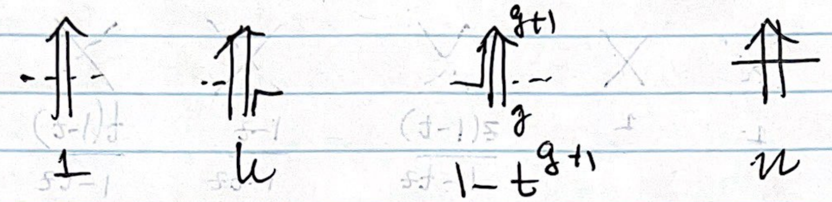
Board (Before !)

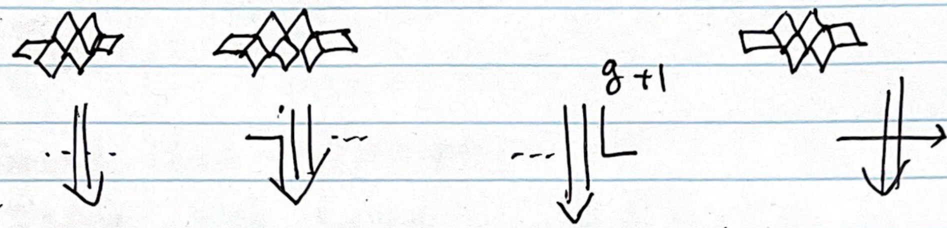
Board 1 ⊕ ice

Red W_u



$1 \quad u \quad 1-t^{g+1} \quad u$

Blue W^*_v

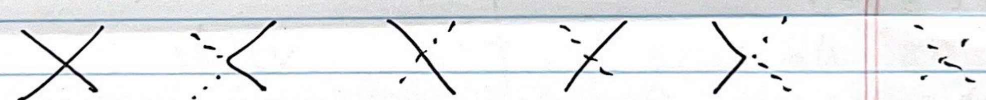


$1 \quad 1 \quad v(1-t^{g+1}) \quad v$

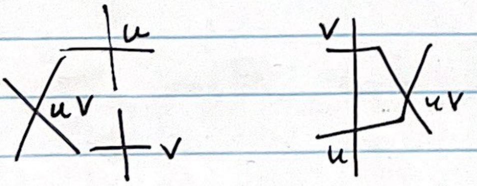
μ μ μ
back

Board 2

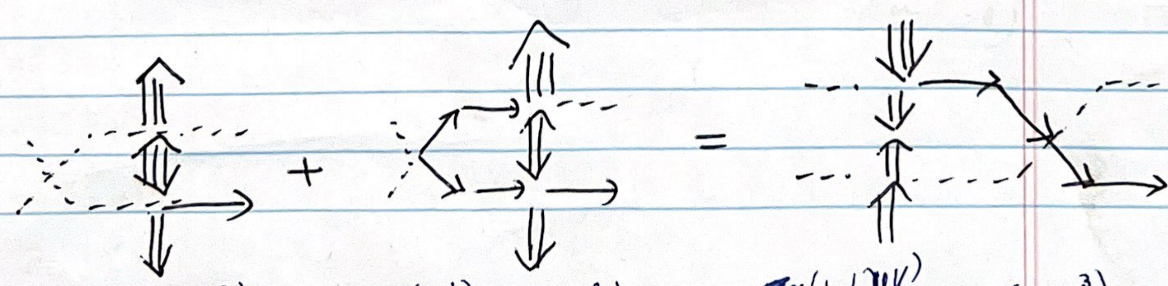
Crosses.



$t \quad \frac{z(1-t)}{1-z} \quad \underbrace{\frac{1-tz}{1-z}} \quad \frac{1-t}{1-z} \quad 1$



Ex. YBE



$$\underbrace{v(1-t^3)} + \frac{uv(1-t)}{1-uv}\cdot\underbrace{(1-t^3)v} = \frac{(1-t\,uv)}{1-uv}\cdot 1\cdot \underbrace{v(1-t^3)}$$

(Not for board) Red cross, $z = u/v$

1	1	$\frac{z(1-t)}{1-tz}$	$\frac{1-z}{1-tz}$	$\frac{t(1-z)}{1-tz}$	$\frac{1-t}{1-tz}$

integr

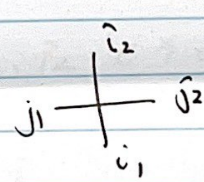
Integrable lattice vertex models $\mathbb{Z}^2$

- Random Tilings (lozenge & domino)
- Particle systems (TASEP / ASEP)
- Symm. f. (Schur / HL / qW / Macdonald / factorial Schur & many new), + almost all properties

many ways to get vertex

- Square Ice / ASM
- Ising, I guess (skip).

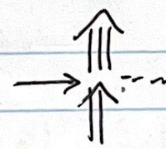
I. Def

Vertex 

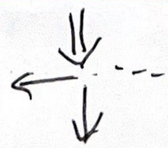
$i_1+j_1=i_2+j_2$

all ≥ 0

$j_1, j_2 \in \{0,1\}$



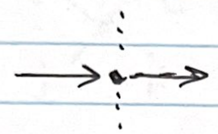
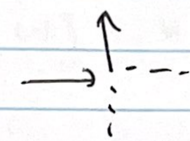
$W_z(i_1, j_1; i_2, j_2)$, z = several params,
global (t)
local (~~x_i, y_j~~ u_x, v_y dep. on ij coord.)

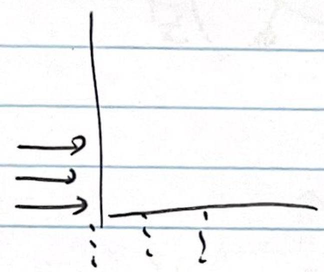
- colored
- fermionic restrictions
- flips like 

Ia. Stoch. 6V. (Remark)

$a_1 = a_2 = 1$

$b_i + c_i = 1$ all ≥ 0

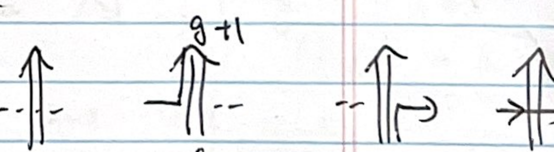
 or  random

 sample as M.C. (tomorrow!)

III. Vertex models for Symm. f.

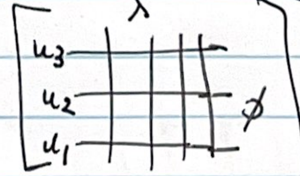
$\lambda = (\lambda_1 \geq \dots \geq \lambda_N \geq 0) \rightsquigarrow$ vertex config

<u>Def.</u> Weights W_u



$1 \qquad 1-t^{g+1} \qquad u \qquad u$

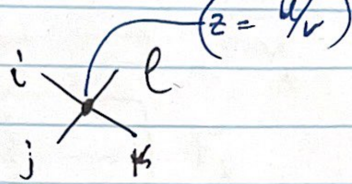
(save time)

$F_\lambda(u_1 .. u_N) = Z$ 

- $\Longleftrightarrow$ tableau formula for HL (proportional to)
- $t=0$ Schur $s_\lambda \longleftrightarrow$ <u>tilings!</u>

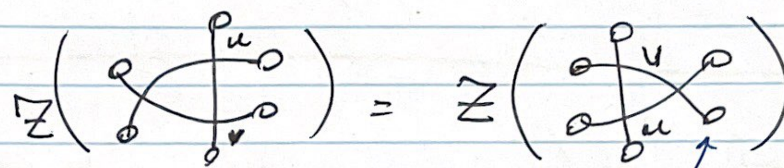
Symmetry. (sketch)

(Symm. f.!)

$\exists$ 

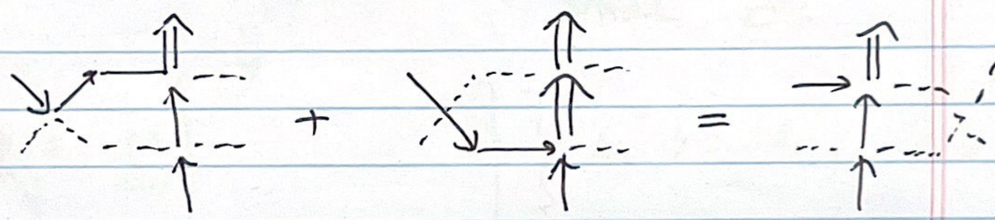
$i+j = k+l$, $\quad i\,j\,k\,l \in \{0,1\}$

s.t.

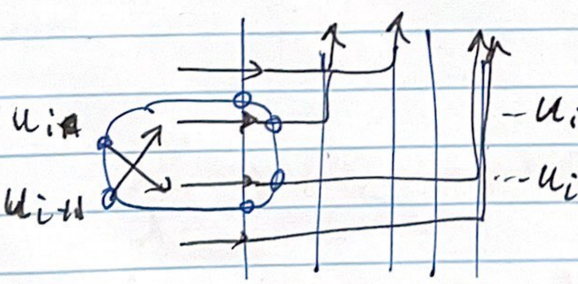


fixed b/c (6 pcs)

Ex.

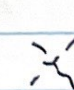


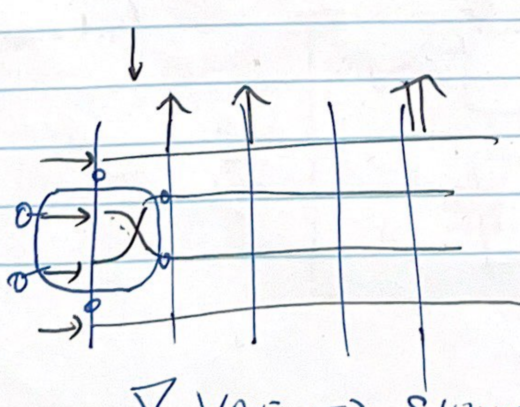
Then: train / zipper / surgery

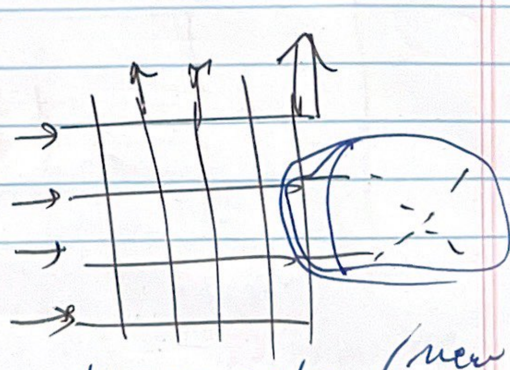


1) Attach , pay weight (here = 1)

2) move one by one

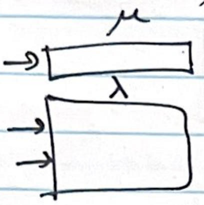
3) Detach , pay wght





final step

! YBE $\Rightarrow$ Symm. or Hecke action (new symm. funct. here) (optional) ~~Pieri~~ Branching / Pieri?



$F_{\mu/\lambda}$

F_λ

$$\sum F_\lambda F_{\mu/\lambda} = F_\mu$$

IV. Cauchy id.

Dual wts, W_v^* $\Downarrow$

Dual Z:

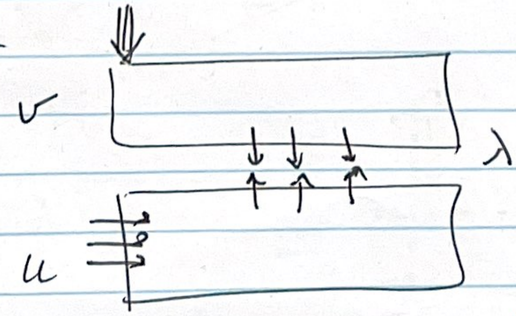
$$G_\lambda(v_1 \ldots v_M) = Z\Big[\overset{\Downarrow\, 0^N}{\qquad\qquad}\Big]\Big\}\, M \text{ rows},\ v_1 \ldots v_M$$

$\lambda = (\lambda_1 \ldots \lambda_N)$

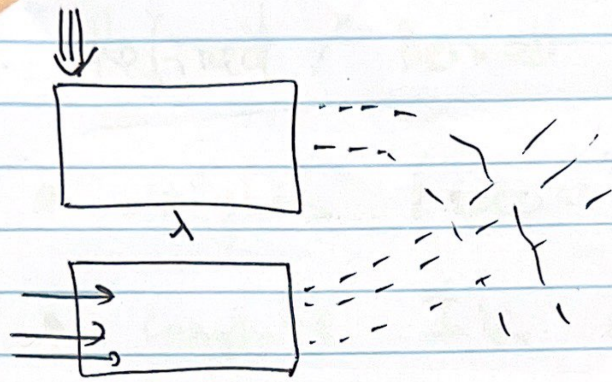
W_u, W_v^* satisfy YBE.

$$\sum_{\substack{\lambda \\ \| \\ (\lambda_1 \ldots \lambda_N)}} F_\lambda(u_1 \ldots u_N)\, G_\lambda(v_1 \ldots v_M) =$$

$|u_i v_j| < 1.$ $\forall i j$

$= Z\Big[$  $\Big]$

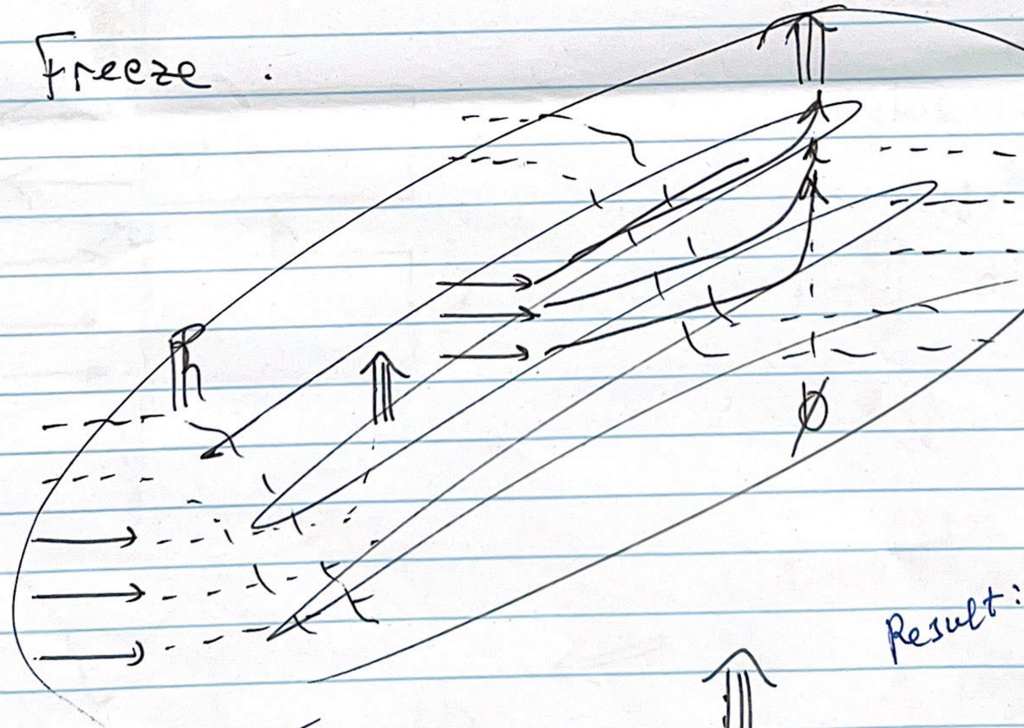
unbounded

1) 

add MN crosses & pay wt 1.

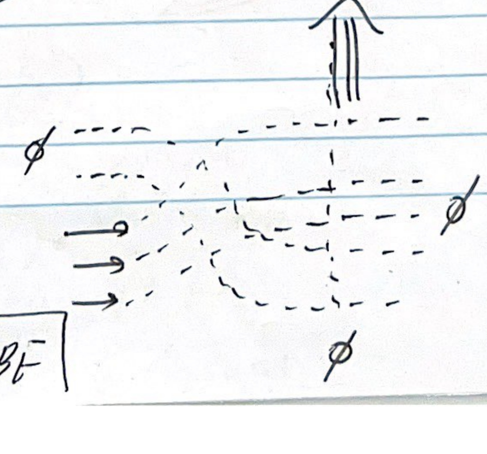
2) move left

3) Freeze.



Result: $\prod_{ij} \frac{1-tu_iv_j}{1-u_iv_j}$

& $(1-t)(1-t^2)\dots(1-t^N)$ harmless



Tomorrow: probab. consequences.

NB: RSK directly follows from some YBE

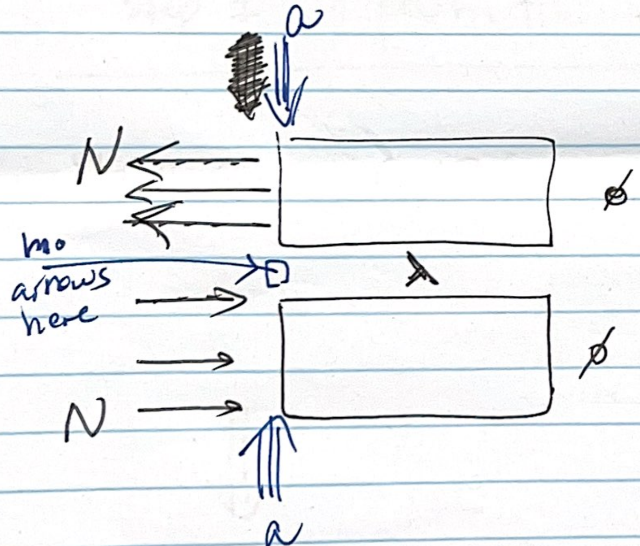
$\square$

V. Refined / Boxed identity

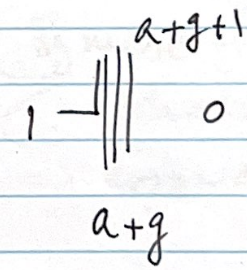
- weights (special) + YBE
- compute IK det
- set $\gamma = t^{-K}$ & observe a true boxed identity

(slight variation)

$t^a = \gamma$ (analytic parameter)



weights

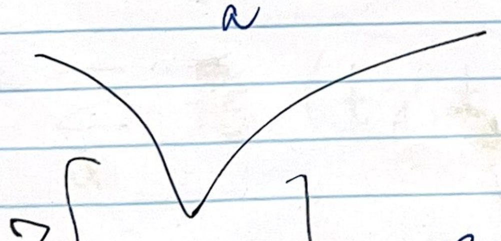


$$1 - t^{a+g+1} = 1 - \gamma t^{g+1}$$

$$= (1 - t^{g+1}) \, \frac{1 - \gamma t^{g+1}}{1 - t^{g+1}}$$

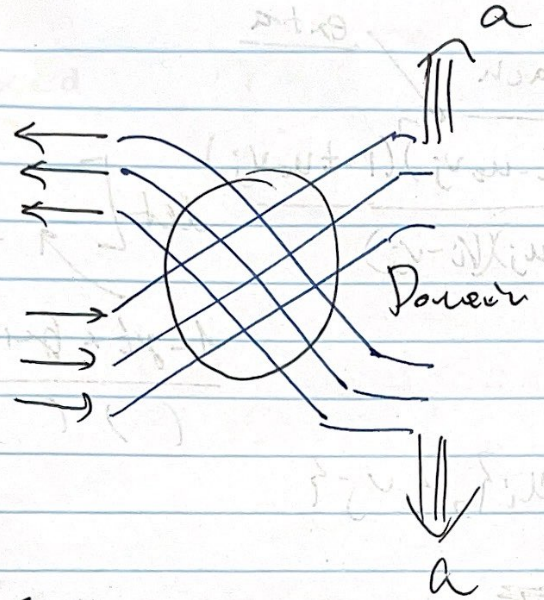
old weight

refinement



$$\Longleftarrow$$

$$= \sum_{\lambda} F_\lambda(\vec{u}) \, \overbrace{\underbrace{F^*_\lambda(\vec{v})}_{\text{slight variation maybe}}}^{G_\lambda(v)} \cdot \underbrace{\frac{(t\gamma, t)_{m_0}}{(t,t)_{m_0}}}_{\text{refinement.}}$$



Domain wall 6V +
decoration

step 1, partit. f.

step 2, PWBC (Izergin - Korepin)
(Symmetry +
(reduction $u_i = v_j^{-1}$
$\Rightarrow$ answer.

$\Downarrow$

$$\sum_\lambda (t\gamma, t)_{m_0(\lambda)} F_\lambda(\vec{u}) G_\lambda(\vec{v})$$

$$= \det\left[\frac{1-\gamma t + (\gamma-1) t u_i v_j}{(1-t u_i v_j)(1-u_i v_j)}\right]_1^N \cdot \frac{\prod(1-t u_i v_j)}{\prod_{i<j}(u_i-u_j)(v_i-v_j)}$$

$\gamma=0$, det simplifies

Izergin – Korepin Approach

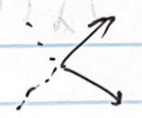
extra

$$F(u_1 .. u_N, v_1 .. v_N) = \frac{\prod (1-u_i v_j)(1-t u_i v_j)}{\prod_{i<j} (u_i-u_j)(v_i-v_j)} \det\Big[\quad\Big]$$

$$\frac{1-\gamma t+(\gamma-1)u_i v_j t}{(\ )(\ )}$$

- Symmetric in $\{u_i\}, \{v_j\}$
- Polynomial in u_n
- $u_N v_N = 1 \Rightarrow$ reduces to $F(u_1 .. u_{N-1}, v_1 .. v_{N-1})$
- $u_1 .. u_N = 0 \Rightarrow$ simplifies
- $N = 1$, simplifies

(Same for vertex model

R_z  , multiplied by $(1-z)$

$z = u_i v_j$)

Boxed, $\gamma = t^{-k} \Rightarrow m_0(\lambda) \le k-1$

$$m_0(\lambda) = N - \lambda_1'$$

$$\Rightarrow \lambda_1' \ge N+1-k$$

$$\underbrace{\sum_{\lambda_1' \le N}}_{\text{Cauchy}} = \underbrace{\sum_{\lambda_1' \le N+1-k}}_{\substack{\text{get LIS} \\ \text{distribution} \\ \text{for } \lambda'.}} + \underbrace{\sum_{N+1-k \le \lambda_1' \le N}}_{\text{det}}$$

!