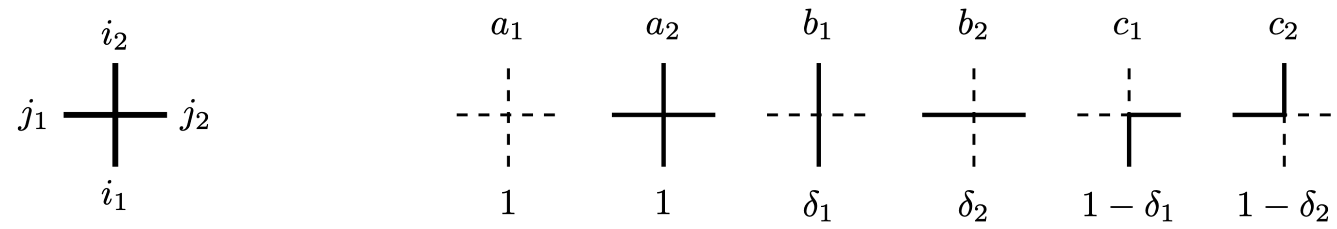
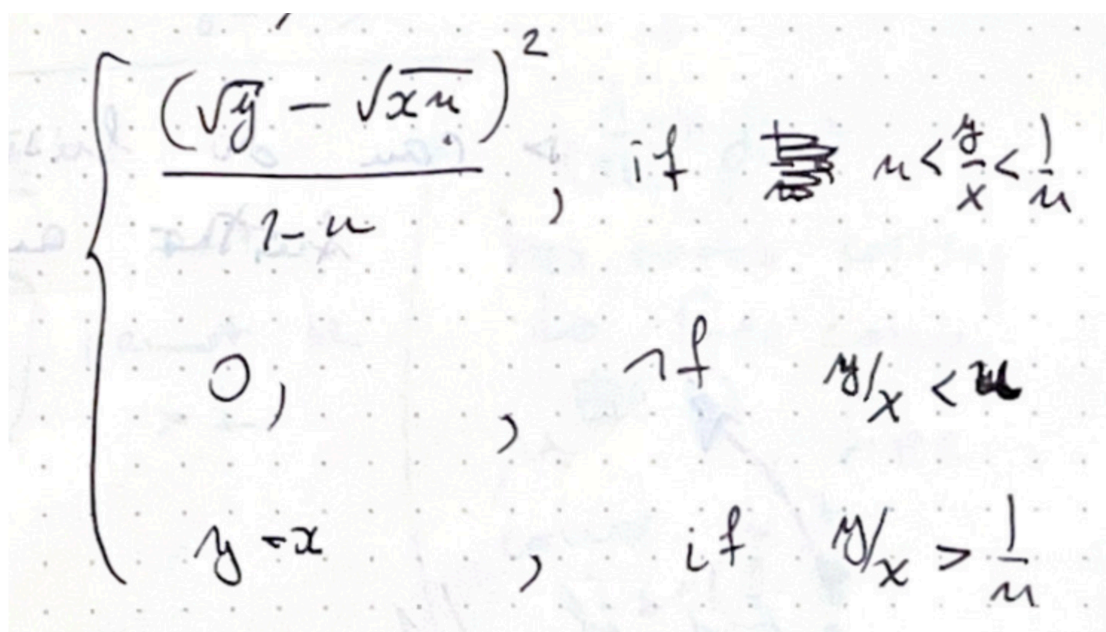
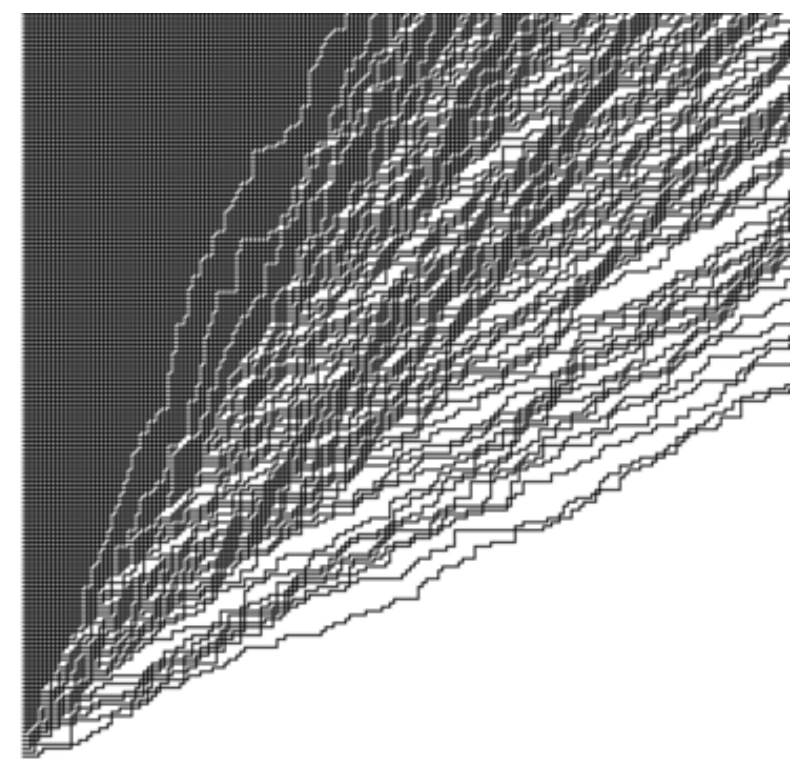
I. Stochastic 6v model



$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad t := \delta_2/\delta_1$$

- Ferroelectric; in boxed setting exhausts all symmetric 6V
- Boxed is very complicated (but arctic curve is known/ conjectured)
- However, there is a stochastic variant [BCG], which has KPZ class fluctuations (TW GUE)





It is also believed to have translation invariant “liquid” Gibbs measures, but this has not been proven. Their local statistics are complicated, not determinantal.

On a cylinder: KPZ phases are “free evolution”, the other phases are constrained to be either too fast, or too slow. The “too fast” is less probable (e^{-N^2}), so these phases do not exist. (**Contrast with dimers**)

II. Stochastic 6v model → ASEP

$$\delta_1 = \frac{1-u}{1-tu}, \delta_2 = \frac{t(1-u)}{1-tu},$$

$$1-\delta_1 = \frac{(1-t)u}{1-tu}, 1-\delta_2 = \frac{1-t}{1-tu}$$

ASEP: also has KPZ fluctuations; very popular model; natural and appeared in mRNA modeling in 1968

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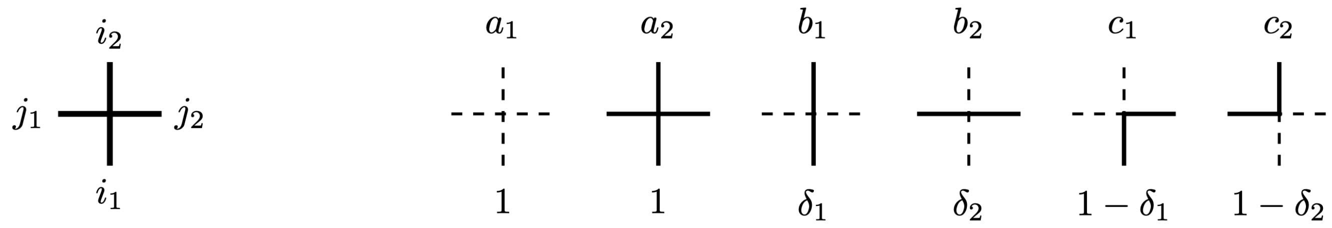

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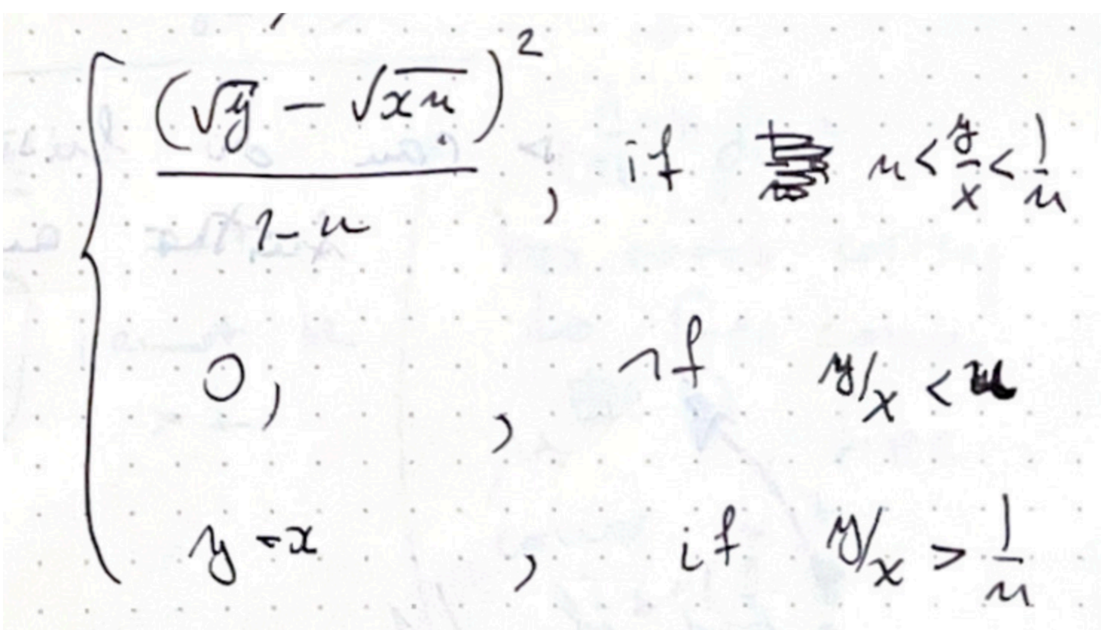

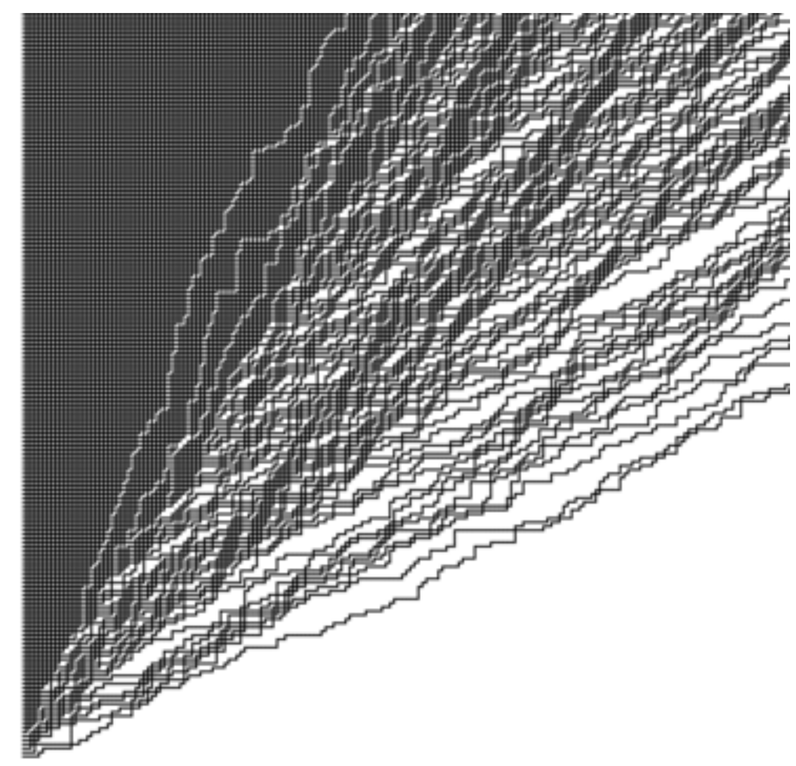

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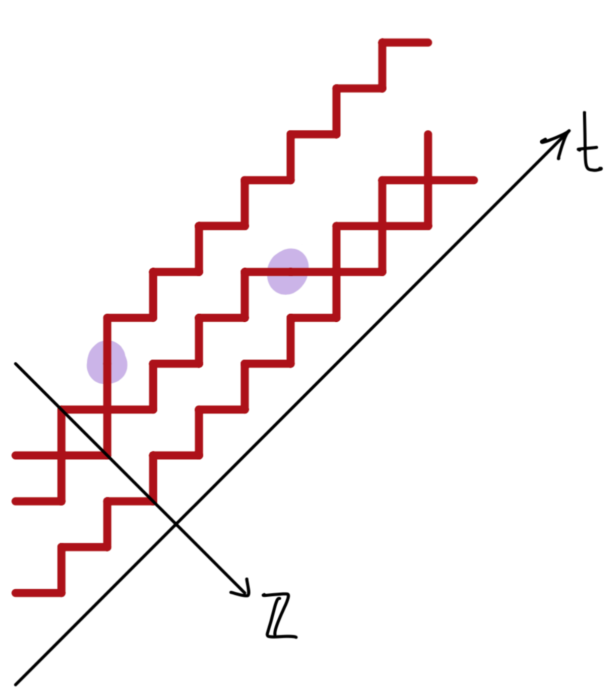
II. Stochastic 6v model → ASEP

$\delta_1, \delta_2 \to 0$ and t stays fixed (so, $u \to 1$)



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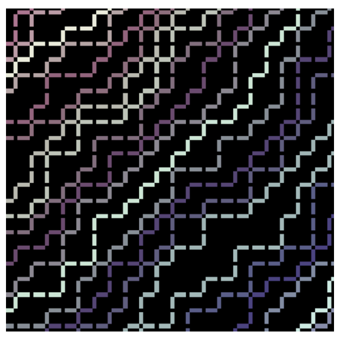
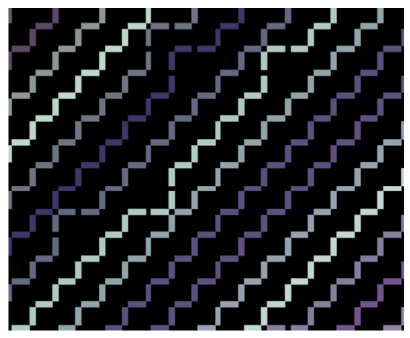
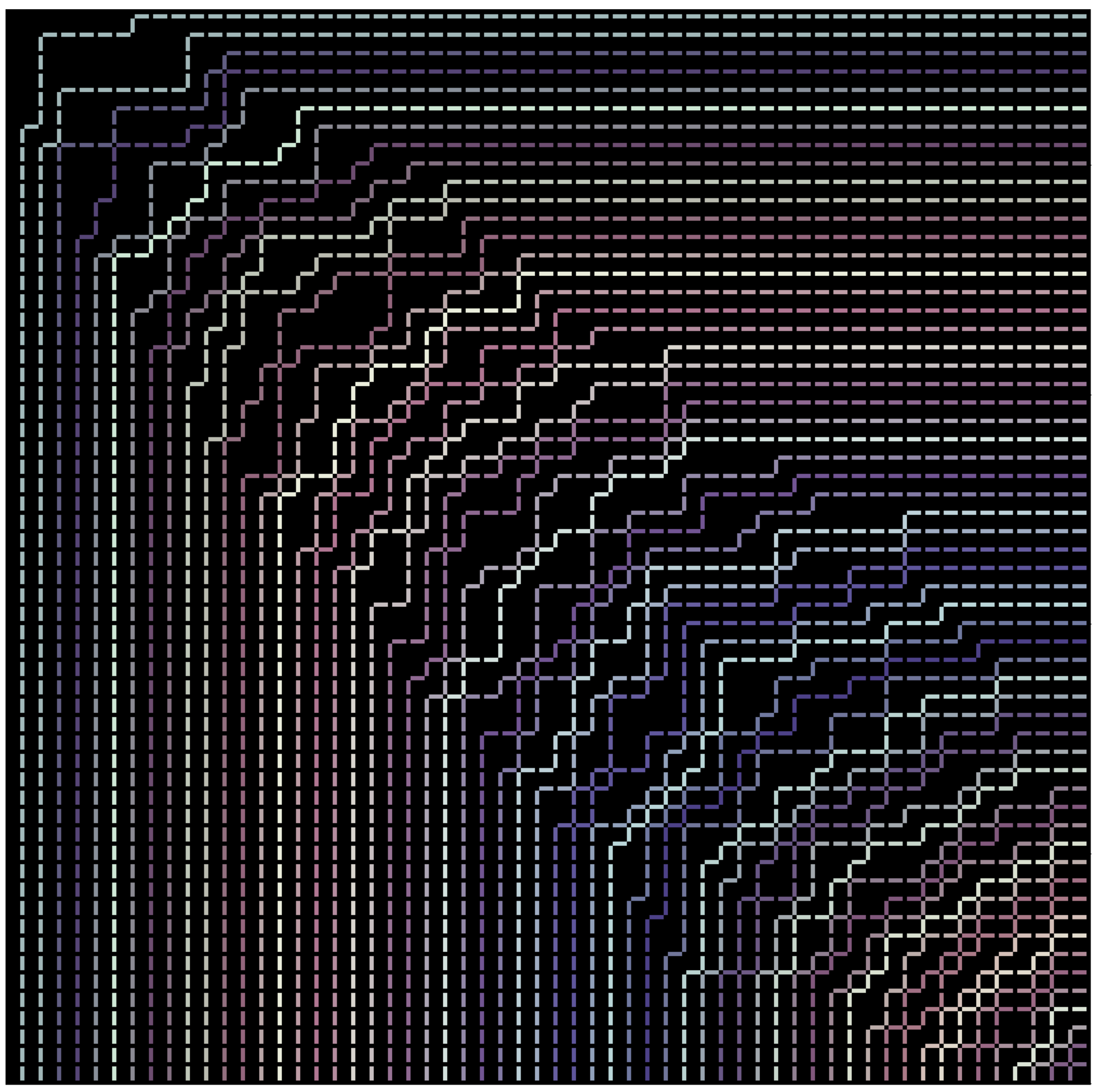
ASEP: also has KPZ fluctuations; very popular model; natural and appeared in mRNA modeling in 1968

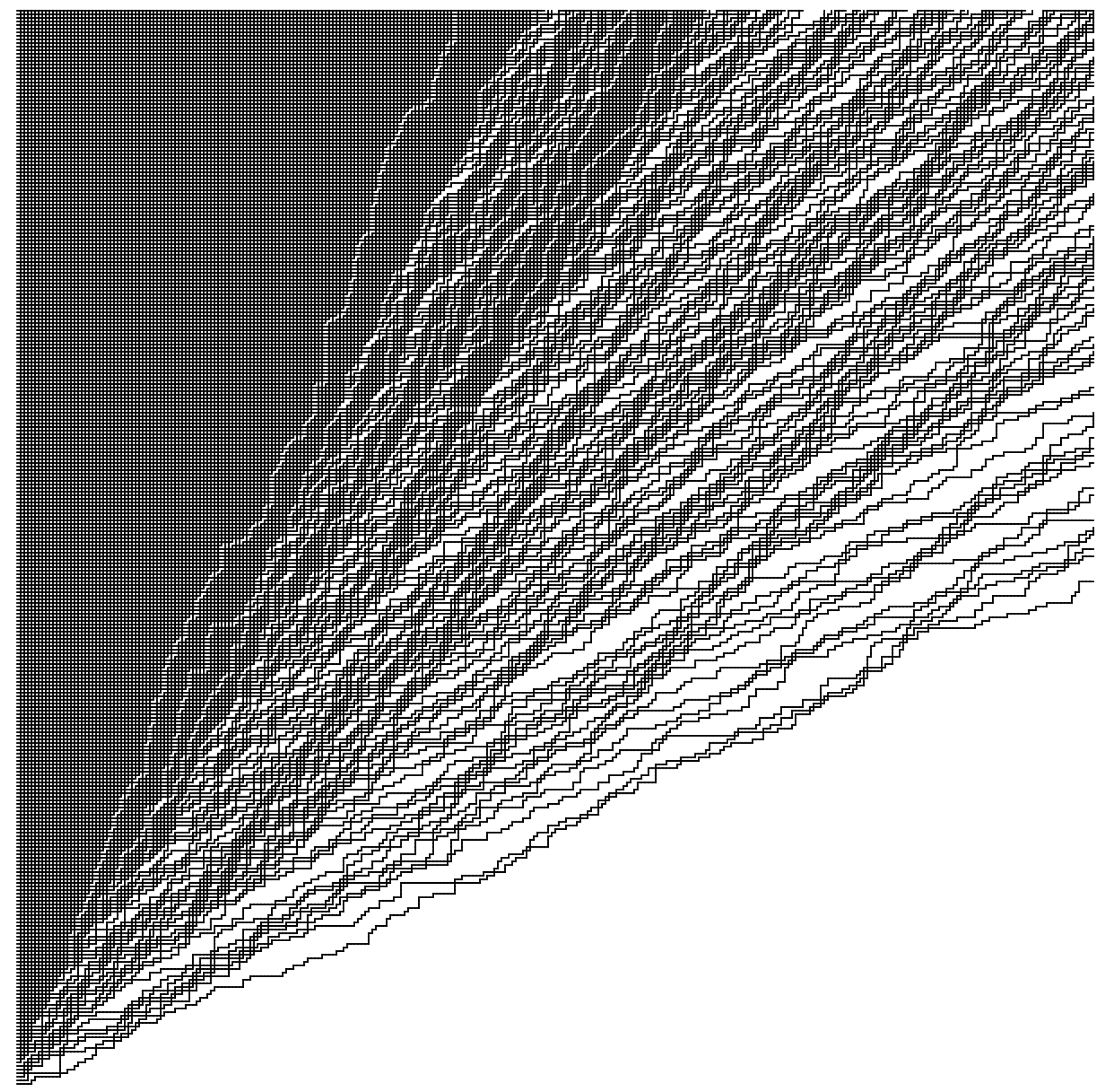




(antiferroelectric - has gas)



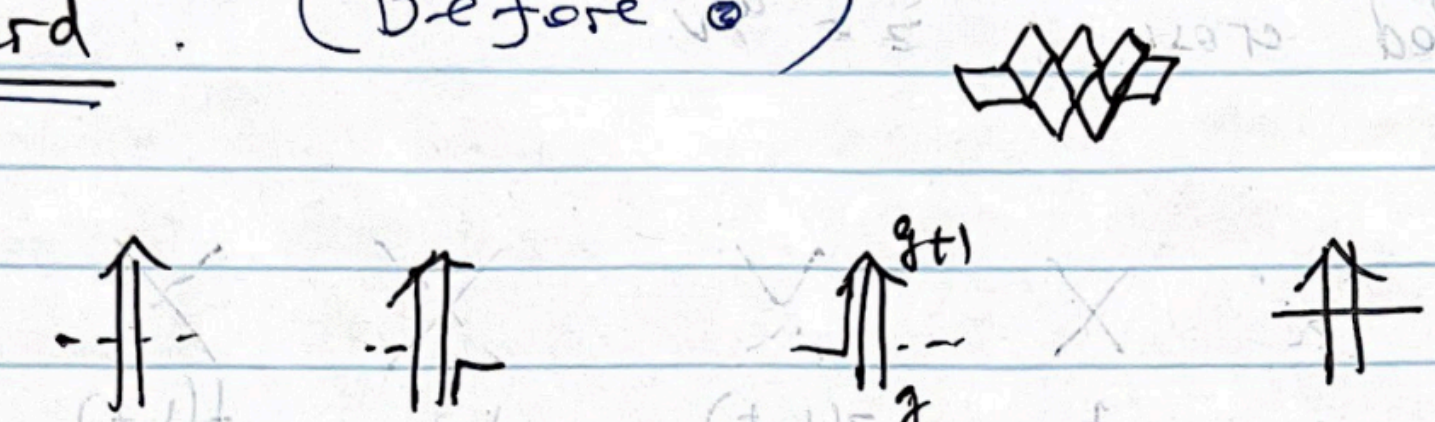
$a_1 = a_2 = b_2 = c_1 = c_2 = 1$, $b_1 = 3$,
domain wall boundary conditions



stochastic six vertex model in a quadrant

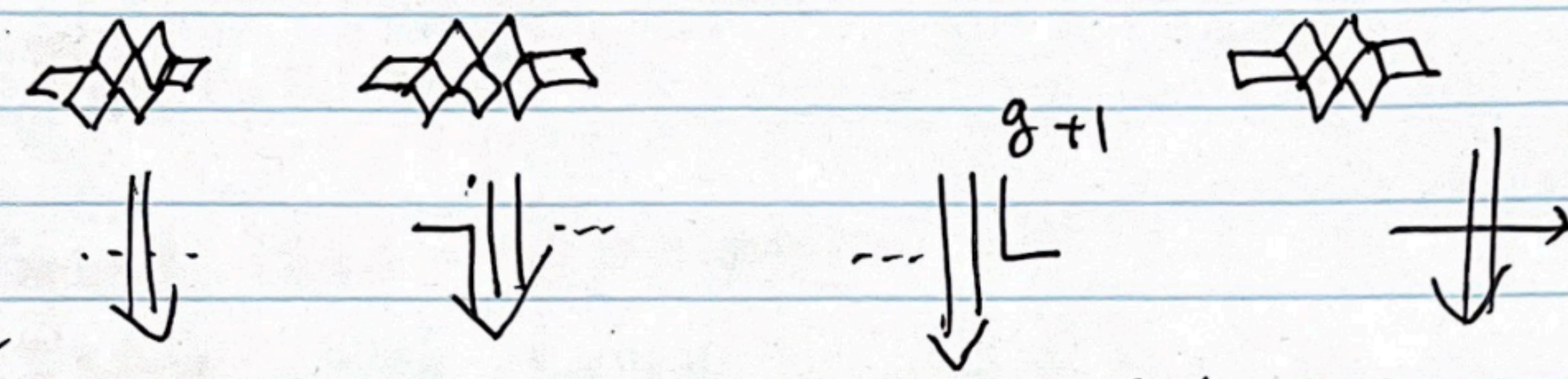
Board (Before !)

Red



W_u: 1 u $1-t^{g+1}$ u

Blue



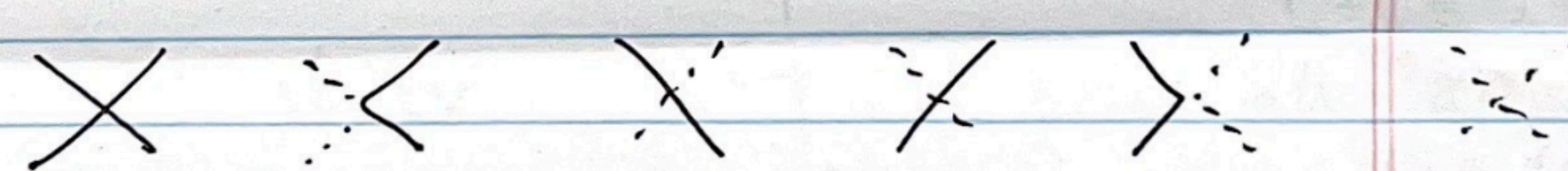
W^*_v: 1 1 $v(1-t^{g+1})$ v

Board 1 ⊕ ice

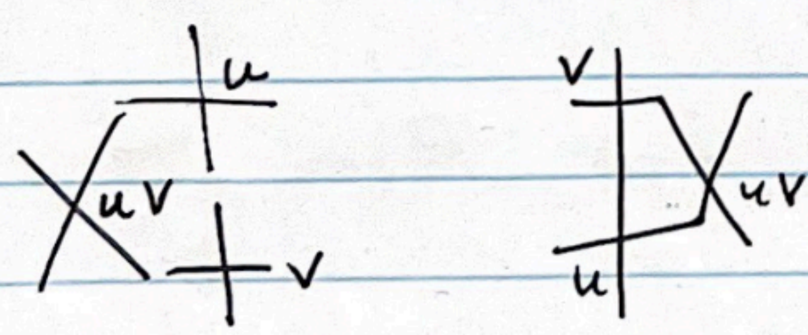
μ μ μ
μ ○ μ ○ ○ μ
○ ○ ○
back

Board 2

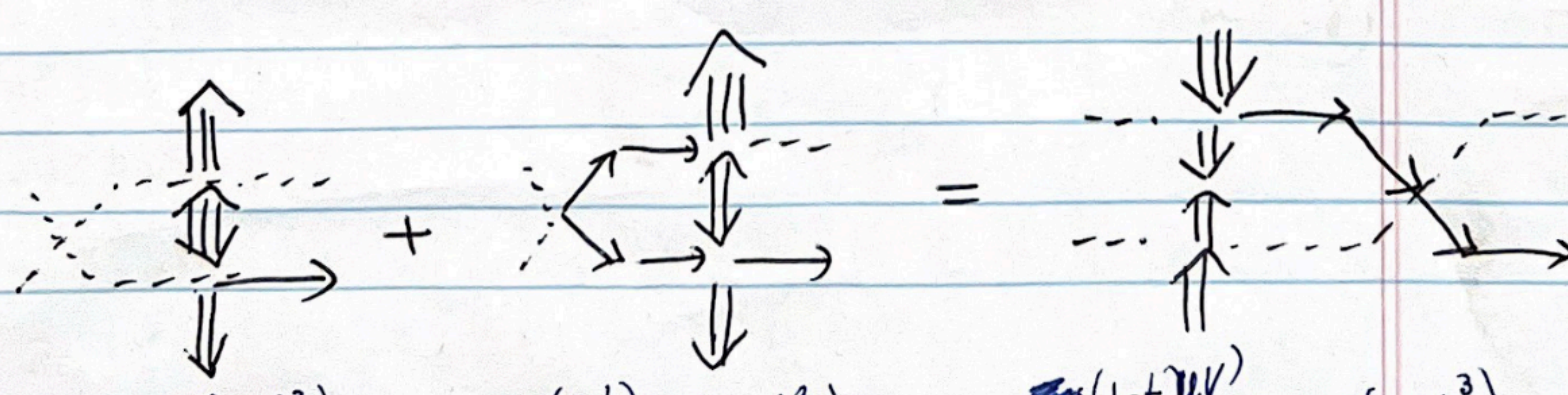
Crosses.



t $\frac{z(1-t)}{1-z}$ $\frac{1-tz}{1-z}$ $\frac{1-t}{1-z}$ 1



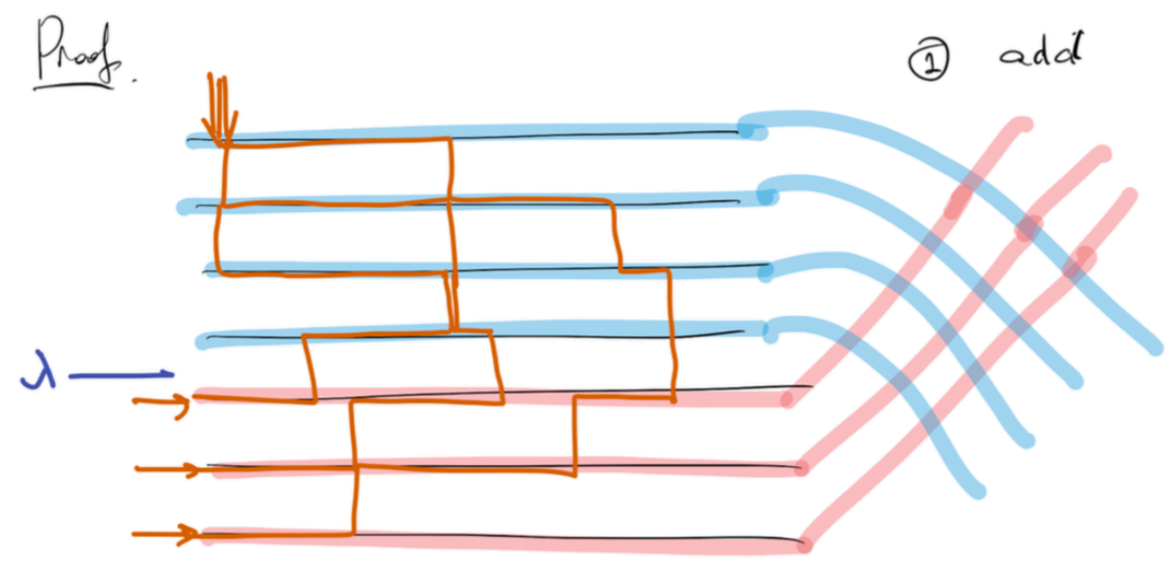
Ex. (YBE)



$$v(1-t^3) + \frac{uv(1-t)}{1-uv}\cdot(1-t^3)v = \frac{(1-t)uv}{1-uv}\cdot 1\cdot v(1-t^3)$$

III. Stochastic 6v model and Hall-Littlewood measures via bijectivisation

Recall the Yang-Baxter proof of the Hall-Littlewood Cauchy identity from yesterday



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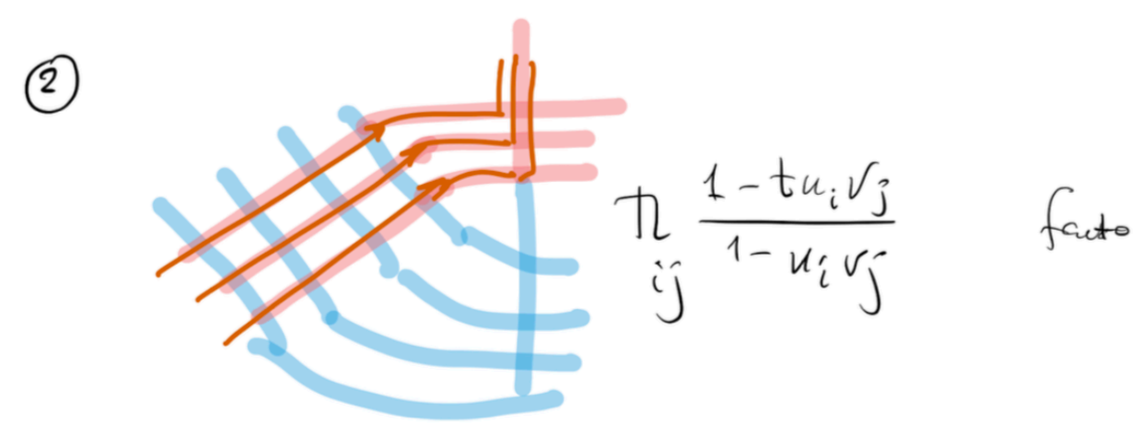

Let A, B be finite sets and $\sum_{a \in A} w(a) = \sum_{b \in B} w(b)$ (with positive terms)

A **bijectivisation** (**coupling**) of this identity is a family of transition probabilities $p(a \to b)$ and $p'(b \to a)$, satisfying

$$w(a)p(a \to b) = w(b)p'(b \to a)$$

for all $a \in A,\ b \in B$.

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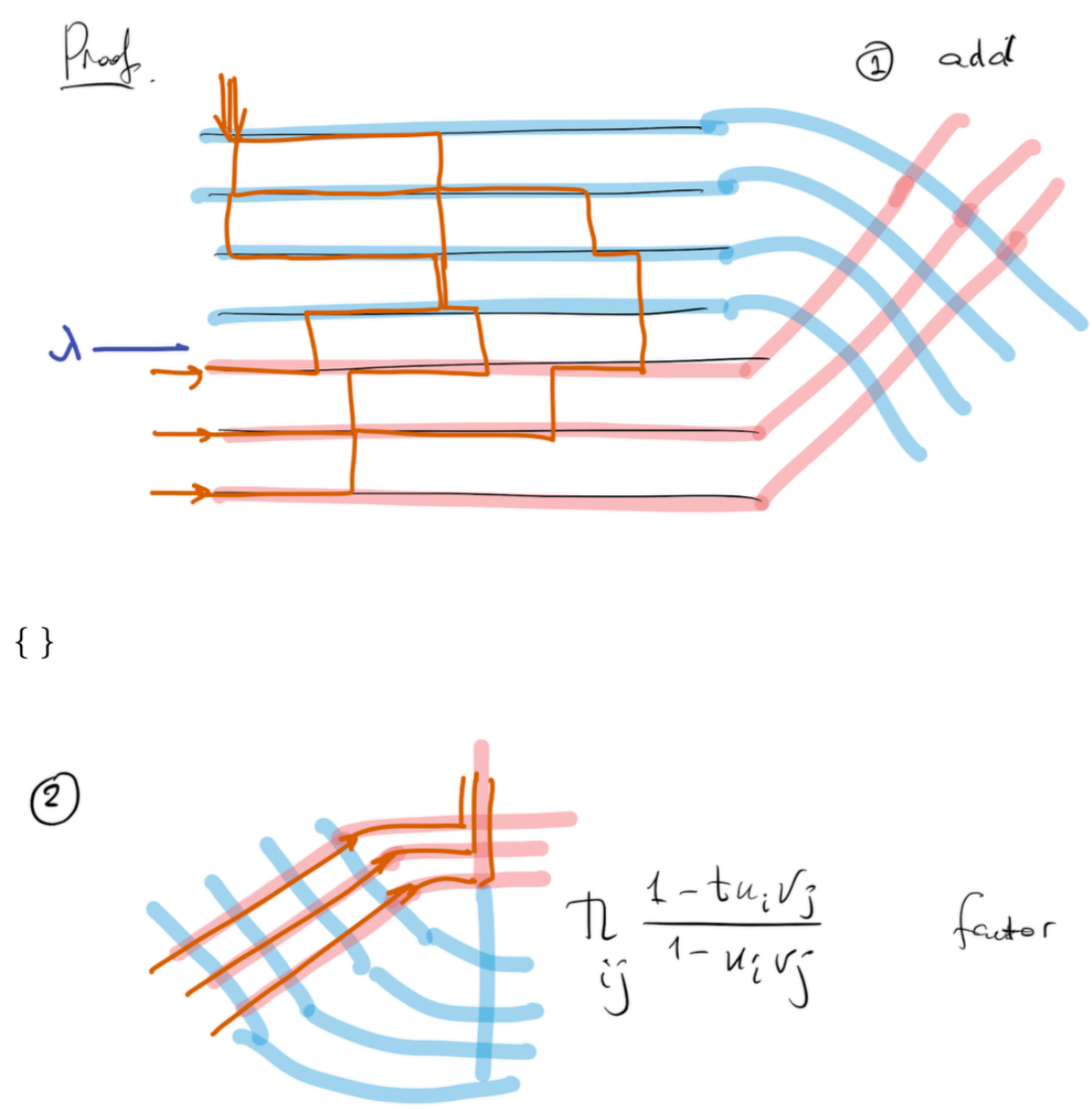

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Example: 1 + 3 = 2 + 2

	2	2
1	1	0
3	1/3	2/3

(maximally dependent)

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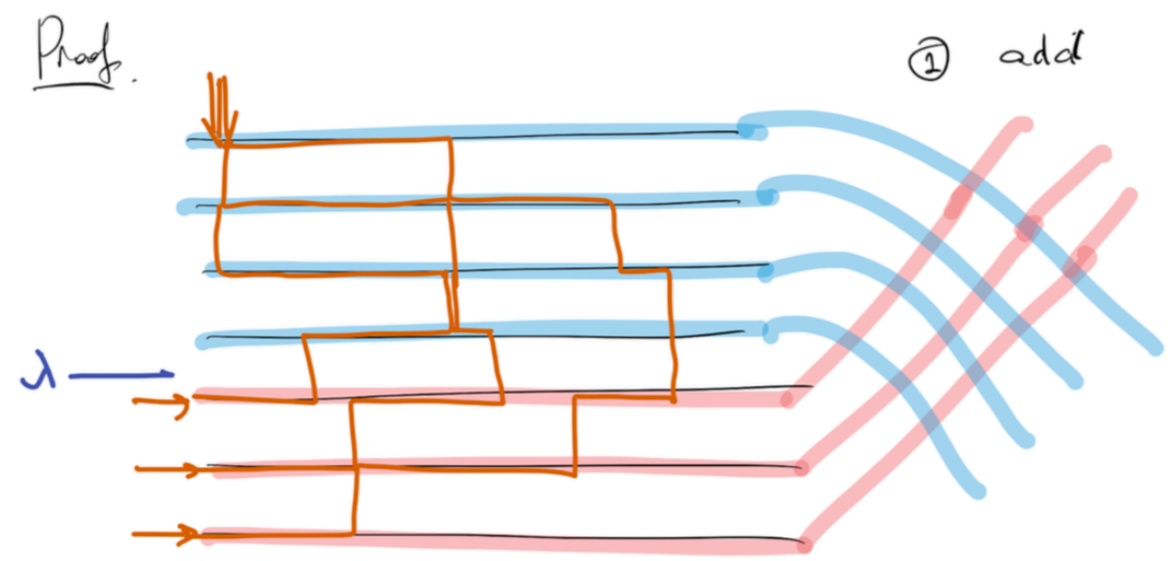

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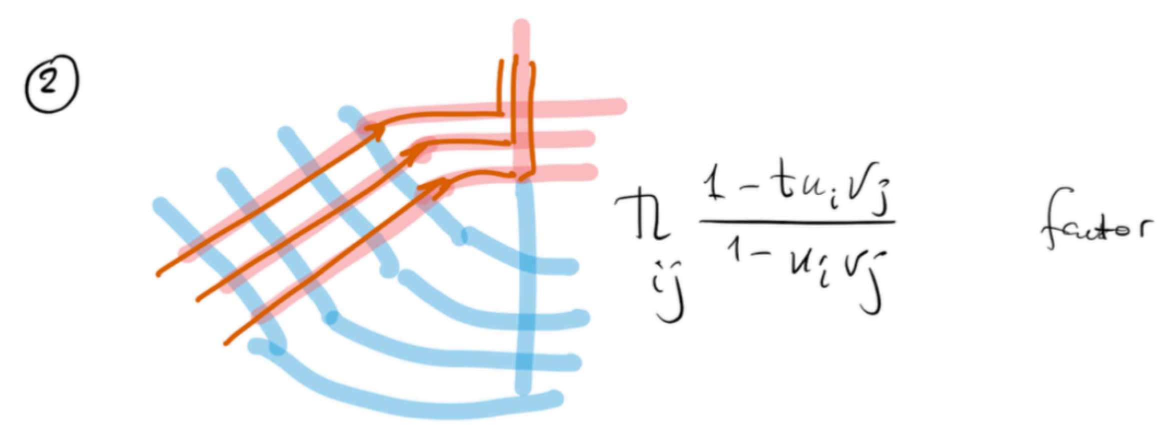

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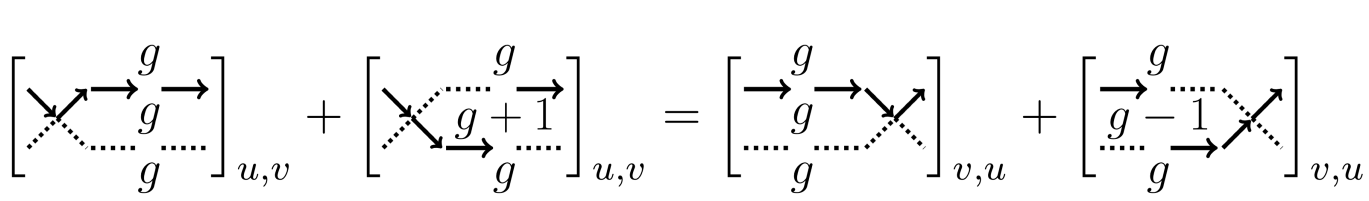
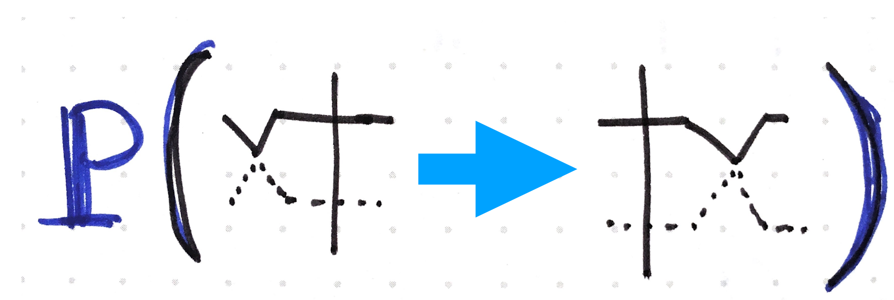
For the Yang-Baxter equation:



leads to



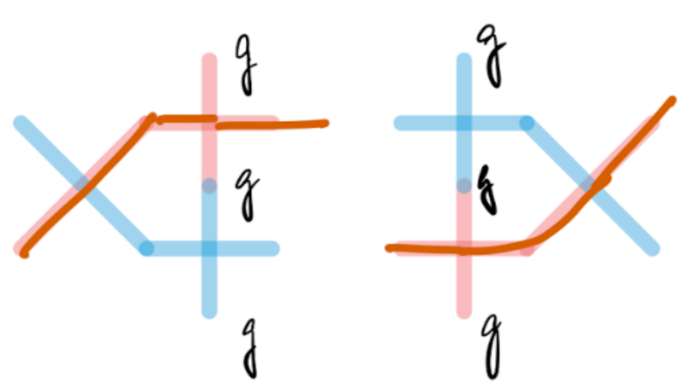
$$w(A) = \tilde{w}(C) + w(\tilde{D}),$$

(unique coupling)

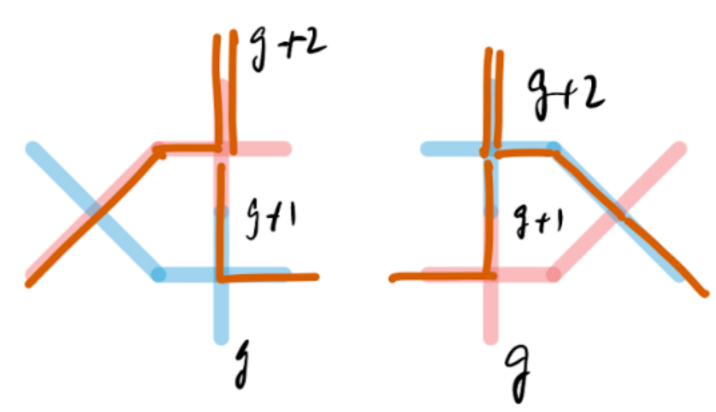
	C	D
A	p_1	$1-p_1$

$$p_1 = \tilde{w}(C)/w(A)$$

$$1-p_1 = \tilde{w}(D)/w(A)$$



prob. = 1



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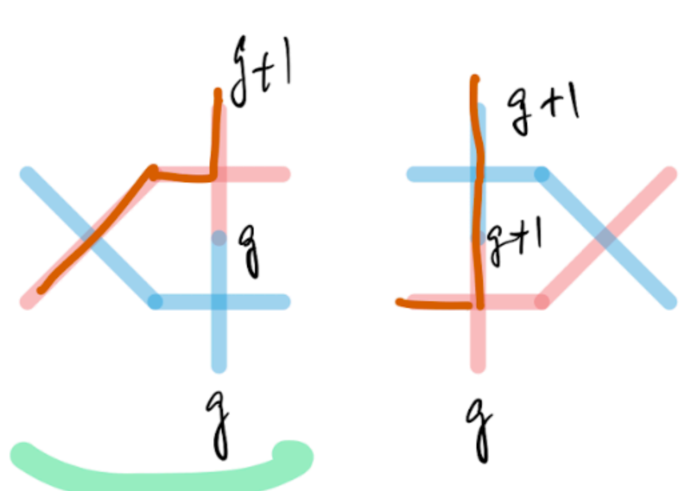
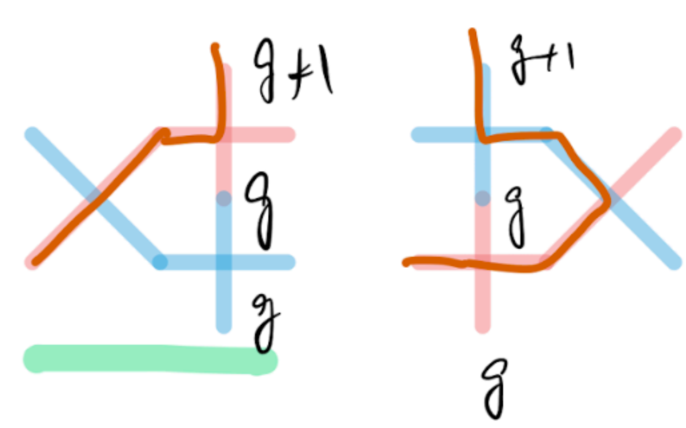
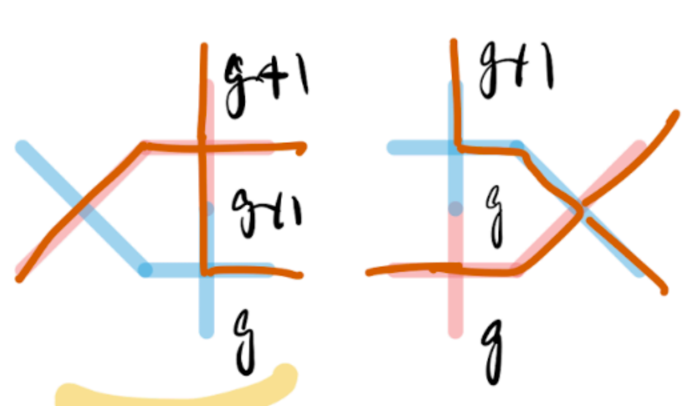


prob = α

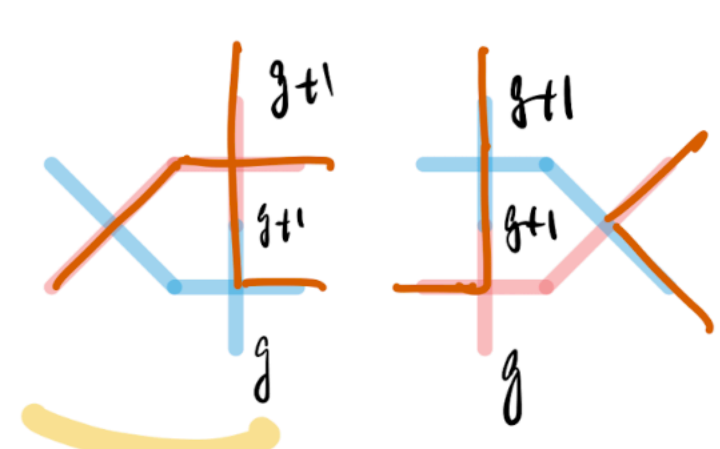


prob = $1-\alpha$

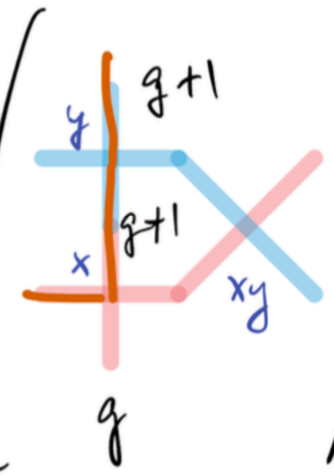
{ }

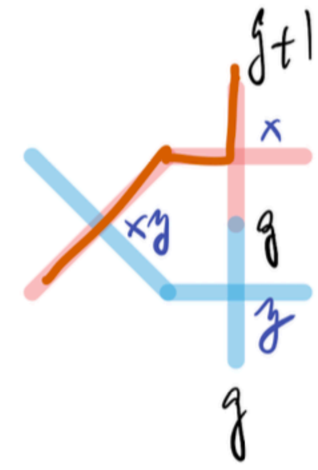


prob = β



prob = $1-\beta$

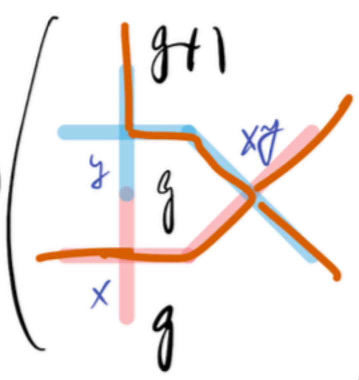
$$\alpha = w\left(\right.$$ 

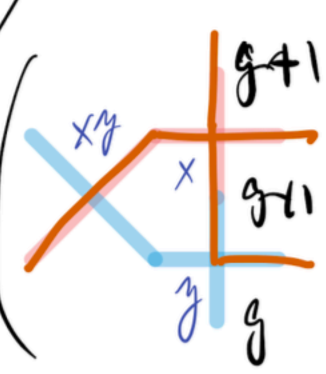
$$\left.\right) \Big/ \; w\left(\right.$$ 

$$\left.\right)$$

$$= \frac{(1-t^{g+1})\cdot}{\frac{1-txy}{1-xy}\cdot(1-t^{g+1})\cdot} = \boxed{\frac{1-xy}{1-txy}}$$

$\uparrow$ $b_1(x,y)$

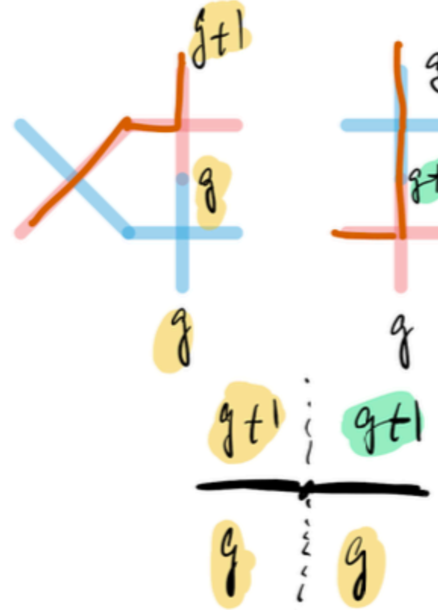
$$\beta = w\left(\right.$$ 

$$\left.\right) \Big/ \; w\left(\right.$$ 

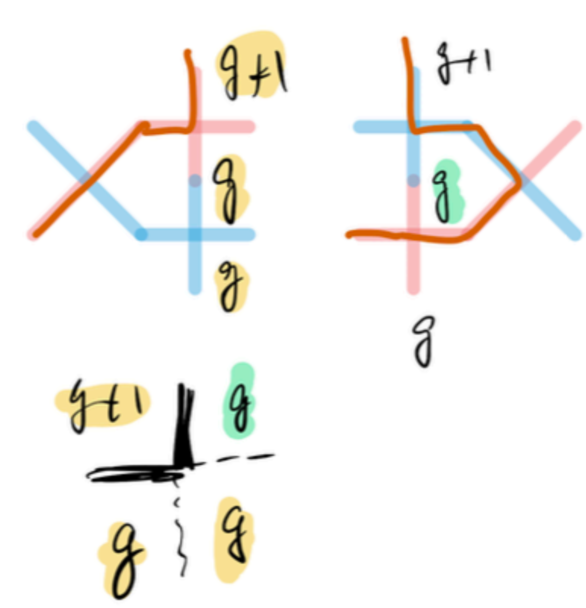
$$\left.\right) = \frac{t^{\circ}\, x\cdot}{\frac{1-txy}{1-xy}\, x}$$

$$b_2(x,y) = t\, b_1(x,y) \longrightarrow \; = \boxed{\frac{t\,(1-xy)}{1-txy}}$$

□



$b_1 = \alpha$



Theorem 5.3.1. S6V to HL coupling. Take the stochastic six vertex model, with inhomogeneous parameters $u_1, u_2, \dots$ along the vertical, and $v_1, v_2, \dots$ along the horizontal directions. The stochastic six vertex model updates the vertex at (x, y) with probabilities

$$b_1(u_y, v_x) = \frac{1 - u_y v_x}{1 - t u_y v_x}, \qquad b_2(u_y, v_x) = t\frac{1 - u_y v_x}{1 - t u_y v_x}.$$

Then the height function of this stochastic six vertex model (with domain wall like boundary conditions in $\mathbb{Z}^2_{\geq 0}$, i.e., paths enters at each site on the left boundary and nothing enters from below) has the following equality in distribution:

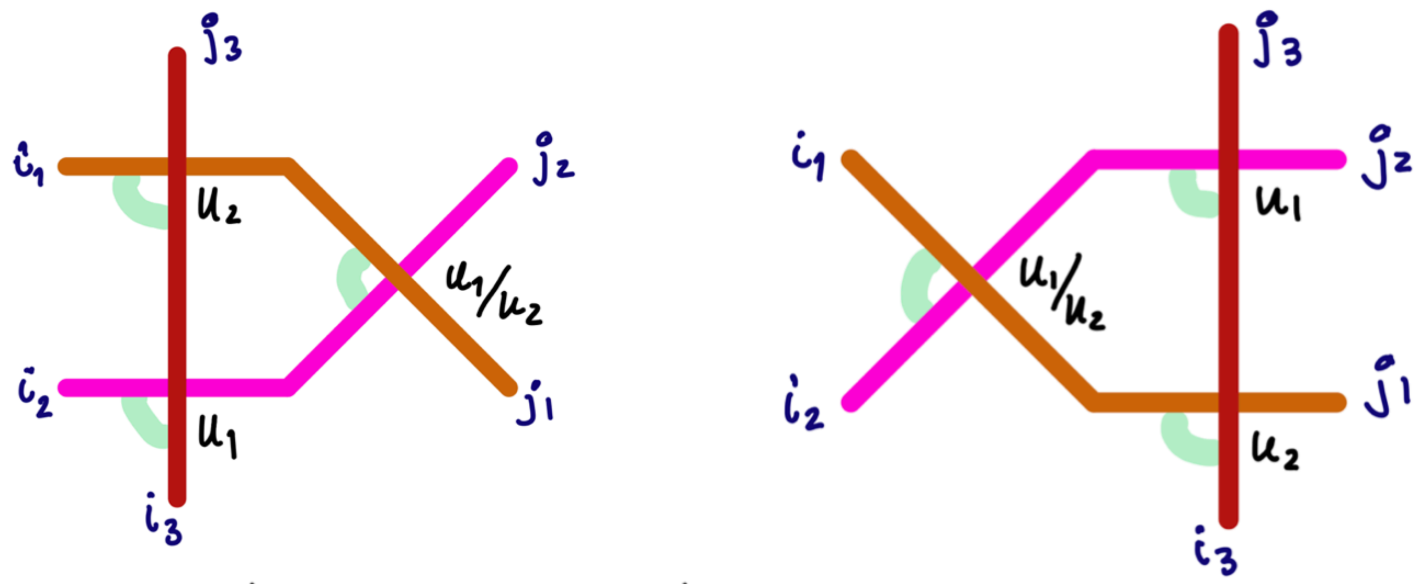
$$h(x, y) \stackrel{d}{=} m_0(\lambda^{(x,y)}) = y - \ell(\lambda^{(x,y)}),$$

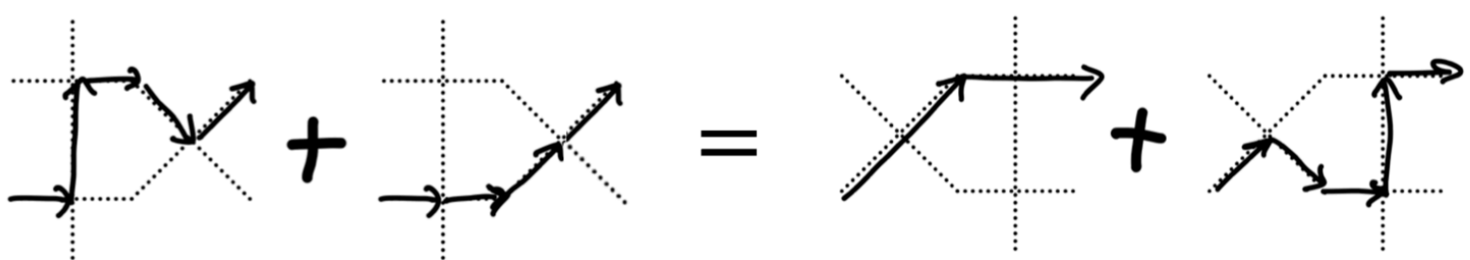
where $\lambda^{(x,y)}$ is the random signature distributed according to the Hall-Littlewood measure

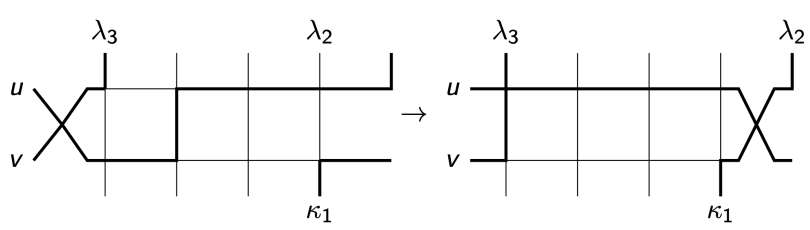
$$\mathrm{Prob}(\lambda) = \prod_{i=1}^{x} \prod_{j=1}^{y} \frac{1 - u_j v_i}{1 - t u_j v_i} P_\lambda(u_1, \dots, u_y) Q_\lambda(v_1, \dots, v_x).$$

Then, there is a whole different story to analyze $m_0(\lambda)$ asymptotically for the Hall-Littlewood measures, but it can be done

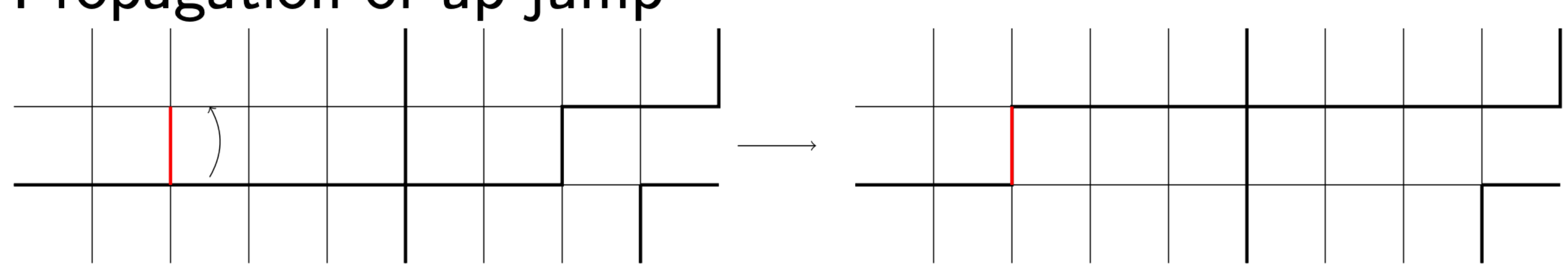
IV. Another application of the same idea: Borodin-Ferrari/Toninelli's dynamics on the six-vertex model



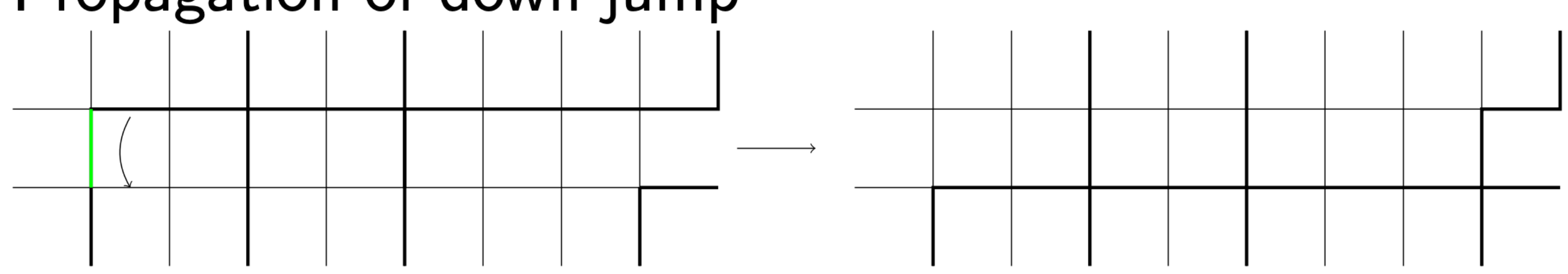




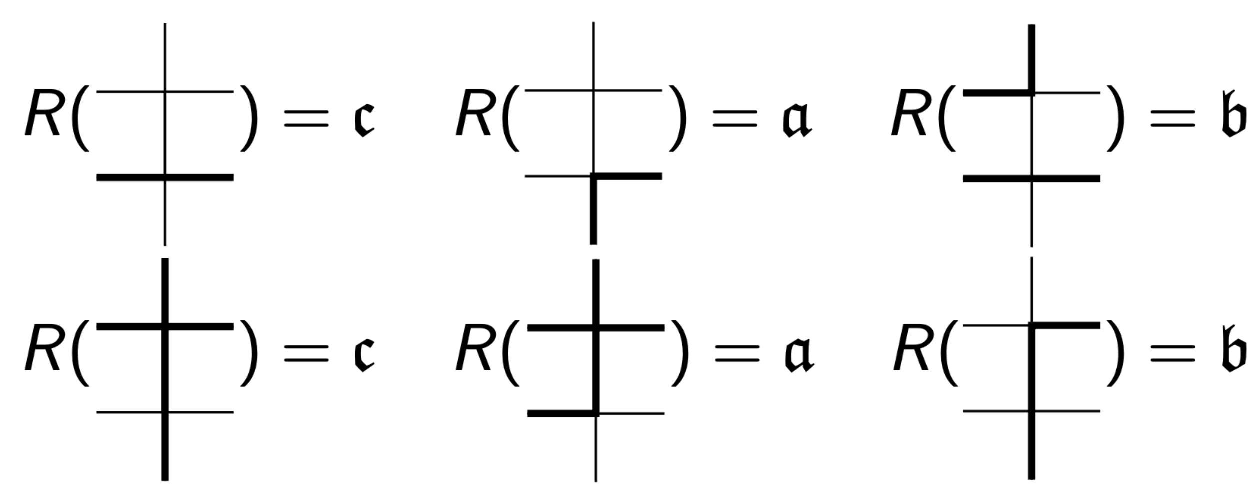
1. Occupied horizontal edges can jump up or down, and this jump propagates.
2. Propagation of up jump



3. Propagation of down jump



$$J(\rho, u) = -\frac{\rho(1-\rho)}{(\rho + u - \rho u)^2} \quad .$$

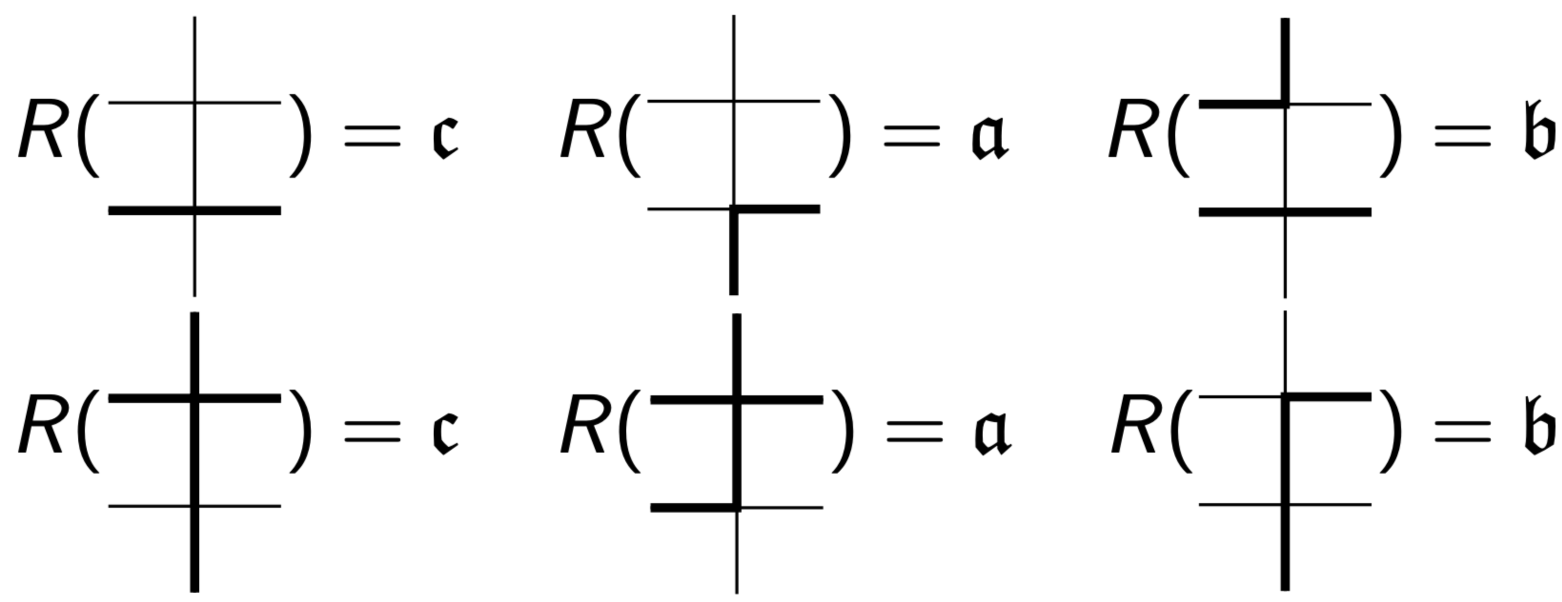


$$\mathfrak{c} := \frac{1-q}{(1-u)\,(1-qu)}, \qquad \mathfrak{b} := \frac{1-qu}{(1-u)\,(1-q)\,u},$$

$$\mathfrak{a} := \frac{(1-u)\,q}{(1-q)\,(1-qu)\,u}.$$

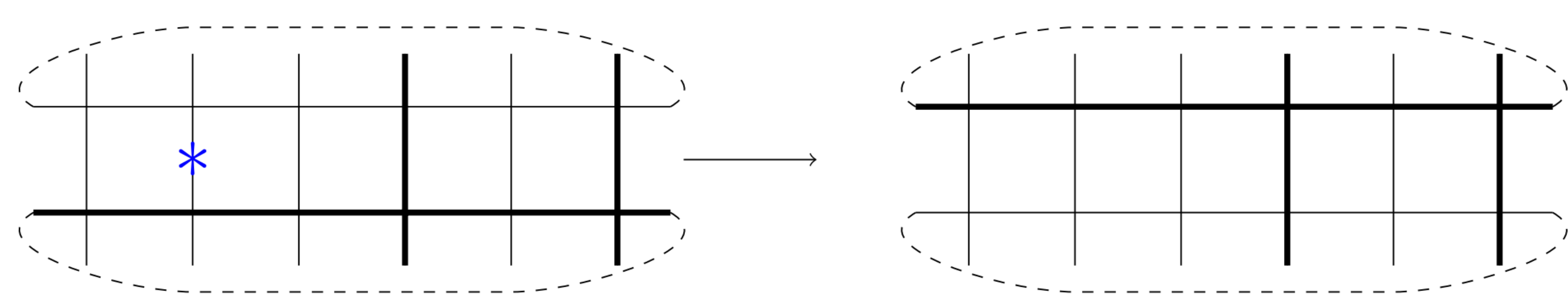
For tilings: $\frac{1}{\pi}\sin\psi_1\left(\frac{\sin\psi_1}{\tan\psi_2} + \sqrt{1 + \frac{\sin^2\psi_1}{\tan^2\psi_2}}\right)$

Specialize to 5-vertex model with r=corner weight, all other weights = 1
(It's not stochastic, but this is fine)



$$\mathfrak{c} = \frac{c_1 c_2}{\sqrt{b_1 b_2 a_1 a_2}}, \qquad \mathfrak{a} = \frac{\sqrt{b_1 b_2}}{\sqrt{a_1 a_2}}, \qquad \mathfrak{b} = \frac{\sqrt{a_1 a_2}}{\sqrt{b_1 b_2}}.$$

Now propagation may "loop all the way around the torus". For example:



Then for $a_2 = 0$ this is rescaled: $\mathfrak{c} = \frac{c_1 c_2}{\sqrt{b_1 b_2}}, \qquad \mathfrak{a} = \sqrt{b_1 b_2}, \qquad \mathfrak{b} = 0.$

For $b_1 = b_2 = 1, c_1 = c_2 = r$ we have $\mathfrak{c} = r^2$, $\mathfrak{a} = 1$.

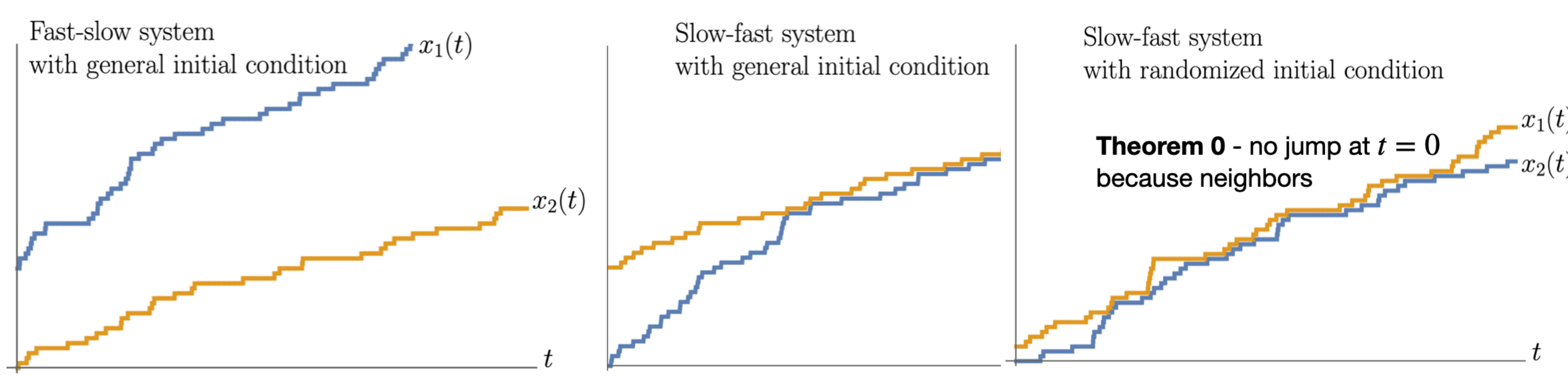
Encoding 5-vertex paths as tilings, we will get a generalization of Toninelli's dynamics

V. Yet another application of the same idea: TASEP with two cars

Two cars (discrete time TASEP with Bernoulli jumps). Randomized initial conditions

Theorem 0. Cars start at 0,1 (*step initial configuration*) ⇒ the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Theorem **fails** when cars are not initially neighbors, $x_1(0) - x_2(0) - 1 > 0$



Theorem 1 (P.-Saenz 2022). "*Be wise - randomize*". Recall $a_1 > a_2 > 0$.
Let $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$. Start SF from $(y_1(0), x_2(0))$.

Then the trajectories of the second particle become **the same in distribution**.

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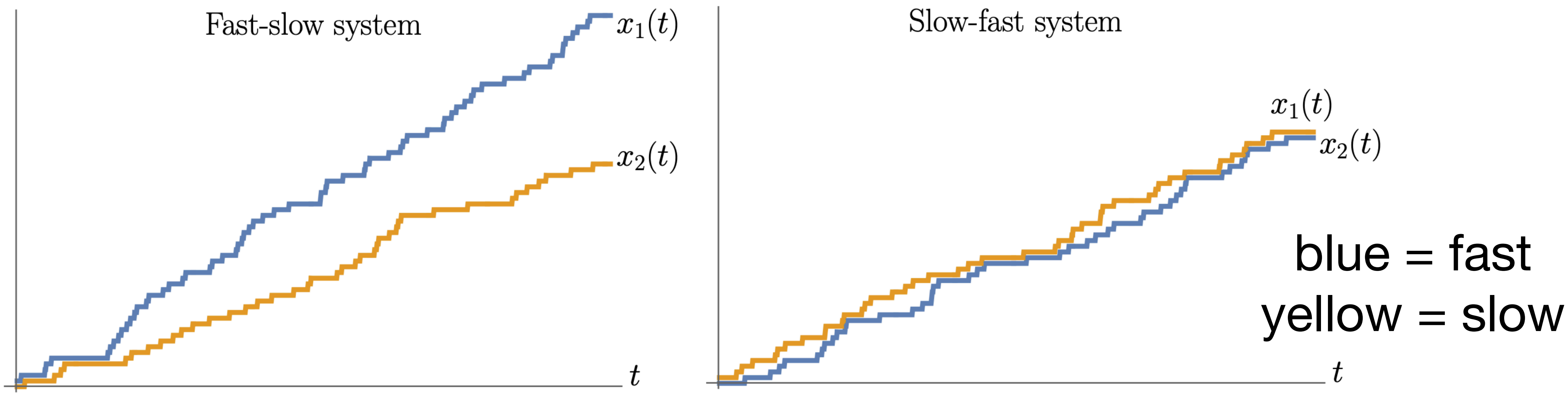

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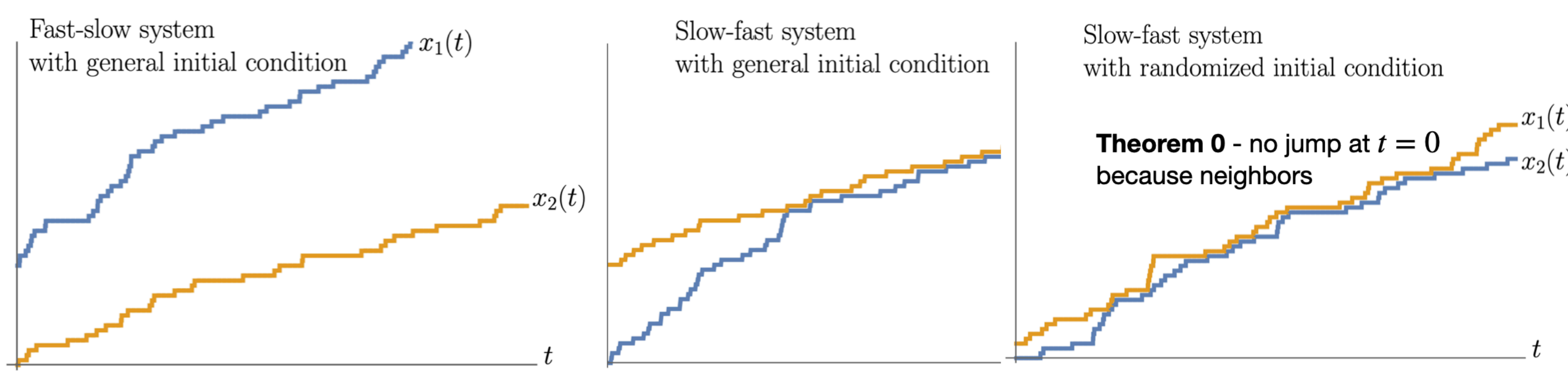

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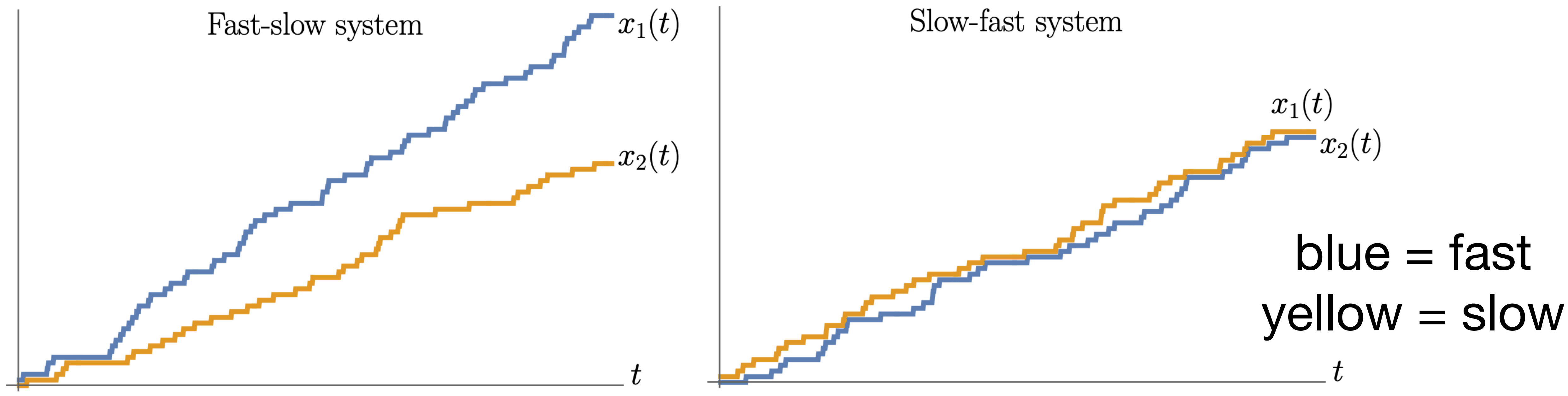
Then the trajectories of the second particle become **the same in distribution**.

V. Yet another application of the same idea: TASEP with two cars



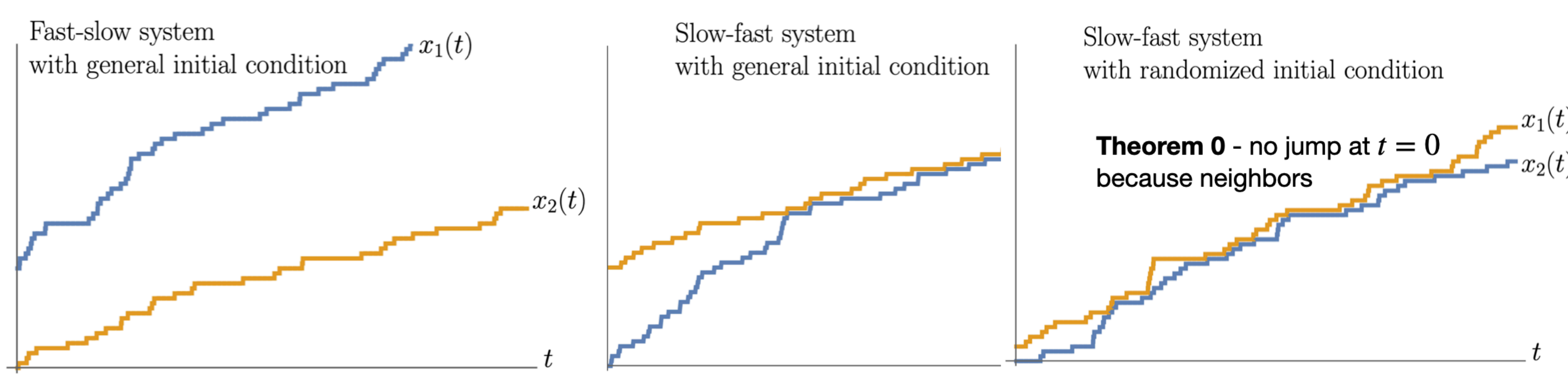
Theorem 0. (Vershik-Kerov ~1981; O'Connell 2003)

Cars start at 0,1 (*step initial configuration*)
⇒ the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Two cars (discrete time TASEP with Bernoulli jumps). Randomized initial conditions

Theorem 0. Cars start at 0,1 (*step initial configuration*) ⇒ the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Theorem **fails** when cars are not initially neighbors, $x_1(0) - x_2(0) - 1 > 0$



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