# Symmetric functions II: applications, extensions and open problems

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# Longest Increasing Subsequence

Given a permutation $w : [1, \ldots, n] \to [1, \ldots, n]$,

$$lis(w) := \max\{k : \exists i_1 < i_2 < \ldots < i_k : w(i_1) < w(i_2) < \cdots < w(i_k)\}.$$

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RSK: $w \to (P, Q)$:

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$$lis(w) \sim \lambda_1, \qquad \mathbb{P}(\lambda) = \frac{(f^\lambda)^2}{n!}$$

(Plancherel measure)

# Asymptotics of SYT

**Standard Young Tableaux** of shape $\lambda$: $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$

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**Hook-length formula** [Frame-Robinson-Thrall]:

$$\#\{\text{SYTs of shape } \lambda\} = f^{\lambda} = \frac{|\lambda|!}{\prod_{u \in \lambda} \lambda_i - i + \lambda'_j - j + 1} = \frac{5!}{4 * 3 * 2 * 1 * 1}$$

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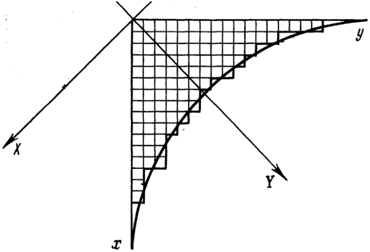


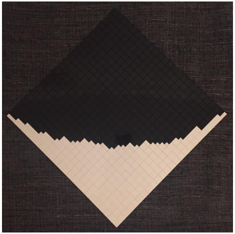

$$\sum_{\lambda\vdash n} (f^\lambda)^2 = n!$$

**Theorem**[Vershik-Kerov, Logan-Shepp 1977]
Under the Plancherel measure $Pr[\lambda] = \frac{(f^\lambda)^2}{n!}$, the typical partition $\lambda \vdash n$ looks like the picture to the right and for them $f^\lambda = \sqrt{n!}e^{-O(\sqrt{n})}$.
Moreover, there exist $c_1, c_0$, such that

$$e^{-c_1\sqrt{n}}\sqrt{n!} \leq \max_{\lambda\vdash n} f^\lambda \leq e^{-c_0\sqrt{n}}\sqrt{n!}.$$

# Limit shapes under Plancherel measure

**Theorem**[Vershik–Kerov–Logan–Shepp] If $\lambda^{(n)}$ is chosen wrt to the Plancherel measure $\mathbb{P}(\lambda) = \frac{(f^\lambda)^2}{n!}$, then its limit shape , in the sense of



$$\left| \frac{1}{\sqrt{n}} \lambda_i^{(n)} - \varphi(i\sqrt{n}) \right| < C\, n^{-1/6} \quad \text{for some} \quad C > 0,$$

is given by $\varphi : [0,2] \to [0,2]$ – the 135° rotation of $\big(x, y(x)\big)$:

$$y(x) := \frac{2}{\pi} \left( x \arcsin \frac{x}{\sqrt{2}} + \sqrt{2 - x^2} \right), \quad -\sqrt{2} \le x \le \sqrt{2}.$$

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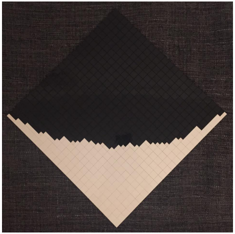

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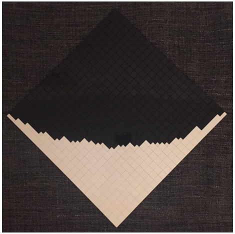

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**Theorem** [Vershik-Kerov, McKay]: The maximal dimension $f^\lambda$ for $\lambda \vdash n$, denoted $D(n)$ satisfies

$$\sqrt{n!}\, e^{-c_1\sqrt{n}(1+o(1))} \le D(n) \le \sqrt{n!}\, e^{-c_2\sqrt{n}(1+o(1))}$$

for some $c_1 > c_2 > 0$. Moreover $\lambda^{(n)}$ satisfies $f^{\lambda^{(n)}} \ge \sqrt{n!}e^{-a\sqrt{n}}$ for some $a$ iff it has the above shape.

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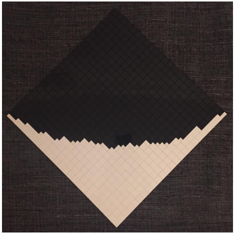

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**Open problem**: Show that asymptotically $c_1 = c_2$, i.e. $D(n) = \sqrt{n!}\, e^{-c\sqrt{n} + o(\sqrt{n})}$.

# LIS and the Tracy-Widom distribution

## Theorem (Baik-Deift-Johansson)

*Let $w^n$ denote a uniformly random permutation of $[1, \ldots, n]$. Then for every $x \in \mathbb{R}$ we have that*

$$\mathbb{P}\left(\frac{lis(w^n) - 2\sqrt{n}}{n^{1/6}} \leq x\right) \to F_2(x) \qquad \text{as } n \to \infty.$$

Here $F_2$ is the Fredholm determinant, aka the Tracy-Widom distribution of the maximal eigenvalue of a GUE matrix.

$$F_2 = 1 \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_t^{\infty} \cdots \int_t^{\infty} \det_{i,j=1}^{n} [A(x_i, x_j)] dx_1 \cdots dx_n,$$

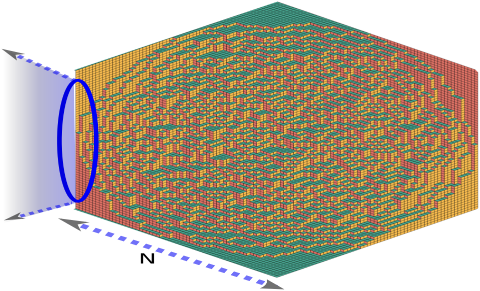
where

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos(1/3t^3 + xt) dt$$

is the Airy function and

$$A(x,y) = \begin{cases} \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}, & x \neq y, \\ Ai'(x)^2 - xAi(x)^2 & x = y. \end{cases}$$

# Behavior near the flat boundary:



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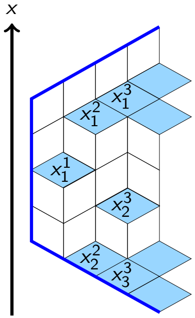


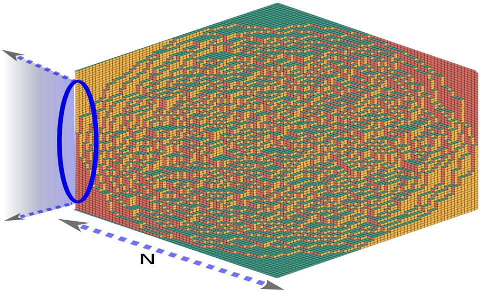

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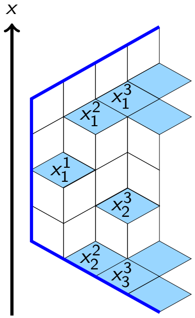


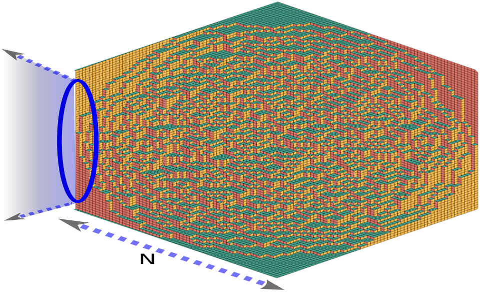


Horizontal lozenges near a flat boundary:

$$x_2^2 \le x_1^1 \le x_1^2$$
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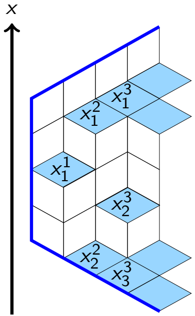


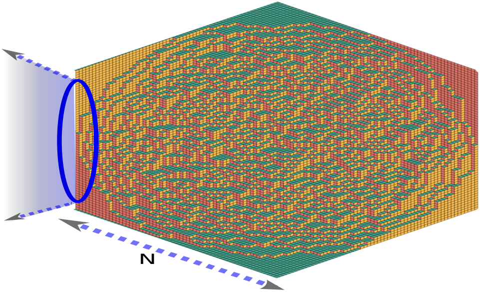


Horizontal lozenges near a flat boundary:

$$\begin{array}{ccccccccc} & & & & x_1^1 & & & & \\ & & & \le & & \le & & & \\ & & x_2^2 & & & & x_1^2 & & \\ & \le & & \le & & \le & & \le & \\ x_3^3 & & & & x_2^3 & & & & x_1^3 \end{array}$$

**Question:** Joint distribution of $\{x_j^i\}_{i=1}^k$ as $N \to \infty$ (rescaled)?

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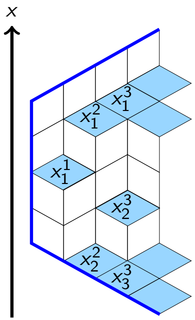


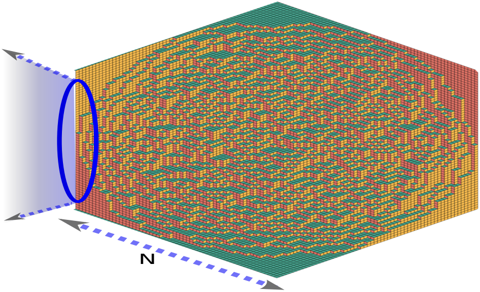


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$$
\begin{array}{ccccc}
 & & x_1^1 & & \\
 & x_2^2 & & x_1^2 & \\
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\end{array}
$$

with $x_2^2 \leq x_1^1 \leq x_1^2$, $x_3^3 \leq x_2^2 \leq x_2^3$, $x_2^3 \leq x_1^2 \leq x_1^3$.

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**Conjecture** [Okounkov–Reshetikhin, 2006]:
Fixed boundary: The joint distribution converges to a *GUE*-corners (aka *GUE*-minors) process: eigenvalues of GUE matrices.

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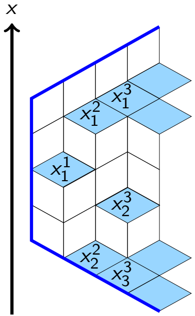


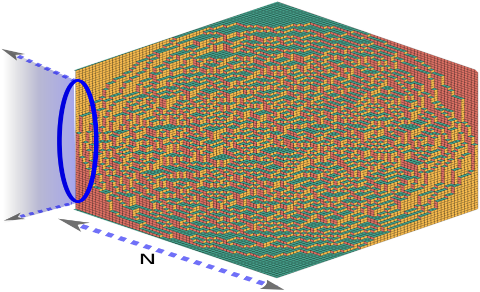


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Proofs: hexagonal domain [Johansson-Nordenstam, 2006], more general domains [Gorin-P,2012], [Novak, 2014], unbounded [Mkrtchyan, 2013], symmetric tilings [P, 2014, 2015], $q^{vol}$ [Mkrtchyan–Petrov, 2016], 6V model [Dimitrov] etc

# Behavior near the flat boundary: GUE

GUE: matrices $A = [A_{ij}]_{i,j}$: $A = \overline{A^T}$

$\mathrm{Re}A_{ij}, \mathrm{Im}A_{ij}$ – i.i.d. $\sim \mathcal{N}(0, 1/2)$, $i \neq j$

$A_{ii}$ – i.i.d. $\sim \mathcal{N}(0, 1)$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$(x_1^k \leq x_2^k \leq \cdots \leq x_k^k)$ – eigenvalues of $[A_{i,j}]_{i,j=1}^k$

*Interlacing condition:* $x_{i-1}^j \leq x_{i-1}^{j-1} \leq x_i^j$

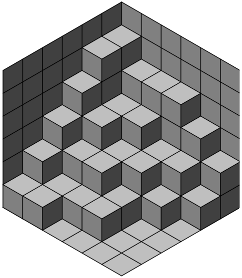
$$\begin{array}{ccccccc} x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ \leq & & x_1^2 & & x_2^2 & & \geq \\ & & & x_1^1 & & & \end{array}$$

The *joint distribution* of $\{x_i^j\}_{1 \leq i \leq j \leq k}$ is the
*GUE–corners (also, GUE-minors) process*, $=: \mathbb{GUE}_k$ .

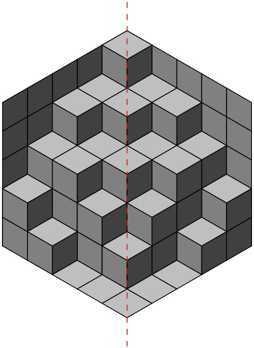
# Unrestricted (uniform) vs symmetric

Tilings of the hexagon $a \times b \times c \times a \times b \times c$, s.t.

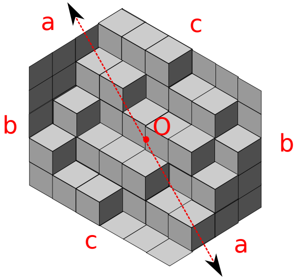
Unrestricted



Vertically symmetric



Centrally symmetric

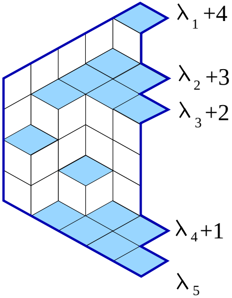



Limit behavior: fluctuations near the boundary, limit surface, CLT?

# Tilings setup

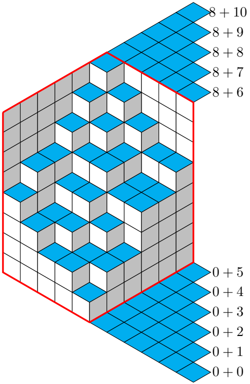
Domain $\Omega_{\lambda(N)}$:
positions of the $N$ horizontal lozenges on right boundary are:

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$




$\lambda(5) = (4, 3, 3, 0, 0)$

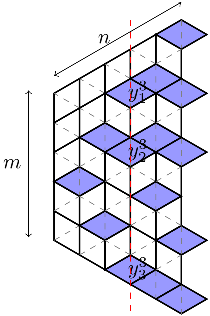
($\frac{1}{N}\Omega_{\lambda(N)}$ is *not necessarily a finite polygon as* $N \to \infty$ , e.g. $\lambda(N) = (N, N-1, \ldots, 2, 1)$)



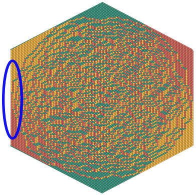

$\lambda = (\underbrace{a, \ldots, a}_{c}, \underbrace{0, \ldots, 0}_{b})$

$\leftrightarrow a \times b \times c$... hexagon.

# Behavior near the flat left boundary




Line $k = 3$



## Theorem

*Let $Y_n^k = (y_1^k, \dots, y_k^k)$ – horizontal lozenges on kth line of a uniformly random tiling $T \in \mathcal{T}_n$. As $n \to \infty$ the collection*

$$\left\{ \frac{Y_n^j - \mu_n}{\sqrt{n\sigma_n}} \right\}_{j=1}^k \to \mathbb{GUE}_k$$

*weakly as RVs, where*

- *$\mathcal{T}_n$ – all tilings of a hexagon [Johansson-Nordenstam].*
- *$\mathcal{T}_n = \Omega_{\lambda(n)}$ – $\mu_n = E(f)$, $\sigma_n = S(f)$, "$f(t) = \lim_{n\to\infty} \frac{\lambda(n)_{nt}}{n}$" [Gorin-P, 2013].*
- *$\mathcal{T}_n$ – vertically symmetric lozenge tilings of a $n \times m \times n..$ hexagon, $a = \lim_{n\to\infty} m/n$, $\mu_n = m/2$, $\sigma_n = \frac{a^2+2a}{8}$ [P, 2014].*
- *$\mathcal{T}_n$ – centrally-symmetric tilings of a $a \times b \times c$... hexagon with $a = 2qn$, $b = 2pn$, $c = 2(1-q)n$: $\mu_n = 2pqn$ and $\sigma_n = 2pq(1-q)(1+p)$ [P, 2015+].*

# Limit shape (surface)

## Theorem (P)

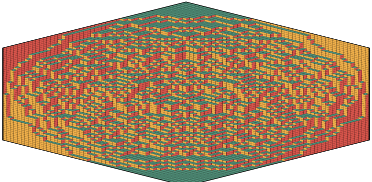
*Let $H_n(u, v)$ – height function of a uniformly random tiling from a set $\mathcal{T}_n$, i.e.*

$$H_n(u, v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v,$$

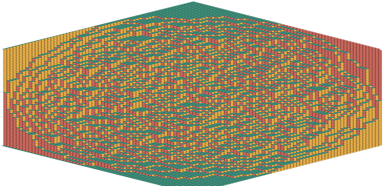
*where $y^k_i$ is the vertical height of the $i$th horizontal lozenge on the $k$th vertical line (left to right). For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u, v)$ converges uniformly in probability to a deterministic function $L(u, v)$ ("the limit shape"), which can be computed explicitly... when $\mathcal{T}_n$ is*

- *$\mathcal{T}_n$ – polygonal domain [Cohn, Kenyon, Larsen, Propp, Okounkov]*
- *$\mathcal{T}_n = \Omega_{\lambda(n)}$ for "nice" family $\lambda(n)$ [Bufetov-Gorin].*
- *$\mathcal{T}_n$ – symmetric tilings [P, 2014].*
- *$\mathcal{T}_n$ – centrally symmetric tilings [P, 2015].*

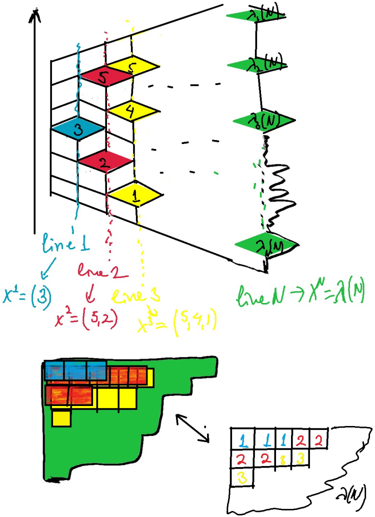
Symmetric:



General:



# Tilings probability: combinatorics and SSYTs




Lozenge tilings with right boundary $\lambda(N)$

$\Longleftrightarrow$

Semi-Standard Young Tableaux $T$ of shape $\lambda(N)$ and entries $1, \ldots, N$.

Tilings with horizontal lozenges on vertical line $k$ at positions $x^k = \eta_1, \ldots, \eta_k$

$\Longleftrightarrow$

SSYTs $T$ whose entries $1..k$ have shape $\eta$

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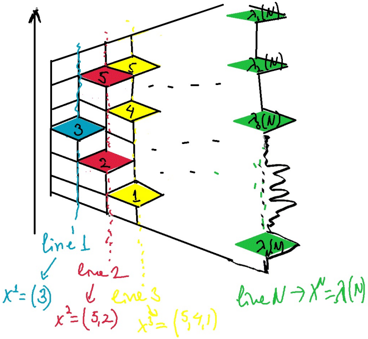


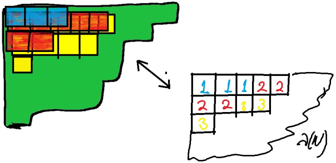

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Number of SSYTs of shape $\nu$, entries $1...\ell$ = $s_\nu(\underbrace{1, \ldots, 1}_{\ell})$.

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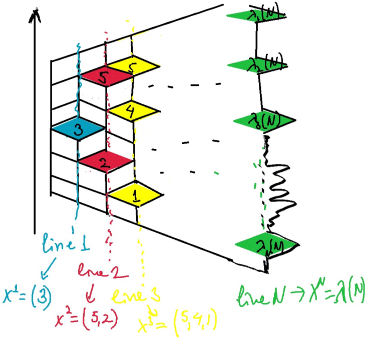


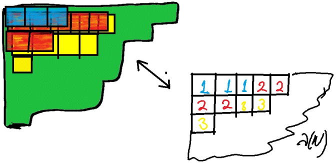


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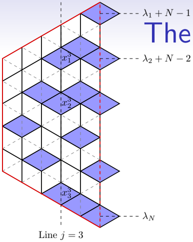
**Proposition**[Gorin-P] For any variables $y_1,\ldots,y_k$, the **Schur Generating Function** of $x^k$ is

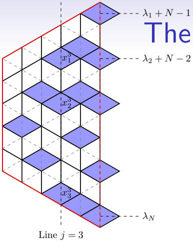
$$\mathbb{E}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right) = \frac{s_\lambda(y_1,\ldots,y_k,\overbrace{1,\ldots,1}^{N-k})}{s_\lambda(\underbrace{1,\ldots,1}_{N})} =: S_\lambda(y_1,\ldots,y_k).$$

# The explicit Schur Generating Functions[1]




$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line $j$ of $T \in \mathcal{T}_n$

[1] from [Gorin-P], [P, 2014, 2015]

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$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line $j$ of $T \in \mathcal{T}_n$

$$\text{MGF:} \qquad \mathbb{E}\left[ \frac{s_{x^k(T)}(y_1,\ldots,y_k)}{s_{x^k(T)}(\underbrace{1,\ldots,1}_{k})} \,\middle|\, T \sim Unif(\mathcal{T}_n) \right] = \sum_{\nu} \frac{s_\nu(y_1,\ldots,y_k)}{s_\nu(1^k)} \Pr(x^k(T)=\nu) = \ldots$$

- $= S_{\lambda(n)}(y_1,\ldots,y_k) = \frac{s_{\lambda(n)}(y_1,\ldots,y_k,1^{n-k})}{s_{\lambda(n)}(1^n)}$ for $\mathcal{T}_n = \Omega_{\lambda(n)}$.
- $= \prod_i y_i^{m/2} \cdot \frac{\mathfrak{so}_{(\frac{m}{2})^n}(y_1,\ldots,y_k,1^{n-k})}{\mathfrak{so}_{(\frac{m}{2})^n}(1^n)}$ for $\mathcal{T}_n$ – symmetric tilings of $n \times m \times n$....
- $= S_{(\frac{b}{2})^{a/2}}(y_1,\ldots,y_k)^2$ for $\mathcal{T}_n$ – centrally symmetric tilings of $a \times b \times c$... hexagon.

[1] from [Gorin-P], [P, 2014, 2015]

# Tilings probability III: MGF asymptotics

## Proposition (Gorin-P)

$$\mathbb{E}\left[\frac{s_{\nu-\delta_k}(y_1,\ldots,y_k)}{s_{\nu-\delta_k}(\underbrace{1,\ldots,1}_{k})} \quad \nu \sim \mathbb{GUE}_k\right] = \exp\left(\frac{1}{2}(y_1^2+\cdots+y_k^2)\right),$$

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Compare:

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*For any $k$ real numbers $h_1,\ldots,h_k$ and $\lambda(N)/N \to f$ we have:*

$$\lim_{N\to\infty} S_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\ldots,e^{\frac{h_k}{\sqrt{NS(f)}}}\right) e^{\left(-\frac{E(f)}{\sqrt{NS(f)}}\sum_{i=1}^k h_i\right)} \quad = \quad \exp\left(\frac{1}{2}\sum_{i=1}^k h_i^2\right).$$

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**Theorem.** Let $\Upsilon^k_{\lambda(N)} = \{x^k, x^{k-1},\ldots\}$ –collection of positions of the horizontal lozenges on lines $k, k-1,\ldots,1$ of tiling from $\Omega_{\lambda(N)}$, then

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$ (GUE-corners process of rank $k$).

# The limit surface

Counting measure:

$$m[\mu] := \frac{1}{L} \sum_{i=1}^{L} \delta\left(\frac{\mu_i + L - i}{L}\right),$$

Random measure on $\mu$s: $\rho^n(\mu)$ (e.g. $= \mathrm{Prob}\{x^k(T) = \mu\}$ for $T \in \mathcal{T}_n$), $m[\rho]$ – pushforward.

$$S_\rho(u_1, \ldots, u_k) := \sum_{\mu} \rho(\mu) \frac{s_\mu(u_1, \ldots, u_k)}{s_\mu(1^k)} = \mathbb{E}\left[ \frac{s_{x^k(T)}(y_1, \ldots, y_k)}{s_{x^k(T)}(\underbrace{1, \ldots, 1}_{k})} \,\middle|\, T \sim Unif(T_n) \right]$$

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**Theorem**[Bufetov-Gorin,2014] Suppose that $\rho^N$ is s.t. for every $r$

$$\lim_{N\to\infty}\frac{1}{N}\ln\left(S_{\rho^N}(u_1,\ldots,u_r,1^{N-r})\right) = Q(u_1)+\cdots+Q(u_r),$$

uniformly in a $\mathbb{C}$ nbhd of $(1^r)$, $Q$ – analytic. Then the random measures $m[\rho^N]$ converge, as $N\to\infty$, in probability to a deterministic measure $M$ on $\mathbb{R}$ with moments

$$\int_{\mathbb{R}} t^p M(dt) = \sum_{\ell=0}^{p}\binom{p}{\ell}\frac{1}{(\ell+1)!}\frac{\partial^\ell}{\partial u^\ell}u^p Q'(u)^{p-\ell}\Bigg|_{u=1}$$

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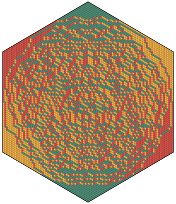
Our cases: MGF = normalized Schur $S_{\lambda(n)}$, $SO$ characters, etc.
**Asymptotics** using [Gorin-P, 2013] for fixed $r$:

$$\lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_1,\ldots,u_r) = \sum_{i=1}^{r} \lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_i) = \sum_{i=1}^{r} \Phi(u_i)$$

# Limit surface for symmetric tilings



## Theorem (P, 2014)

*Let $n, m \in \mathbb{Z}$, such that $m/n \to a$ as $n \to \infty$, where $a \in (0, +\infty)$. Let $H_n(u,v)$ – height function of a symmetric tiling of $n \times m \times n\ldots$ hexagon,i.e.*

$$H_n(u,v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v.$$

*For all $1 \geq u \geq v \geq 0$, as $n \to \infty$:*
*$H_n(u,v)$ converges unif. in prob. to a deterministic function $L(u,v)$ ("the limit surface").*

For any fixed $u \in (0,1)$, $L(u,v)$ is the distribution function of the measure $\mathbf{m}$, given by its moments:

$$\int_{\mathbb{R}} t^r \mathbf{m}(dt) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{(\ell+1)!} u^{-r+\ell} \frac{\partial^\ell}{\partial z^\ell} z^p \Phi'_a(z)^{p-\ell} \Big|_{z=1},$$

where $\Phi_a(e^y) = y\frac{a}{2} + 2\phi(y;a) - 2$ and...

$$h(y) = \frac{1}{4}\left( (e^y+1) + \sqrt{(e^y+1)^2 + 4(a^2+a)(e^y-1)^2} \right)$$

$$\phi(y;a) = (\frac{a}{2}+1)\ln\left(h(y) - (\frac{a}{2}+1)(e^y-1)\right) - (\frac{a}{2}+\frac{1}{2})\ln\left(h(y) - (\frac{a}{2}+\frac{1}{2})(e^y-1)\right)$$
$$+ \frac{a}{2}\ln\left(h(y) + \frac{a}{2}(e^y-1)\right) - (\frac{a}{2}-\frac{1}{2})\ln\left(h(y) + (\frac{a}{2}-\frac{1}{2})(e^y-1)\right)$$

## Theorem (P, 2015)

*The scaled height function $H_n(u,v)$ of a centrally symmetric tiling of an $a \times b \times c\ldots$ hexagon converges uniformly in probability to a deterministic function $L(u,v)$ – the limit surface, as $n \to \infty$, where $n = \frac{a+c}{2}$ and $a/n, b/n$ – approx constant.*
*The limit surface coincides with the limit surface for the uniformly random tilings of the hexagon (without symmetry constraints).*

# Asymptotics of Schur functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})} \qquad (\textit{similarly, othercharacters}$$

**Theorem** [Gorin-P] For any partition $\lambda$ and any $x \in \mathbb{C} \setminus \{0,1\}$ we have

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi\mathbf{i}} \oint_C \frac{x^z}{\prod_{i=1}^N (z-(\lambda_i+N-i))} dz,$$

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**Theorem**[Gorin-P] If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ [under certain convergence conditions], for all fixed $y \neq 0$:

$$\lim_{N\to\infty} \frac{1}{N} \ln S_{\lambda(N)}(e^y; N, 1) = yw_0 - \mathcal{F}(w_0) - 1 - \ln(e^y - 1),$$

where $\mathcal{F}(w; f) = \int_0^1 \ln(w - f(t) - 1 + t)dt$, $w_0$ – root of $\frac{\partial}{\partial w}\mathcal{F}(w; f) = y$. If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ ["other" conv. cond.], for any fixed $h \in \mathbb{R}$:

$$S_{\lambda(N)}(e^{h/\sqrt{N}}; N, 1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

where $E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 (f(t) - t + 1/2)^2 dt - 1/6 - E(f)^2.$

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**Multivariate:** [Gorin-P ] Let $D_{i,1} = x_i \frac{\partial}{\partial x_i}$ , $\Delta-$ Vandermonde det, then

$$S_\lambda(x_1,\ldots,x_k;N) = \prod_{i=1}^{k} \frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times \frac{\det\left[D_{i,1}^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1,\ldots,x_k)} \prod_{j=1}^{k} S_\lambda(x_j;N,1)(x_j-1)^{N-1}.$$

**Corollary**[Gorin-P]

If $\frac{\ln\left(S_{\lambda(N)}(x;N,1)\right)}{N} \to \Psi(x)$ unif. on a compact $M \subset \mathbb{C}$. Then for any $k$

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x_1,\ldots,x_k;N,1)\right)}{N} = \Psi(x_1)+\cdots+\Psi(x_k)$$

uniformly on $M^k$. More informally, under various regimes of convergence for $\lambda(N)$ we have $S_{\lambda(N)}(x_1,\ldots,x_k) \sim S_{\lambda(N)}(x_1)\cdots S_{\lambda(N)}(x_k)$.

# Factorial Schur functions

**Factorial Schur functions:**

$$s_\mu(x_1,\ldots,x_k|a_1,\ldots,a_n) = \frac{\det[(x_i-a_1)(x_i-a_2)\cdots(x_i-a_{\mu_j+k-j})]_{i,j=1}^k}{\prod_{i<j}(x_i-x_j)}$$

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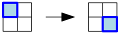
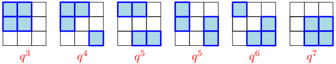
**Excited diagrams:**

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from $\mu$ via  $\}$



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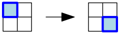
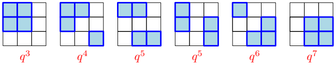
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from $\mu$ via  $\}$




## Theorem (Ikeda-Naruse, Kreiman+Knutson-Tao, Knutson-Miller-Yong)

*Let $\mu \subset \lambda \subset d \times (n-d)$. Let $v(n-d+1-i) = \lambda_i + (n-d+1-i)$ and $v(j) = d + j - \lambda'_j$. Then*

$$s_\mu^{(d)}(y_{v(1)},\ldots,y_{v(d)}|y_1,\ldots,y_{n-1}) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

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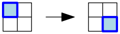
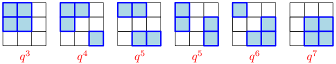
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from $\mu$ via  $\}$

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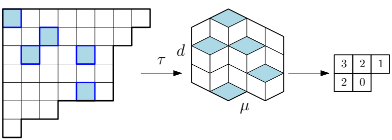
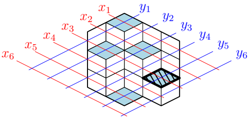
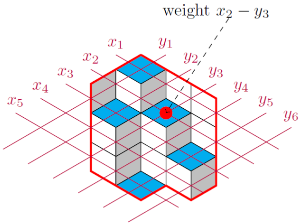


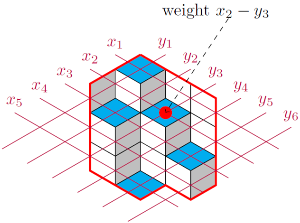
# Multivariate local weights



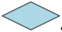

$$\text{Total weight} = \prod_{\lozenge \, at \, (i,j)} (x_i - y_j)$$

$$(x_1 - y_1)(x_2 - y_3)(x_3 - y_5)(x_3 - y_2)(x_5 - y_5).$$

# Multivariate local weights

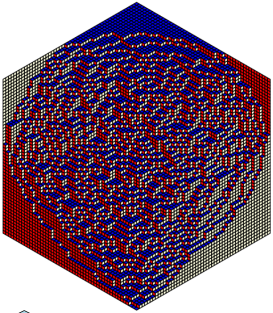


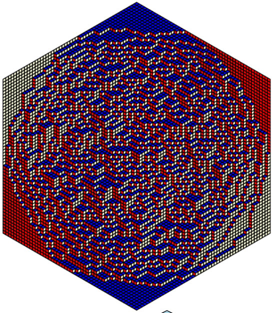

Total weight $= \prod_{\text{at } (i,j)} (x_i - y_j)$

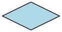
 at $(i,j)$

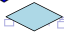
$(x_1 - y_1)(x_2 - y_3)(x_3 - y_5)(x_3 - y_2)(x_5 - y_5).$

Simulation: base $\mu = N^N$:

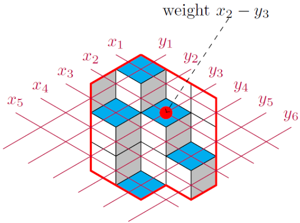


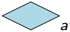


 at $(i,j) = 2N - (i+j)$

 $= 1$

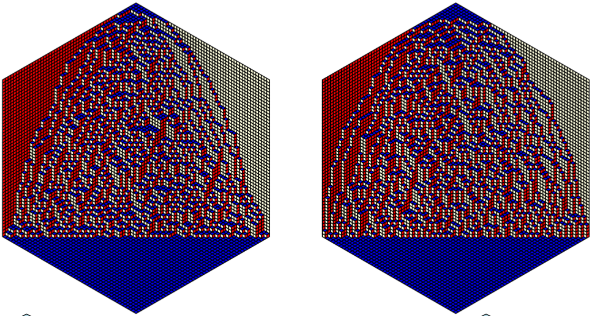
# Multivariate local weights

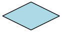


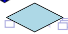

$$\text{Total weight} = \prod (x_i - y_j)$$
 at $(i,j)$

$$(x_1 - y_1)(x_2 - y_3)(x_3 - y_5)(x_3 - y_2)(x_5 - y_5).$$

Simulation 2: base $\mu = \delta_n$



 at $(i,j) = 4n - (i+j)$

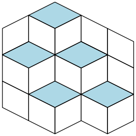
 $= 1$

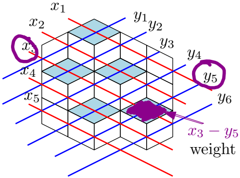
# Lozenge tilings with multivariate weights

Plane partitions with base $\mu$, height $d$

weights of horizontal lozenges $= x_i - y_j$

| | | |
|---|---|---|
| 3 | 2 | 1 |
| 2 | 1 | |





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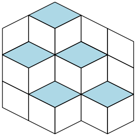
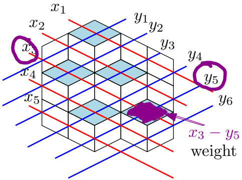





## Theorem (Morales-Pak-P)

*Consider tilings with base $\mu$ and height $d$, we have that*

$$\sum_{T \in \Omega_{\mu,d}} \prod_{(i,j)\in T} (x_i - y_j) = \det[A_{i,j}(\mu,d)]_{i,j=1}^{d+\ell(\mu)},$$

*where*

$$A_{i,j}(\mu,d) := \begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{d+\ell(\mu)-j})}{(x_i - x_{i+1})\cdots(x_i - x_{d+\ell(\mu)})}, & \text{when } j = \ell(\mu)+1,\ldots,\ell(\mu)+d, \\ \frac{(x_i - y_1)\cdots(x_i - y_{\mu_j+d})}{(x_i - x_{i+1})\cdots(x_i - x_{d+j})}, & \text{when } j = i-d,\ldots,\ell(\mu), \\ 0, & \text{when } j < i-d. \end{cases}$$

## Corollary (Krattenthaler, Stanley etc)

*Consider the set $PP(\mu, d)$ of plane partitions of base $\mu$ and entries less than or equal to $d$. Then their volume generating function is given by the following determinantal formula*

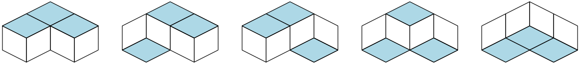
$$\sum_{P \in PP(\mu,d)} q^{|P|} = q^{\sum_r r\mu_r} \det[C_{i,j}]_{i,j=1}^{\ell+d},$$

*where*

$$C_{i,j} = \begin{cases} \frac{(-1)^{d+\ell-i} q^{(d-i)(d+\ell-j) - \frac{(d-i+\ell)(d-i-\ell-1)}{2}}}{(q;q)_{d+\ell-i}}, & \text{when } j = \ell+1, \ldots, \ell+d, \\ \frac{(-1)^{d+j-i} q^{(d-i)(\mu_j+d) - \frac{(d+j-i)(d-i-j-1)}{2}}}{(q;q)_{d+j-i}}, & \text{when } j = i-d, \ldots, \ell, \\ 0, & \text{when } j < i-d, \end{cases}$$

*where $(q;q)_m = (1-q)\cdots(1-q^m)$ is the $q-$Pochhammer symbol.*

E.g. $\mu = (2,1)$, $d = 1$:



$$\sum_{P \in PP((2,1),1)} q^{|P|} = q^0 + q^1 + 2q^2 + q^3$$

# Counting skew SYTs: formulas

Outer shape $\lambda$, inner – $\mu$, e.g. for $\lambda = (5,4,4,2,1), \mu = (2,2,1)$ :

Skew SYT:

| | | | | |
|---|---|---|---|---|
| | | 2 | 3 | 6 |
| | | 5 | 8 | |
| | 1 | 7 | 10 | |
| 4 | 9 | | | |
| 11 | | | | |

Skew SSYT of content $(1^3, 2^2, 3^3, 4^3)$:

| | | | | |
|---|---|---|---|---|
| | | 1 | 1 | 3 |
| | | 2 | 3 | |
| | 2 | 4 | 4 | |
| 1 | 3 | | | |
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| 1 | 3 | | | |
| 4 | | | | |

**Jacobi-Trudi**[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[ \frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)} .$$

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$\lambda/\mu = \delta_{n+2}/\delta_n$:

| | | | | |
|---|---|---|---|---|
| | | | 5 | 6 |
| | | 1 | 9 | |
| | 2 | 7 | | |
| 3 | 4 | | | |
| 8 | | | | |

$\leftrightarrow 8 > 3 < 4 > 2 < 7 > 1 < 9 > 5 < 6$

$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$ – Euler numbers: $2, 5, 16, 61 \ldots$.:

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

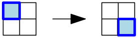
# Hook-Length formula for skew shapes

**Theorem (Naruse, SLC, September 2014)**

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

*where* $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* $\lambda/\mu$.

**Excited diagrams:**

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from $\mu$ via  $\}$

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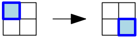
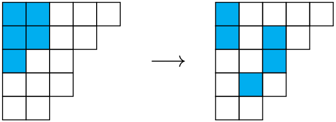
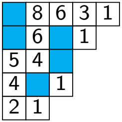
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from $\mu$ via  $\}$



Hook lengths inside $\lambda$: 

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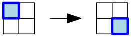
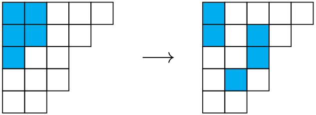
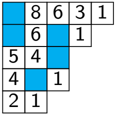
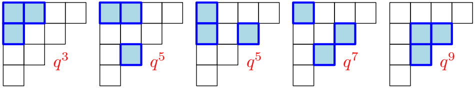
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from $\mu$ via  $\}$

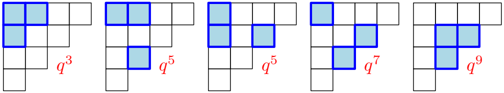


Hook lengths inside $\lambda$: 


$$f^{(4321/21)} = 7! \left( \frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

# Hook-Length formula for skew shapes




$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

## Theorem (Morales-Pak-P)

*For skew SSYTs, we have that*

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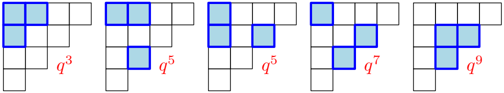


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$$s_{(3,2)/(1)}(1,q,q^2,\cdots) = \underset{\begin{array}{cc} & 0\;0 \\ 0\;1 & \end{array}}{q^{0+0+0+1}} + \underset{\begin{array}{cc} & 0\;1 \\ 0\;1 & \end{array}}{q^{0+1+0+1}} + \cdots + \underset{\begin{array}{cc} & 1\;3 \\ 0\;3 & \end{array}}{q^{1+3+0+3}} + \underset{\begin{array}{cc} & 1\;1 \\ 2\;3 & \end{array}}{q^{1+1+2+3}} + \cdots$$

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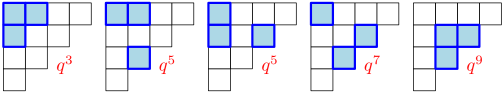


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## Theorem (Morales-Pak-P)

*For (reverse) plane partitions of skew shape $\lambda/\mu$:*

$$\sum_{\pi\in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S\in PD(\lambda/\mu)} \prod_{u\in S} \left[\frac{q^{h(u)}}{1-q^{h(u)}}\right].$$

*where* $PD(\lambda/\mu) := \{S\subset[\lambda] : S\subset[\lambda]\setminus D, \text{ for some } D\in\mathcal{E}(\lambda/\mu)\}$.

Other proofs by [M. Konvalinka], other new results in [Naruse-Okada, Grinberg-Korniichuk- Molokanov-Khomych]

# Proofs of NHLF

- Equivaraint Schubert Calculus [Naruse, generalized in MPP1] via Schubert class localization formulas at Grassmannian permutations, i.e. certain evaluation of Schubert polynomials = Factorial Schur functions.

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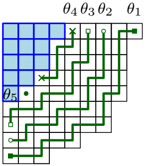
- Bijection: Hillman-Grassl ( generalized RSK) on nonnegative integer arrays of certain shapes. [MPP2]
- Non-intersecting lattice paths.

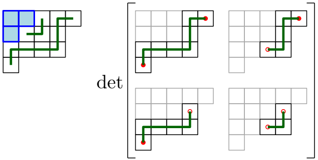
# Non-intersecting lattice paths

**Theorem**[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1, \ldots, \theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of $\lambda/\mu$, then

$$s_{\lambda/\mu} = \det \left[ s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k}.$$

where $s_{\varnothing} = 1$ and $s_{\theta_i \# \theta_j} = 0$ if the $\theta_i \# \theta_j$ is undefined.
$\theta_1$ – border strip following the inner border of $\lambda$;
$\theta_i$ – inner border of $\lambda \setminus (\theta_1 \cup \cdots \cup \theta_{i-1})$ etc until $\mu$ is hit,
then – border strips from each connected part etc.
Ordering: corners.
Strip $\theta_i \# \theta_j$ := shape of $\theta_1$ between the diagonals of the endpoints of $\theta_i$ and $\theta_j$.



# NHLF for border strips

## Lemma (MPP)

*For a border strip $\theta = \lambda/\mu$ with end points $(a, b)$ and $(c, d)$ we have*

$$s_\theta(1, q, q^2, \ldots, ) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}.$$

$s$  $(1, q, q^2, \ldots) = \dfrac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)}$

$\gamma=(3,1),(3,2),(2,2),(2,3),(1,3)$: 

$+ \dfrac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$ 

$+ \dfrac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$ 

$+ \dfrac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2}$ 

$+ \dfrac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$ 

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

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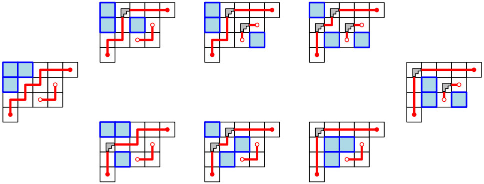
$$s_\theta(1,q,q^2,\ldots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1-q^{h(i,j)}}.$$

**Excited diagrams** for $\lambda/\mu \leftrightarrow$ Non-Intersecting Lattice Paths:



# NHLF for border strips

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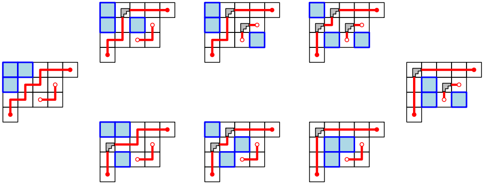

$$s_{\lambda/\mu} \underset{\text{Lascoux-Pragacz}}{=} \det\left[ s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k} \underset{s_{\text{Border Strip}}}{=} \det\left[ \sum_{\gamma:(a_i,b_i)\to(c_j,d_j)} \prod_{u\in\gamma} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \right]$$

$$\underset{Lindstrom-Gessel-Viennot}{=} \sum_{NILP:\gamma_1,\ldots} \prod_{u\in\gamma_1\cup\ldots} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \underset{\mathcal{E}(\lambda/\mu)=NILP}{=} \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in D} \frac{q^{\cdots}}{1-q^{h_u}}$$

# Other interesting polynomials

1. Hall-Littlewood polynomials:

$$P_\lambda(x_1,\ldots,x_n;t) = \frac{1}{\prod_{i=1}^n [m_i(\lambda)]!_t} \sum_{w \in S_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

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$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(t) P_\mu(x;t),$$

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Set $w_0 := n(n-1)\ldots 21$, adjacent transpositions $s_i = (i, i+1)$:

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**Special case:**
$w$ – Grassmannian, i.e. $\exists! d$, s.t. $w_d > w_{d+1}$

$$\Longrightarrow \mathfrak{S}_w(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_d) \text{ where } \lambda_i = w_{d+1-i} + i - (d+1)$$

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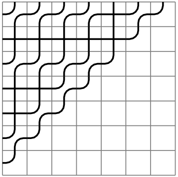
**Pipe dreams (RC graphs):**



$$\mathfrak{S}_w = \sum_{RC(i \to w_i)} \prod_i x_i^{\#(i,j)=+}$$

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**Some formulas:**
**Macdonald's identity**

$$\mathfrak{S}_w(1, 1, \ldots, 1) = \frac{1}{\ell!} \sum_{(r_1,\ldots,r_\ell) \in R(w)} r_1 r_2 \cdots r_\ell.$$

where $R(w) = \{(r_1, \ldots, r_\ell) : s_{r_1} \cdots s_{r_\ell} = w\}$ with $\ell(w) = inv(w)$.

**Cauchy identity:**

$$\sum_{w \in S_n} \mathfrak{S}_w(x)\, \mathfrak{S}_{ww_0}(y) = \prod_{i+j \leq n} (x_i + y_j)$$

$$u(n) := \max_{w \in S_n} \mathfrak{S}_w(1^n)$$

## Conjecture (Stanley, "Schubert Shenanigans")

*There is a limit*

$$\lim_{n\to\infty} \frac{1}{n^2} \log u(n) \qquad .$$

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## Theorem (Stanley)

$$\frac{1}{4} \leq \liminf_{n \to \infty} \frac{\log_2 u(n)}{n^2} \leq \limsup_{n \to \infty} \frac{\log_2 u(n)}{n^2} \leq \frac{1}{2}$$

Proof:

$$\sum_{u \in S_n} \mathfrak{S}_u(1^n)\, \mathfrak{S}_{uw_0}(1^n) = 2^{\binom{n}{2}}.$$

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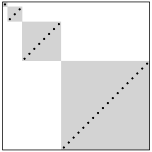
**Layered (Richardson) permutations $L_n$:**

$$w(b_k, b_{k-1}, \ldots, b_1) := \big(w_0(b_k), b_k + w_0(b_{k-1}), \ \ldots \ , n - b_1 + w_0(b_1)\big)$$



## Theorem (Morales-Pak-P)

*Let* $v(n) := \max_{w \in L_n} \mathfrak{S}_w(1^n)$. *Then there is a limit*

$$\lim_{n\to\infty} \frac{1}{n^2} \log_2 v(n) = \frac{\gamma}{\log 2} \approx 0.2932362762,$$

*where* $\gamma \approx 0.2032558981$ *, and the maximum value* $v(n)$ *is achieved at*

$$w(\ldots, b_2, b_1), \quad \textit{where} \quad b_i \sim \alpha^{i-1}(1-\alpha)n \quad \textit{as} \quad n \to \infty,$$

*for every fixed* $i$*, and where* $\alpha \approx 0.4331818312$ *is a universal constant.*

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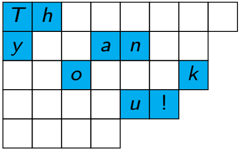
## Proposition (Cor to Lam-Lee-Shimozono, Weigandt, 2018–2019)

*We have the following upper bound:*

$$u(n) \leq \#\mathrm{ASM}_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \approx 2^{0.3774n^2 + O(\log(n))}$$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