MULTINOMIAL DIMER MODEL

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based on earlier work with Cosmin Pohoata (Emory)

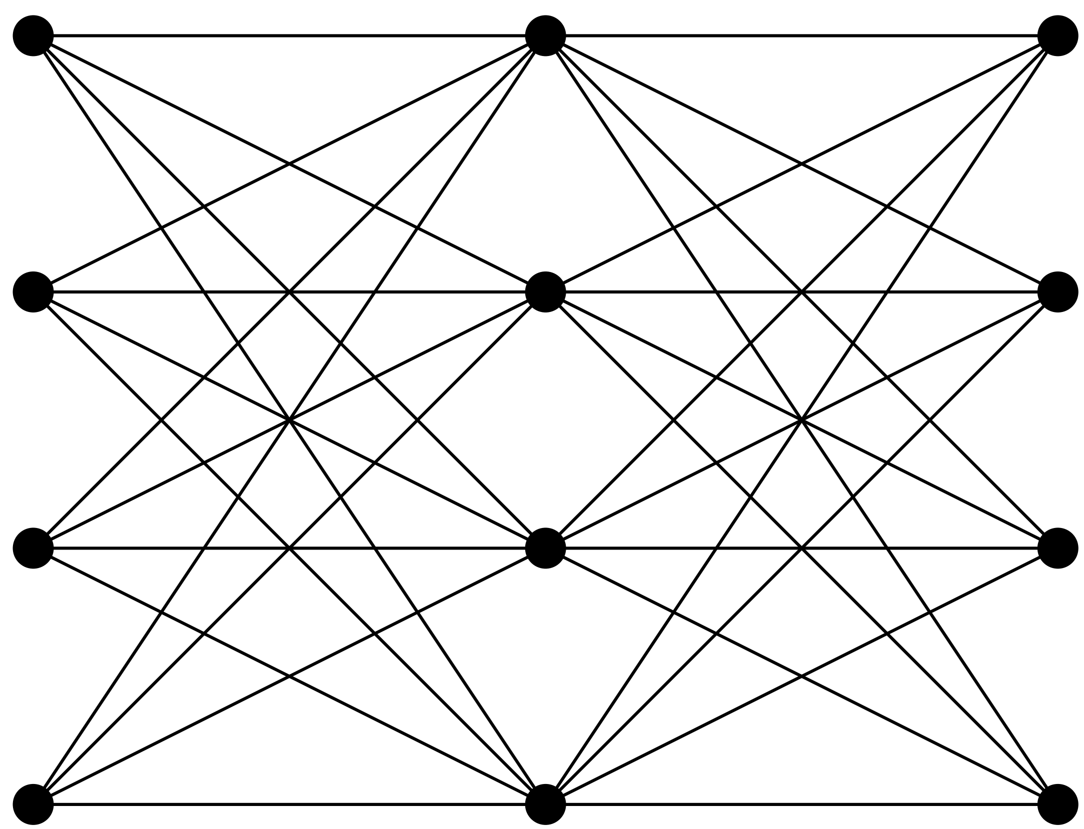
1. Definitions
2. Counting
3. Asymptotics
4. Limit shapes in 2D and 3D
5. Fluctuations

$G = (V, E)$ is a finite graph

G_n has vertices $V \times \{1, 2, \ldots, n\}$

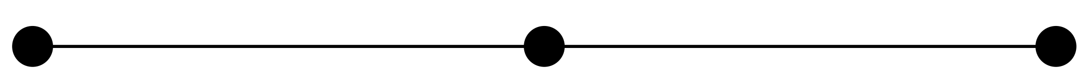
G_n has edges $(u, i) \sim (v, j)$ whenever $u \sim v$.

G_4

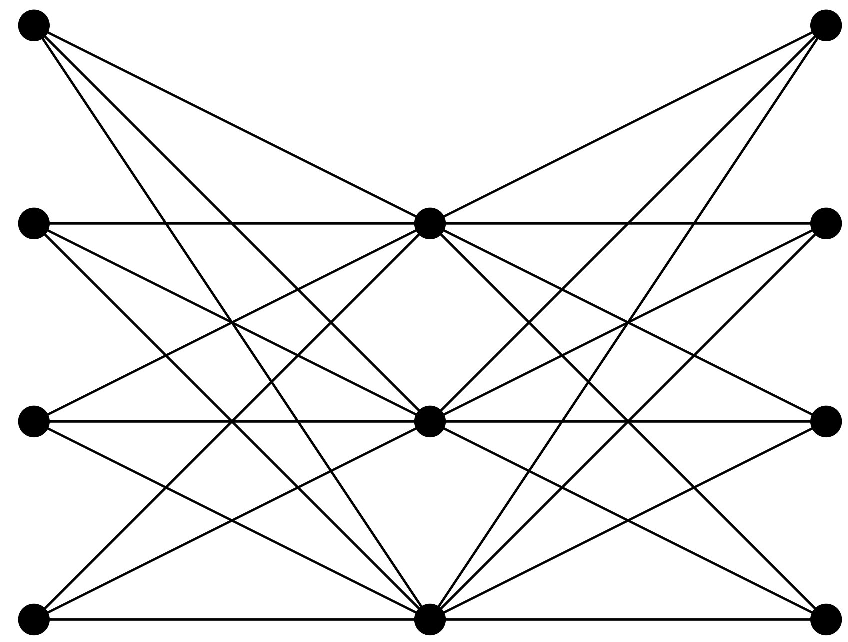


"blow-up" graph of G

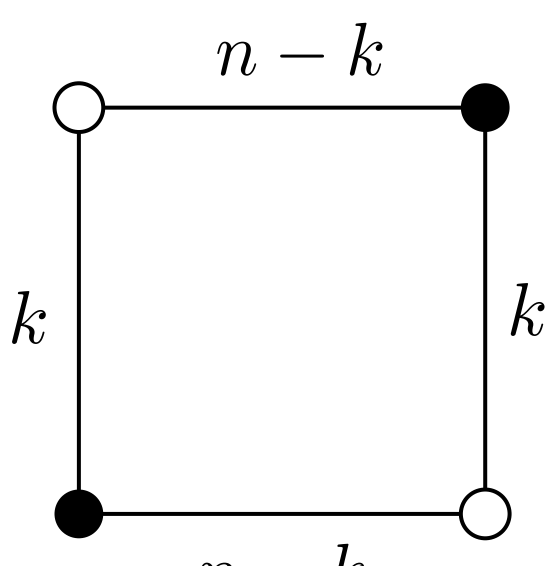
G



We can also let n vary from vertex to vertex: $\mathbf{n} = (n_1, \ldots, n_V)$.



Let $Z(\mathbf{n})$ be the number of dimer covers of $G_{\mathbf{n}}$.



$$Z(n,n,n,n) = \sum_{k=0}^{n} \binom{n}{k}^4 k!^2 (n-k)!^2 = \sum_{k=0}^{n} \frac{n!^4}{k!^2 (n-k)!^2}$$

$$= K! e^{cK + o(K)}$$

where K is the number of dimers ($K = 2n$).

Let x_v a variable for each vertex v of G.

Let $P(\mathbf{x}) = \sum_{uv \in E} x_u x_v$ be the "edge polynomial".

Thm [K'-Pohoata 2021]:

$$Z := \sum_{\mathbf{n} \geq 0} Z(\mathbf{n}) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = e^{P}.$$

Asymptotics

Let K = number of dimers $= \frac{1}{2}\sum n_v$.

Suppose $\mathbf{n} \to \infty$ with $\frac{n_v}{K} \to \alpha_v$.

(So α_v is the fraction of tiles covering v.)

Thm[KP]: We have $Z(\mathbf{n}) = K! e^{cK+o(K)}$ where

$$c = \log P(\mathbf{x}) - \sum_v \alpha_v \log(\alpha_v x_v)$$

and where the x_v are the unique positive solution to

$$\frac{x_v P_{x_v}}{P} = \alpha_v.$$

we call $\{x_v\}$ the *critical gauge.*

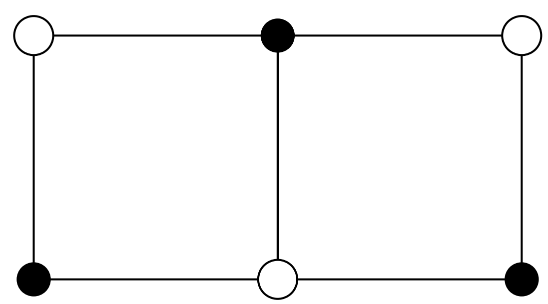
$$\sum_{u\sim v} x_u x_v = \alpha_v P.$$

the critical gauge equation is homogeneous.

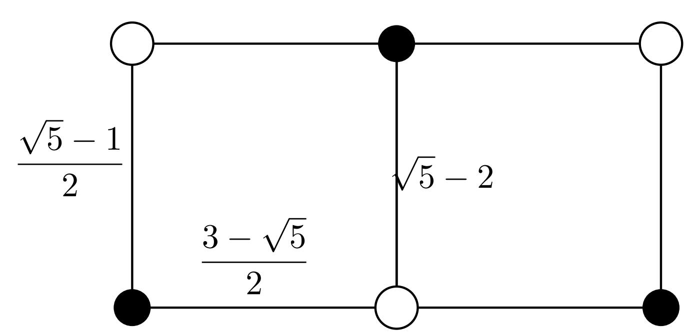
If $\mathbf{n} \equiv n$, we can take $\alpha_v P \equiv 1$, so that the critical gauge is one where the sum of edge weights around each vertex is 1.

Then "dimer probabilities" (edge fractions) are $x_u x_v$.

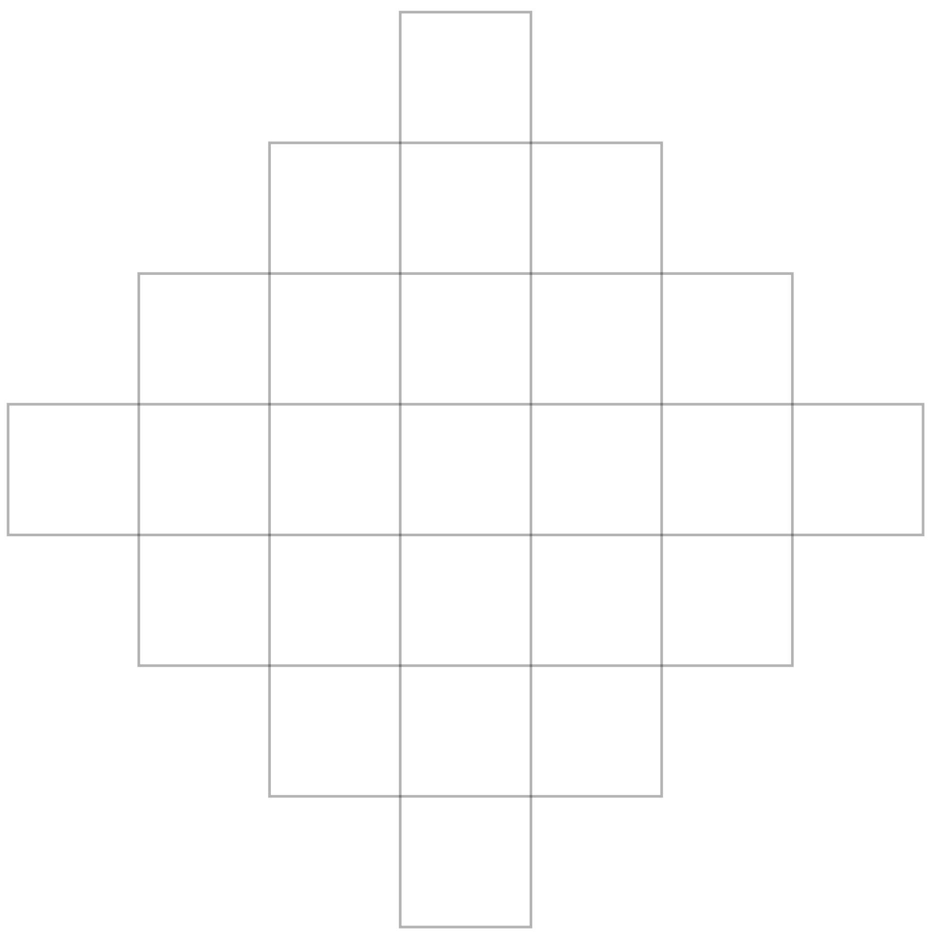
Example.

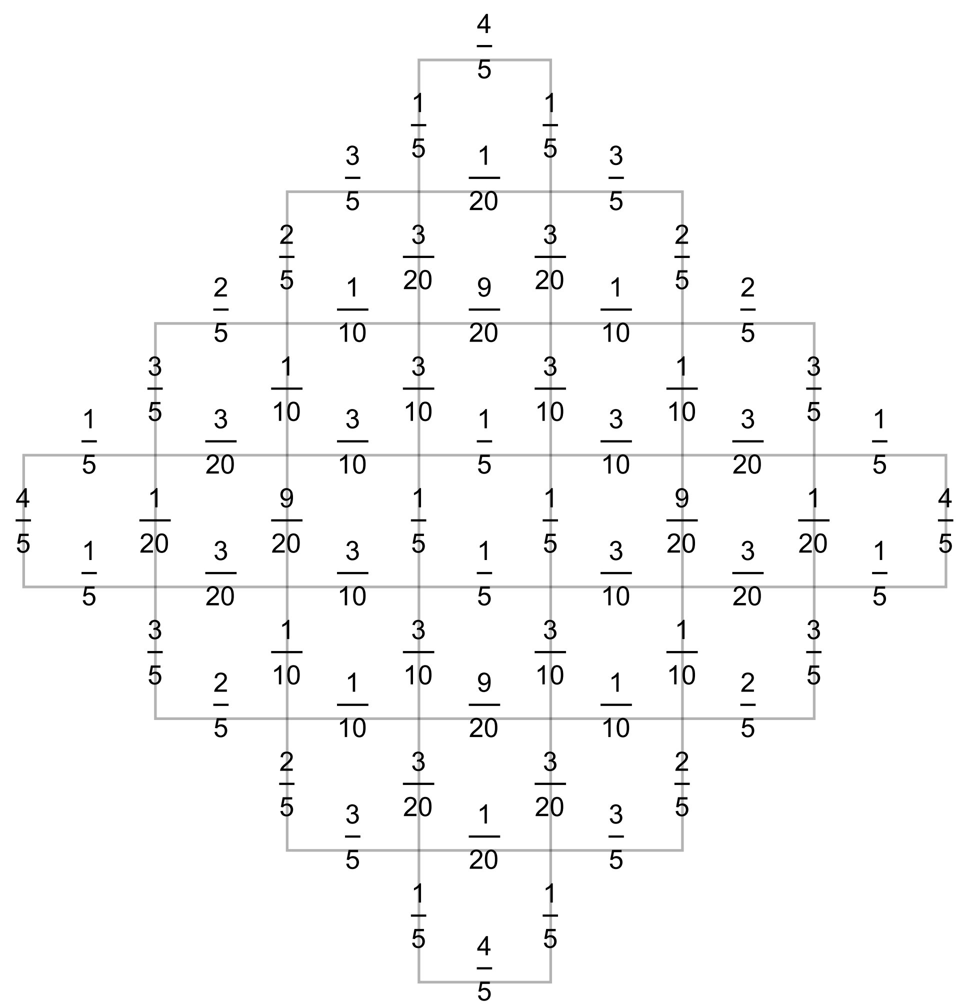


$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ 0 & x_5 & 0 \\ 0 & 0 & x_6 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & \frac{3-\sqrt{5}}{2} & 0 \\ \frac{3-\sqrt{5}}{2} & \sqrt{5}-2 & \frac{3-\sqrt{5}}{2} \\ 0 & \frac{3-\sqrt{5}}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}$$

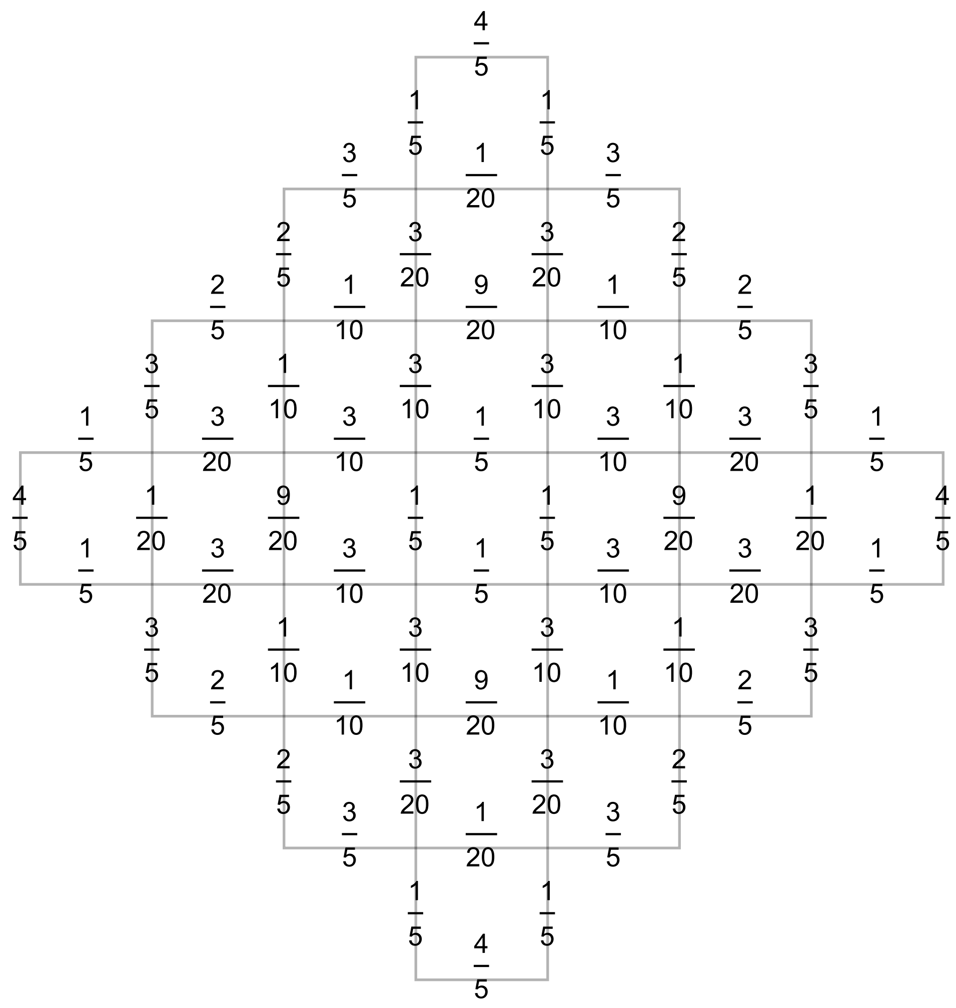


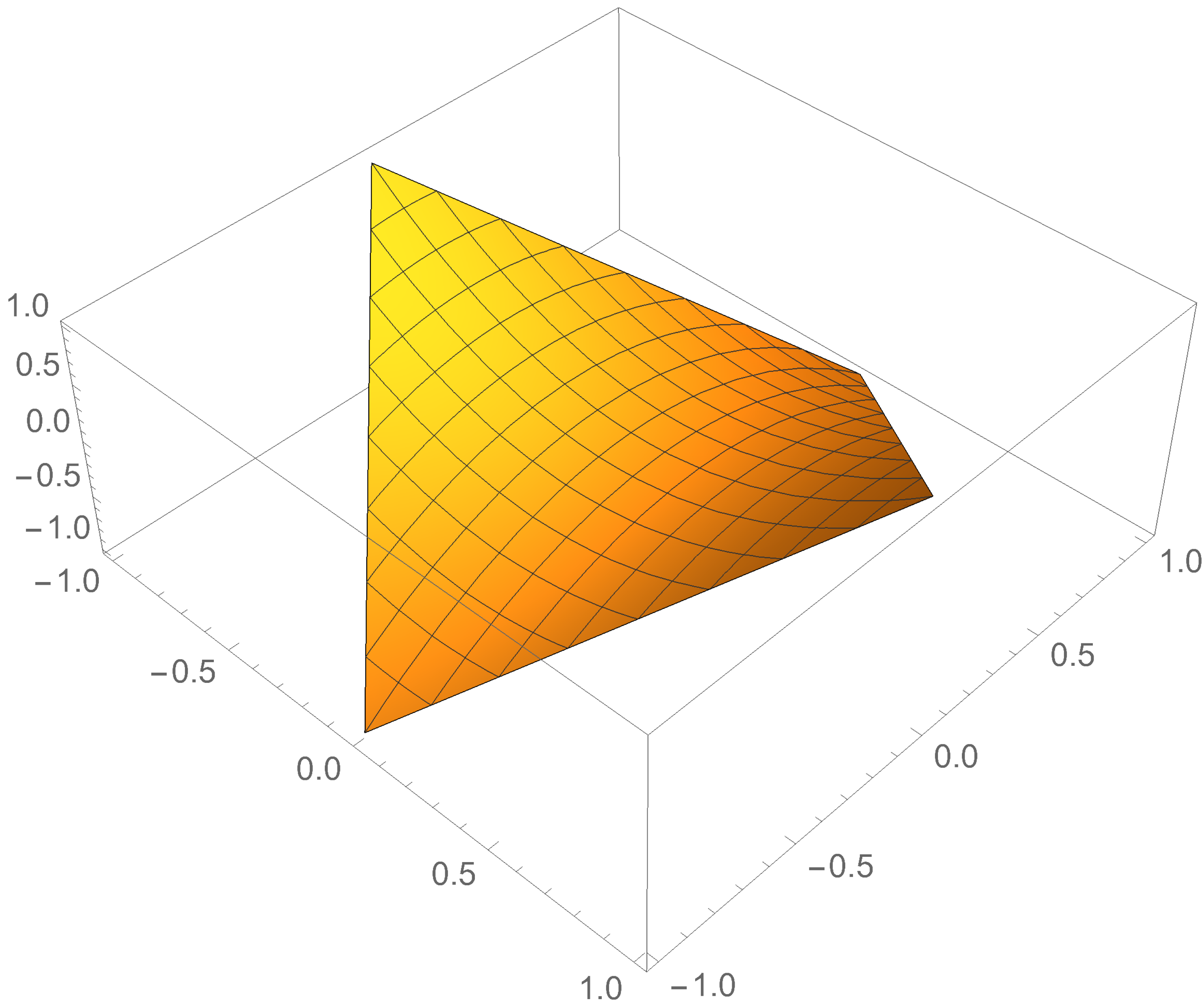
critical gauge for Aztec diamond





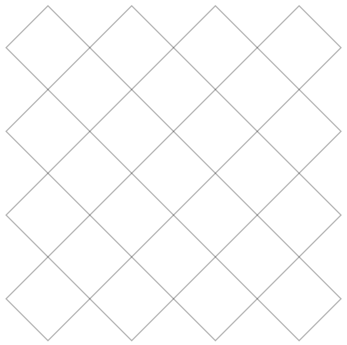
For the critical gauge as above, the tile fractions (edge probabilities) are $x_u x_v$.





The scaling limit height function for the aztec diamond is $h(x, y) = x^2 - y^2$.

Variational principle:



Thm [K-Wolfram]: For multinomial dimers on the scaling limit of (rotated) $\mathbb{Z}^2$, on a domain R with boundary height function $u : \partial R \to \mathbb{R}$, the limit height function h is the unique function with $h|_{\partial R} = u$ maximizing

$$\mathrm{Ent}(h) = \iint_R \sigma(\nabla h)\,dx\,dy$$

where

$$\sigma(s,t) = -\frac{1-s}{2}\log\frac{1-s}{2} - \frac{1+s}{2}\log\frac{1+s}{2} - \frac{1-t}{2}\log\frac{1-t}{2} - \frac{1+t}{2}\log\frac{1+t}{2}.$$

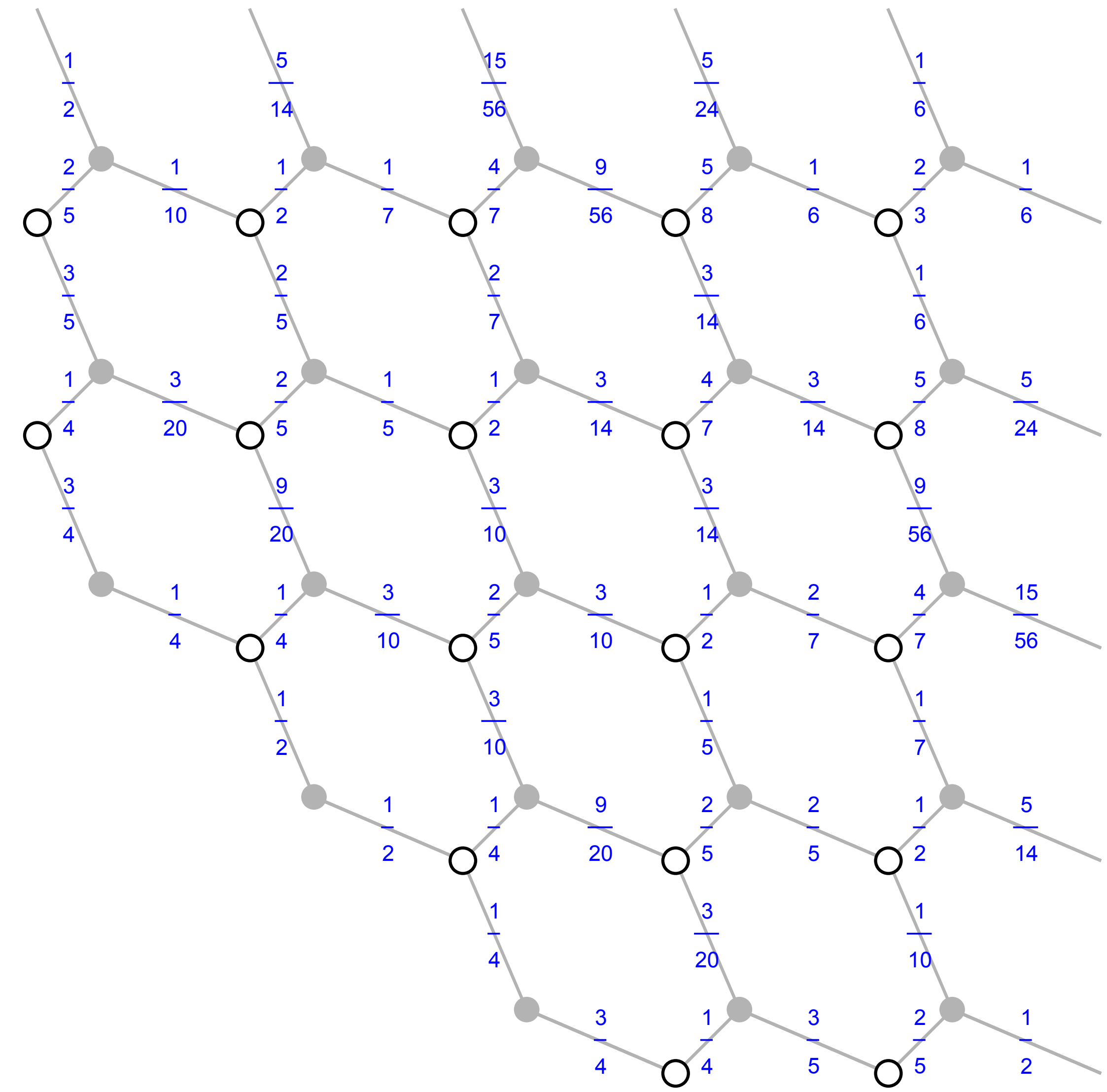
and $(s,t) \in [-1,1]^2$.

The EL equation for the limiting height function is

$$\frac{h_{xx}}{1-h_x^2} + \frac{h_{yy}}{1-h_y^2} = 0.$$

General solutions can be written in terms of ${}_2F_1$'s.

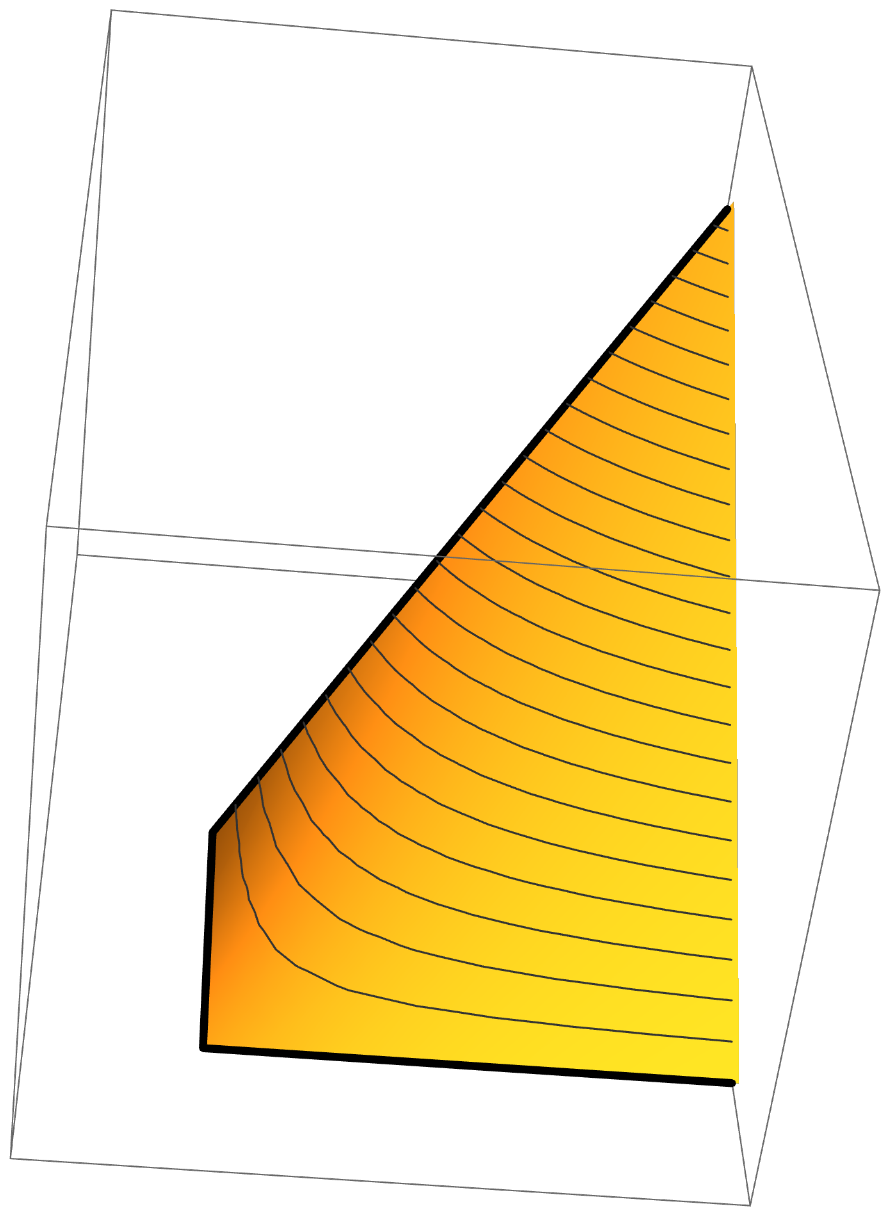
Honeycomb dimers



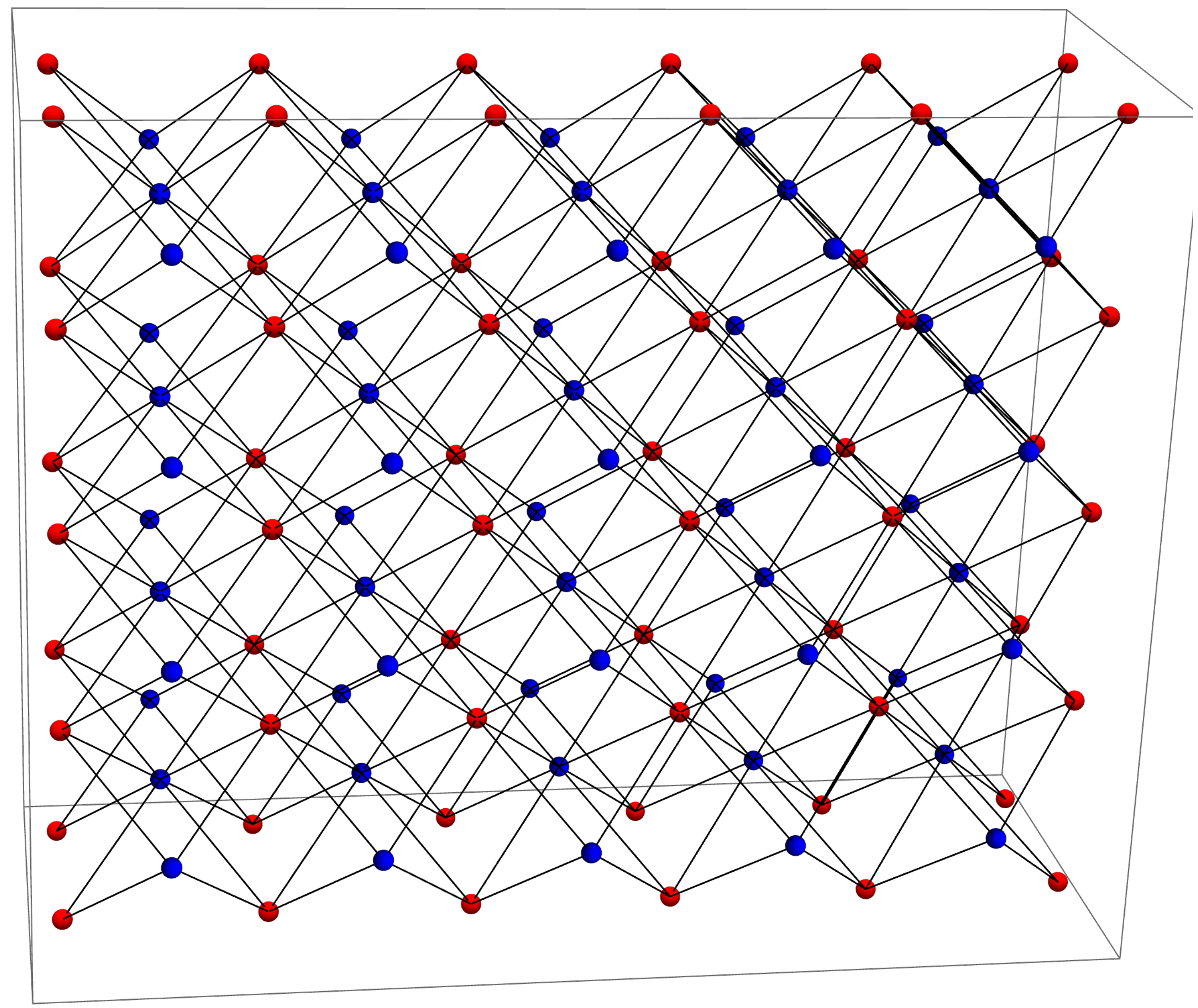
A similar variational principle holds, but with surface tension

$$\sigma(s,t) = s\log s + t\log t + (1-s-t)\log(1-s-t)$$

The EL equation can be solved in general in terms of Bessel functions special solution $$h(x,y) = \frac{(2x-1)(2y+1)}{2(2x+1)}$$



“3D Aztec diamond” (on BCC lattice in $\mathbb{Z}^3$)



Reds: $a \times b \times c$ box

Blues: $(a+1) \times (b-1) \times (c-1)$ box

$$abc = (a+1)(b-1)(c-1)$$

The critical gauge is given by

$$x(i,j,k) = \frac{\binom{a}{i}}{\binom{b}{j}\binom{c}{k}}$$

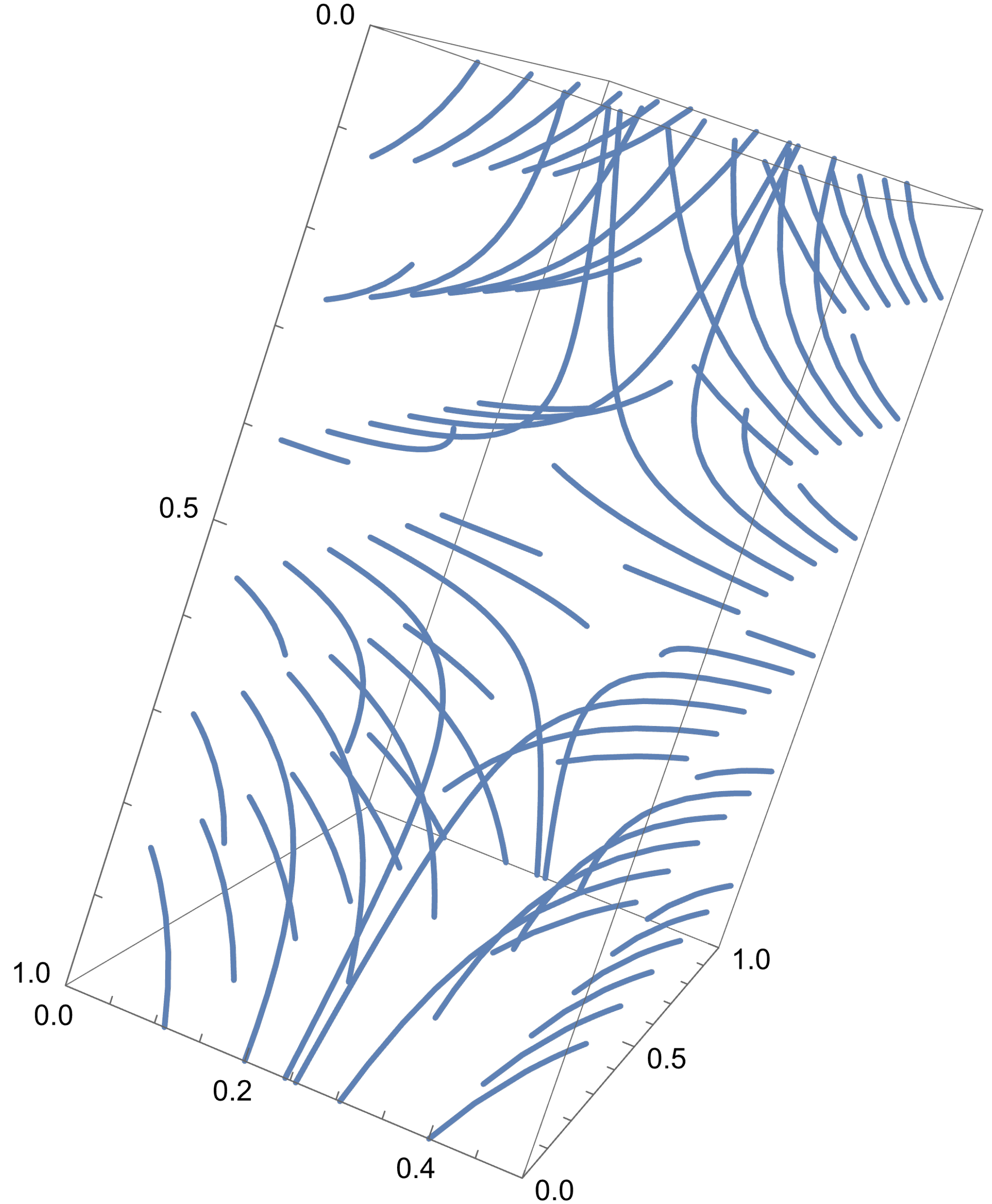
at red vertices and

$$x(i',j',k') = \frac{\binom{b-1}{j}\binom{c-1}{k}}{\binom{a+1}{i+1}} \frac{bc}{(b+1)(c+1)}$$

at blue vertices.

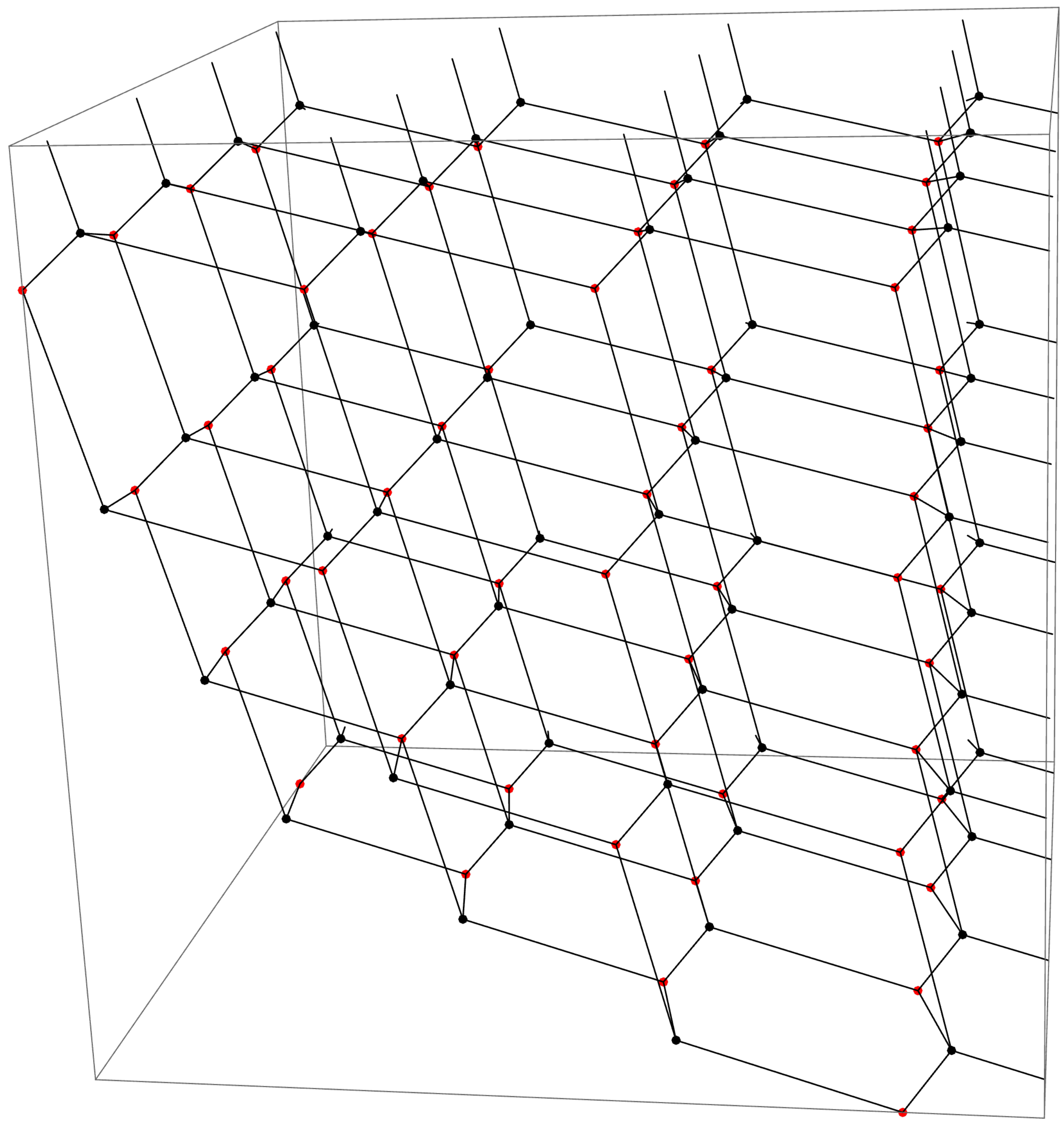
The limit vector field in $[0,\alpha] \times [0,\beta] \times [0,\gamma]$ is

$$(\frac{2x}{\alpha} - 1, 1 - \frac{2y}{\beta}, 1 - \frac{2z}{\gamma})$$



integral curves of the vector field

3D “Honeycomb” model (diamond lattice dimer model)



The EL equations for the vector field (u, v, w) are

$$\frac{u_y}{1-u^2} = \frac{v_x}{1-v^2}$$
$$\frac{v_z}{1-v^2} = \frac{w_y}{1-w^2}$$
$$\frac{w_x}{1-w^2} = \frac{u_z}{1-u^2}.$$

Scaling limit vector field on "truncated orthant" $\{x + y + z > 1\}$:

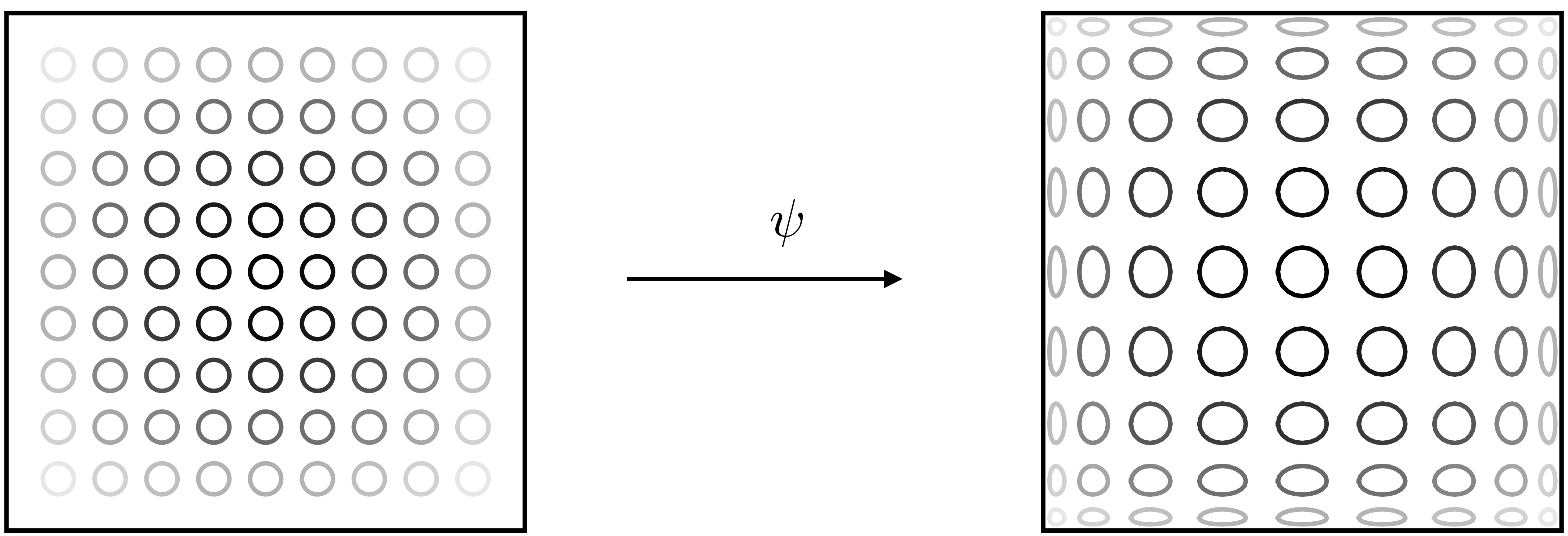
$$(u, v, w) = \left(\frac{x}{(x+y+z)^3}, \frac{y}{(x+y+z)^3}, \frac{z}{(x+y+z)^3}\right)$$

Fluctuations

Thm: (2D Aztec diamond) In the scaling limit, height fluctuations are given by the image of an *inhomogeneous Gaussian Free Field* on $[0, \pi]^2$ (with conductance κ) under a diffeomorphism $\Psi : [0, \pi]^2 \to R$:

$$\psi(u, v) = (\cos u, \cos v),$$

and $\kappa : [0, \pi]^2 \to \mathbb{R}$ is given by $\kappa(u, v) = \frac{1}{\sin u \sin v}$.



GFF with Laplacian $\nabla \cdot \kappa \nabla$

Aztec diamond scaling limit

THANK YOU