsewhere.
- Let $L(\Gamma) \subset T^*M$ be the closure of $d\log f|_W \cup -d\log f|_B$. It is a Lagrangian which retracts onto Γ and is asymptotic to a Legendrian submanifold $\Lambda \subset T^\infty M$ (fiberwise boundary of T^*M).
- When $n=2$, $L(\Gamma)$ is the G&K conjugate surface of Γ (c.f. Shende-Treumann-W.-Zaslow), $\pi(\Lambda)$ are its zig-zag paths, and Λ is the associated link.

Mirror coamoebae for $n > 2$

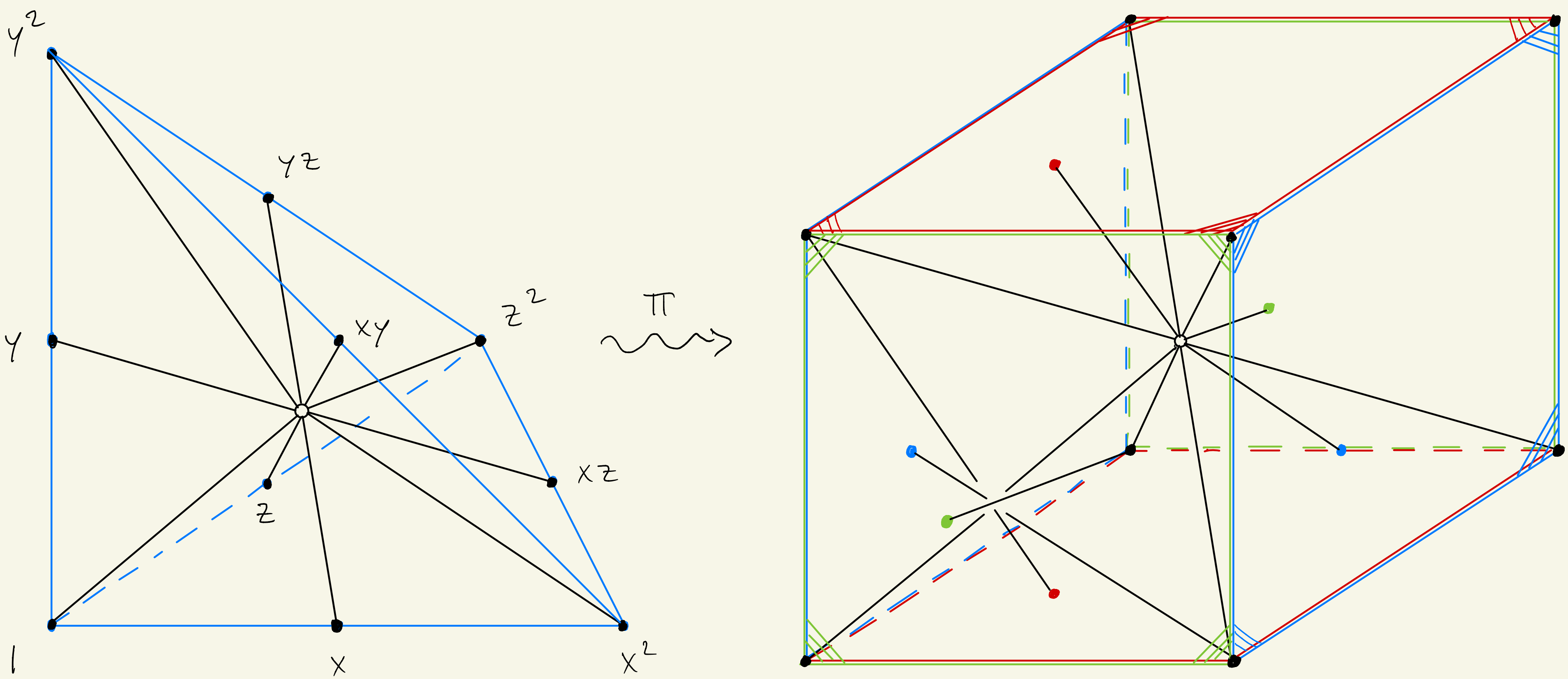
- Now assume $n > 2$ and fix a convex lattice polytope $P \subset \mathbb{R}^n$.



Mirror coamoebae for $n>2$

- Now assume $n>2$ and fix a convex lattice polytope $P\subset\mathbb{R}^n$.
- Write $\pi:\mathbb{R}^n\longrightarrow T^n\simeq\mathbb{R}^n/\mathbb{Z}^n$ and choose a generic $\vec{p}\in\mathrm{Int}(P)$.
- Let $\Gamma_p\subset T^n$ be the graph with

$$\Gamma_0=\{\pi(x),\pi(\vec{0})\}\,,\qquad \Gamma_1=\{\pi([\vec{p},\vec{q}])\}_{\vec{q}\in P\cap\mathbb{Z}^n}$$



- Note: construction fails due to self-intersections if $n=2$.

Mirror coamoebae for $n > 2$

<u>Theorem</u> (Kuo-W.) Let $f \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ be generic w/ Newton polytope P, and let $\Gamma \subset T^n$ and $L(\Gamma) \subset T^*T^n$ be the bipartite graph and Lagrangian given by the above constructions. Then $(L(\Gamma), E_f)$ is mirror to $Z = V(f)$, where E_f is the local system induced by weighting each edge of Γ by the coefficient of the corresponding monomial in f.

- In particular, the mirror of Z can be chosen to have coamoeba ε-close to Γ for any $\varepsilon > 0$.

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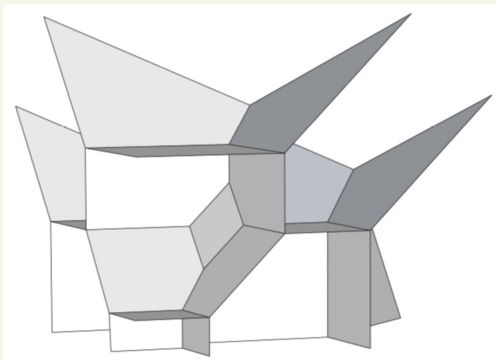
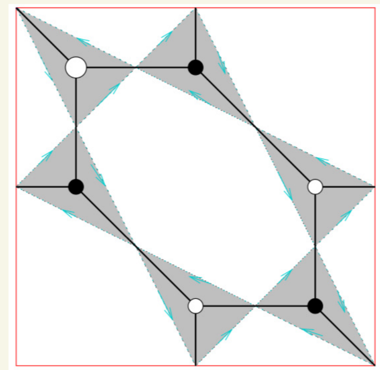
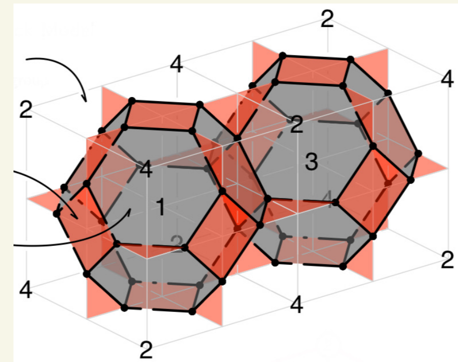
hyper-surface $Z \subset (\mathbb{C}^*)^n$	 tropical varieties	 bipartite graphs	 brane brick models
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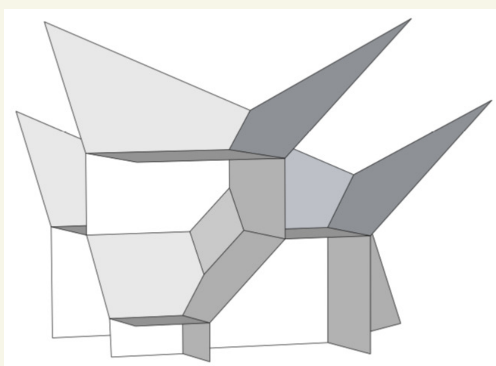
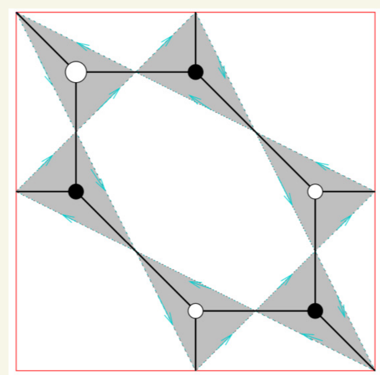
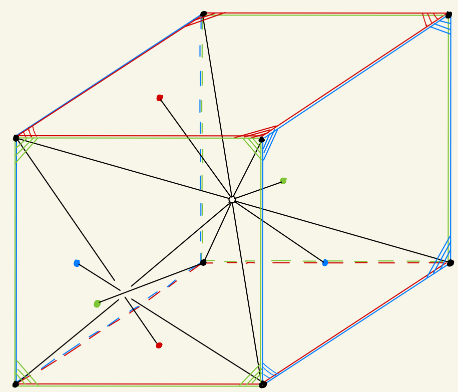
Lagrangian $L \subset (\mathbb{C}^\times)^n$	 tropical varieties	 bipartite graphs	 bipartite graphs