Introduction to symmetric functions

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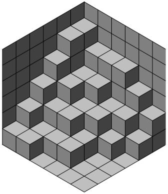
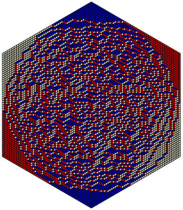
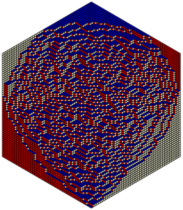
IPAM Program, tutorials, March 2024

The goal

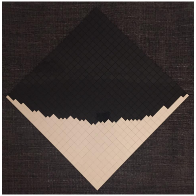
Statistical mechanics: dimer models (lozenge tilings), vertex models etc







Asymptotic Algebraic Combinatorics and Representation Theory: the quest for understanding structure constants (dimensions, Kostka, Littlewood-Richardson, Kronecker coefficients)



(©Dan Betea, at IHP Paris)

Integer partitions

Compositions:
$(a_1, \ldots, a_k) \in \mathbb{Z}^k_{\geq 0}$, such that $a_1 + \cdots + a_k = n$.
e.g. $n = 5, k = 3$: $(1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1)$
How many?

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Integer partitions $\lambda \vdash n : \lambda = (\lambda_1, \ldots, \lambda_\ell)$, s.t
$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$
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Hardy-Ramanujan:

$$p(n) := \#\{\lambda \vdash n\} \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2}{3}n}}$$

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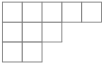
$$\sum_n p(n) q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

Plane partitions

Young diagram of $\lambda = (5,3,2)$: 

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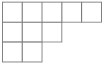
Young diagram of $\lambda = (5, 3, 2)$: 

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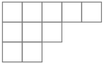
Young diagram of $\lambda = (5, 3, 2)$: 

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Plane partitions

4	4	3	1	1	0	..
4	3	2	1	0	..	
2	2	1	0	..		
1	1	0	..			

$\pi : \mathbb{N}^2 \to \mathbb{Z}_{\geq 0}$, s.t.

$$\pi(i,j) \geq \pi(i+1,j), \pi(i,j+1) \qquad |\pi| := \sum_{i,j} \pi(i,j)$$

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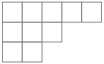
Young diagram of $\lambda = (5, 3, 2)$: 

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MacMahon's generating function:

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{\infty} \frac{1}{(1 - q^i)^i}$$

Plane partitions and dimers

5	4	4	4	3	2
5	3	3	2	2	1
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3	2	2	1		
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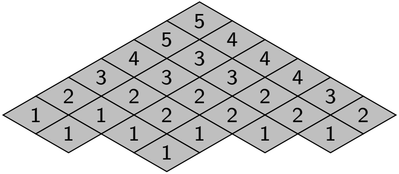

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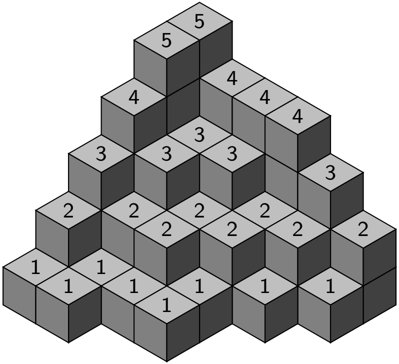

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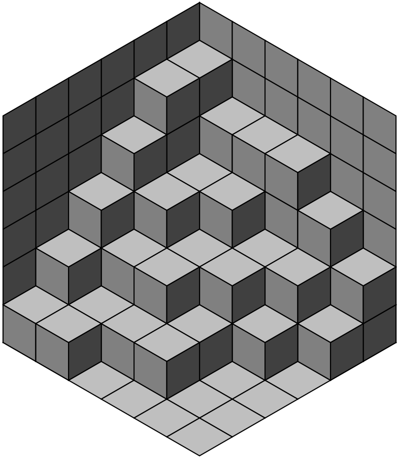

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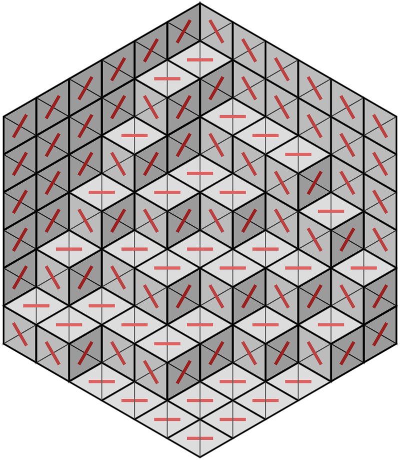

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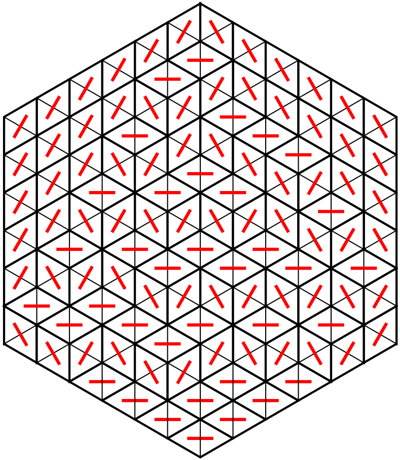

Proof I: bijection

Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

$$RRP \quad \pi = \begin{array}{ccc} 0&1&2\\ 1&1&3\\ 2&& \end{array} \to \begin{array}{ccc} 0&1&2\\ 1&1&3\\ 1&& \end{array} \to \begin{array}{ccc} 0&0&1\\ 0&0&3\\ 0&& \end{array} \to \begin{array}{ccc} 0&0&1\\ 0&0&2\\ 0&& \end{array} \to \begin{array}{ccc} 0&0&1\\ 0&0&1\\ 0&& \end{array}, \begin{array}{ccc} 0&0&0\\ 0&0&0\\ 0&& \end{array}$$

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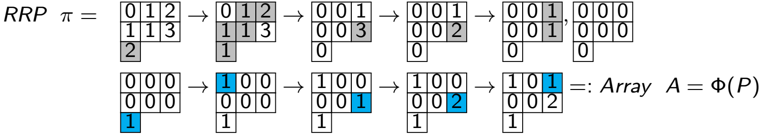

$$Weight(\pi) = |\pi| = 0+1+2+1+1+3+2 = 10 =$$

$$= \sum_{i,j} A_{i,j} hook(i,j) = 1*5 + 1*2 + 2*1 + 1*1 =: weight(A)$$

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$$\sum_{\pi \in RPP(\lambda)} q^{|\pi|} = \sum_{A: Array(\lambda)} \prod_{(i,j)\in\lambda} q^{h(i,j) * A_{i,j}} = \prod_{(i,j)\in\lambda} \frac{1}{1 - q^{h(i,j)}}$$

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Corollary: MacMahon's formula

$$\sum_{\pi \in RPP(a^b)} q^{|\pi|} = \prod_{i=1}^{a}\prod_{j=1}^{b} \frac{1}{1-q^{i+j-1}}$$

The ring of symmetric functions Λ

Λ_n = Formal power series in $x_1, x_2, \ldots$ of degree n, s.t. $f(x_1, x_2, \ldots) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots)$ for all permutations σ.

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$$\begin{aligned} m_{(2,1,1)}(x_1, x_2, x_3, x_4, x_5) &= x_1^2x_2x_3 + x_2^2x_1x_3 + \cdots + x_5^2x_3x_4 \\ &= m_{(2,1,1)}(x_1, \ldots, x_4) + x_5 m_{(2,1)}(x_1 \ldots, x_4) + x_5^2 m_{(1,1)}(x_1, \ldots, x_4) \end{aligned}$$

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Power sums:

$$p_n(x_1, \ldots) := x_1^n + x_2^n + \cdots \qquad p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots$$

$$\begin{aligned} p_2(x_1, \ldots) &= x_1^2 + x_2^2 + \cdots \\ p_{(2,1)}(x_1, \ldots) &= (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots) \\ &= m_3(x_1, \ldots) + m_{(2,1)}(x_1, \ldots) \end{aligned}$$

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Homogeneous:

$$h_n(x_1, \ldots, x_N) := \sum_{a_1+\cdots+a_N=n} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} = \sum_{\lambda \vdash n} m_\lambda(x_1, \ldots, x_N)$$

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Elementary:

$$e_n(x_1,\ldots,x_N) := \sum_{1\leq i_1<i_2<\cdots<i_n\leq N} x_{i_1}\cdots x_{i_n}$$

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e.g. $e_n(\underbrace{1,\ldots,1}_{N}) = \binom{N}{n}$

Relations

$$p_\lambda = \sum_\mu P(\lambda;\mu) m_\mu,$$

where $P(a;b) =$ number of set partitions $(B_1, B_2, \ldots, B_k)$ of $B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k = \{1, \ldots, \ell\}$, such that $\sum_{j \in B_i} \lambda_j = \mu_i$.

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$$e_\lambda = \sum_\mu N_0(\lambda,\mu) m_\mu, \qquad h_\lambda = \sum_\mu N(\lambda,\mu) m_\mu$$

where $N_0(\lambda,\mu) =$ number of $0-1$ matrices A, such that $\sum_i A_{i,j} = \lambda_j$ and $\sum_j A_{i,j} = \mu_i$ (*binary contingency tables*) and $N(\lambda,\mu)$ is the number of nonnegative integer matrices A with same constraints. (*contingency tables*)

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$$\sum_\lambda m_\lambda(x_1,\ldots) h_\lambda(y_1,\ldots) = \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x_1,\ldots) p_\lambda(y_1,\ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j},$$

where $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ where $\lambda = (\ldots, \underbrace{2, \ldots, 2}_{m_2}, \underbrace{1, \ldots, 1}_{m_1})$.

$$\sum_\lambda m_\lambda(x_1,\ldots) e_\lambda(y_1,\ldots) = \sum_\lambda \frac{(-1)^{|\lambda| - \ell(\lambda)}}{z_\lambda} p_\lambda(x_1,\ldots) p_\lambda(y_1,\ldots) = \prod_{i,j} (1 + x_i y_j)$$

The Schur functions

Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C}) \to GL(V)$:

Weyl modules V_λ (aka $\mathcal{W}_\lambda$), indexed by highest weights λ, $\ell(\lambda) \leq N$.

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Characters or representations $\rho : G \to GL(V)$: $\chi_V(g) = \mathrm{Tr}(\rho(g))$
$\{\chi_V : V \in Irr(G)\}$ -orthonormal basis of central functions on G(const on conjugacy classes), $\chi_V \longleftrightarrow V$.

$$s_\lambda(x_1, \ldots, x_N) = \chi_{V_\lambda}\left(\begin{bmatrix} x_1 & 0 & \cdots \\ 0 & x_2 & \cdots \\ \vdots & \ddots & \ldots \end{bmatrix}\right)$$

Special cases:

$$s_{(n)} = h_n \qquad s_{(1^n)} = e_n$$

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Weyl character formula:

$$s_\lambda(x_1, \ldots, x_N) := \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}$$

The Schur functions

Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C}) \to GL(V)$:

Weyl modules V_λ (aka $\mathcal{W}_\lambda$), indexed by highest weights λ, $\ell(\lambda) \leq N$.

Characters or representations $\rho : G \to GL(V)$: $\chi_V(g) = \mathrm{Tr}(\rho(g))$
$\{\chi_V : V \in Irr(G)\}$ -orthonormal basis of central functions on G(const on conjugacy classes), $\chi_V \longleftrightarrow V$.

$$s_\lambda(x_1, \ldots, x_N) = \chi_{V_\lambda}\left(\begin{bmatrix} x_1 & 0 & \cdots \\ 0 & x_2 & \cdots \\ \vdots & \ddots & \cdots \end{bmatrix}\right)$$

Special cases:

$$s_{(n)} = h_n \qquad s_{(1^n)} = e_n$$

Weyl character formula:

$$s_\lambda(x_1, \ldots, x_N) := \frac{\det\left[x_i^{\lambda_j + N - j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}$$

$$s_{\underbrace{(k-1, k-2, \ldots, 1)}_{\delta_k}}(x_1, \ldots, x_k) = \frac{\det\left[x_i^{2(k-j)}\right]_{i,j=1}^k}{\prod_{i<j}(x_i - x_j)} = \prod_{i<j}(x_i + x_j)$$

Schur functions, continued

Jacobi-Trudi identity:

$$s_{\lambda_1,\ldots,\lambda_k} = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & \ddots & h_{\lambda_i+k-j} & \vdots \end{bmatrix}_{i,j=1}^{k}$$

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Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1,x_2,x_3) = \underset{\begin{smallmatrix}1&1\\2&2\end{smallmatrix}}{x_1^2x_2^2} + \underset{\begin{smallmatrix}1&1\\3&3\end{smallmatrix}}{x_1^2x_3^2} + \underset{\begin{smallmatrix}2&2\\3&3\end{smallmatrix}}{x_2^2x_3^2} + \underset{\begin{smallmatrix}1&1\\2&3\end{smallmatrix}}{x_1^2x_2x_3} + \underset{\begin{smallmatrix}1&2\\2&3\end{smallmatrix}}{x_1x_2^2x_3} + \underset{\begin{smallmatrix}1&2\\3&3\end{smallmatrix}}{x_1x_2x_3^2}.$$

MacMahon, second time

SSYT shape $\lambda = (a^b)$ and entries $0, 1, 2, \ldots, b + c - 1$:

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline 2&3&3&4&5\\\hline 4&4&5&6&6\\\hline\end{array} - \begin{array}{|c|}\hline 0\\\hline 1\\\hline 2\\\hline\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline 1&2&2&3&4\\\hline 2&2&3&4&4\\\hline\end{array} = \text{ RPP entries } 0, 1, \ldots, c$$

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$$\sum_{\pi \in RPP(a \times b \times c)} q^{|\pi|} = q^{-\binom{b}{2}a} s_{a^b}(1, q, q^2, \ldots, q^{b+c-1})$$

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$$= q^{-\binom{b}{2}a} \frac{\det[q^{(b+c-i)(\lambda_j+b+c-j)}]_{i,j=1}^{b+c}}{\prod_{i<j}(q^{b+c-i} - q^{b+c-j})} = q^{-\binom{b}{2}a} \prod_{i<j} \frac{(q^{\lambda_i+b+c-i} - q^{\lambda_j+b+c-j})}{q^{b+c-i} - q^{b+c-j}} = \ldots$$

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$$\ldots = \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{a+j+k-1}}{1 - q^{j+k-1}}$$

Standard Young Tableaux (SYT)

SYT of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$:
$T : \lambda \xrightarrow{\sim} \{1, \ldots, n\}$
and
$T_{i,j} < T_{i,j+1}, T_{i+1,j}$

$$T = \begin{array}{|c|c|c|c|} \hline T_{1,1} & T_{1,2} & \cdots & T_{1,\lambda_1} \\ \hline T_{2,1} & \cdots & T_{2,\lambda_2} & \\ \cline{1-3} \vdots & \ddots & \cdots & \\ \cline{1-3} T_{k,1} & \cdots & T_{k,\lambda_k} & \\ \cline{1-3} \end{array}$$

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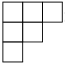
$T =$

$T_{1,1}$	$T_{1,2}$	$\cdots$	T_{1,λ_1}
$T_{2,1}$	$\cdots$	T_{2,λ_2}	
$\vdots$	$\ddots$	$\cdots$	
$T_{k,1}$	$\cdots$	T_{k,λ_k}	

$\lambda =$ :

1	2	3
4	5	
6		

1	2	3
4	6	
5		

1	2	4
3	5	
6		

1	2	4
3	6	
5		

1	2	5
3	4	
6		

1	2	5
3	6	
4		

1	2	6
3	4	
5		

1	2	6
3	5	
4		

$+$ *all transposed*

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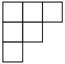
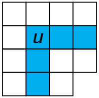
$\lambda =$  :

$$\begin{array}{lll} 1&2&3\\4&5\\6 \end{array} \quad \begin{array}{lll} 1&2&3\\4&6\\5 \end{array} \quad \begin{array}{lll} 1&2&4\\3&5\\6 \end{array} \quad \begin{array}{lll} 1&2&4\\3&6\\5 \end{array} \quad \begin{array}{lll} 1&2&5\\3&4\\6 \end{array} \quad \begin{array}{lll} 1&2&5\\3&6\\4 \end{array} \quad \begin{array}{lll} 1&2&6\\3&4\\5 \end{array} \quad \begin{array}{lll} 1&2&6\\3&5\\4 \end{array} \; + \; \textit{all transposed}$$

Hook-length formula [Frame-Robinson-Thrall]:

$$\dim \mathbb{S}_\lambda = \#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u} = \frac{6!}{5*3*3*1*1*1} = 16$$

Hook length of box $u = (i,j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#\blacksquare \in$ 

Semi-standard Young Tableaux (SSYT)

SSYT of shape $\lambda = (\lambda_1, \dots, \lambda_k)$:
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$T_{i,j} \leq T_{i,j+1},\ T_{i,j} < T_{i+1,j}$

$T =$

$T_{1,1}$	$T_{1,2}$	$\cdots$	T_{1,λ_1}
$T_{2,1}$	$\cdots$	T_{2,λ_2}	
$\vdots$	$\ddots$	$\cdots$	
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$\lambda = \begin{array}{|c|c|c|} \hline \ & \ & \ \\ \hline \ & \ & \\ \cline{1-2} \ & & \\ \cline{1-1} \end{array}$, $n = 6$, $N = 3$:

$$\begin{array}{|c|c|c|} \hline 1&1&1 \\ \hline 2&2& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1&1&1 \\ \hline 2&3& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1&1&2 \\ \hline 2&2& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1&1&2 \\ \hline 2&3& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1&1&3 \\ \hline 2&2& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1&1&3 \\ \hline 2&3& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1&2&2 \\ \hline 2&3& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1&2&3 \\ \hline 2&3& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array}$$

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$$s_\lambda(x_1, \ldots, x_N) = \sum_{T \in SSYT(\lambda, N)} \prod_{(i,j) \in \lambda} x_{T_{i,j}}$$

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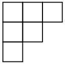
$\lambda =$ , $n = 6$, $N = 3$:

$$\begin{array}{ccc}1&1&1\\2&2&\\3&&\end{array}\quad \begin{array}{ccc}1&1&1\\2&3&\\3&&\end{array}\quad \begin{array}{ccc}1&1&2\\2&2&\\3&&\end{array}\quad \begin{array}{ccc}1&1&2\\2&3&\\3&&\end{array}\quad \begin{array}{ccc}1&1&3\\2&2&\\3&&\end{array}\quad \begin{array}{ccc}1&1&3\\2&3&\\3&&\end{array}\quad \begin{array}{ccc}1&2&2\\2&3&\\3&&\end{array}\quad \begin{array}{ccc}1&2&3\\2&3&\\3&&\end{array}$$

$$s_\lambda(x_1, \ldots, x_N) = \sum_{T \in SSYT(\lambda, N)} \prod_{(i,j) \in \lambda} x_{T_{i,j}}$$

$$s_{(3,2,1)}(x_1, x_2, x_3) = x_1^3 x_2^2 x_3 + x_1^3 x_2 x_3^2 + x_1^2 x_2^3 x_3 + 2x_1^2 x_2^2 x_3^2 + x_1^2 x_2 x_3^3 + x_1 x_2^3 x_3^2 + x_1 x_2^2 x_3^3$$

$$\# SSYT(\delta_k, k) = s_{\delta_k}(1^k) = 2^{\binom{k}{2}}$$

The number of SSYTs

Hook-content formula:

$$\#SSYT(\lambda, N) = \prod_{(i,j)\in\lambda} \frac{N+j-i}{h_{(i,j)}}$$

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Proof:

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Weyl's determinantal formula:

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$$= \frac{\det[(q^{\lambda_i+N-i})^{j-1}]_{i,j=1}^N}{\prod_{i<j}(q^{i-1} - q^{j-1})} = \frac{\prod_{i<j}(q^{\lambda_i+N-i} - q^{\lambda_j+N-j})}{\prod_{i<j}(q^{j-1} - q^{i-1})}$$

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Corollary: Hook-length formula

$$f^\lambda := \#SYT(\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda} h_{(i,j)}}.$$

The number of SSYTs

Hook-content formula:

$$\#SSYT(\lambda, N) = \prod_{(i,j)\in\lambda} \frac{N+j-i}{h_{(i,j)}}$$

Kostka mumbers:

The number of SSYTs of shape λ of type μ – μ_1 1s, μ_2 2s etc:

$$K_{\lambda\mu} := \#SSYT(\lambda;\mu)$$

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu} m_\mu \qquad h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$

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The number of SSYTs of shape λ of type μ – μ_1 1s, μ_2 2s etc:

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$$s_\lambda(x) = \sum_\mu K_{\lambda\mu} m_\mu \qquad h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$

Deciding is $K_{\lambda\mu} > 0$ is in P. However, computing the value of $K_{\lambda\mu}$ is #P-complete when λ, μ-binary and [conjecturally] #P-complete when λ, μ-unary.

RSK

Theorem

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

Theorem (Cauchy's identity)

$$\sum_{\lambda} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Origins: Schur-Weyl duality in representation theory
Combinatorial proof:

RSK

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Origins: Schur-Weyl duality in representation theory
Combinatorial proof:
The **Robinson-Schensted-Knuth (RSK)** bijection:

Let $A \in \mathbb{Z}_{\geq 0}^{m \times n}$.

$$RSK(A) = (P, Q): \; P, Q \in SSYT(\lambda) \text{ for some } \lambda$$

and $type(P) = (\sum_j A_{j,1}, \sum_j A_{j,2}, \ldots)$, $type(Q) = (\sum_j A_{1,j}, \sum_j A_{2,j}, \ldots)$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \xleftrightarrow{RSK} \left(\begin{array}{lllll} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 4 & & \\ 3 & 5 & & & \end{array}, \begin{array}{lllll} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 4 & & \\ 3 & 4 & & & \end{array} \right)$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow \quad w_A = \begin{pmatrix} 3 & 5 & 2 & 4 & 4 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \to \quad w_A = \begin{pmatrix} \downarrow & & & & & & & & & \\ 3 & 5 & 2 & 4 & 4 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

$$(\emptyset, \emptyset) \leftarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow \quad w_A = \begin{pmatrix} 3 & \overset{\downarrow}{5} & 2 & 4 & 4 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

$$(\emptyset, \emptyset) \leftarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \left(\boxed{3}, \boxed{1}\right) \leftarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \to \quad w_A = \begin{pmatrix} 3 & 5 & \overset{\downarrow}{2} & 4 & 4 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

$$(\emptyset, \emptyset) \leftarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \left(\boxed{3}, \boxed{1}\right) \leftarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \left(\boxed{3}\boxed{5}, \boxed{1}\boxed{1}\right) \leftarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow \quad w_A = \begin{pmatrix} 3 & 5 & 2 & \overset{\downarrow}{4} & 4 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

$$(\emptyset, \emptyset) \leftarrow \binom{3}{1} \qquad \left(\boxed{3}, \boxed{1} \right) \leftarrow \binom{5}{1} \qquad \left(\begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \right) \leftarrow \binom{2}{2}$$

$$\left(\begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) \leftarrow \binom{4}{2}$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \to \quad w_A = \begin{pmatrix} 3 & 5 & 2 & 4 & \overset{\downarrow}{4} & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

$$(\emptyset, \emptyset) \leftarrow \binom{3}{1} \qquad \left(\begin{array}{|c|}\hline 3 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline \end{array} \right) \leftarrow \binom{5}{1} \qquad \left(\begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \right) \leftarrow \binom{2}{2}$$

$$\left(\begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \right) \leftarrow \binom{4}{2} \qquad \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \right) \leftarrow \binom{4}{2}$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \to \quad w_A = \begin{pmatrix} 3 & 5 & 2 & 4 & 4 & \overset{\downarrow}{1} & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

$$(\emptyset, \emptyset) \leftarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \left(\boxed{3}, \boxed{1}\right) \leftarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \left(\boxed{3\,|\,5}, \boxed{1\,|\,1}\right) \leftarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right) \leftarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} \qquad \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}\right) \leftarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} \qquad \left(\begin{array}{|c|c|c|}\hline 2 & 4 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}\right) \leftarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

RSK: insertion and recording

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \to \quad w_A = \begin{pmatrix} 3 & 5 & 2 & 4 & 4 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

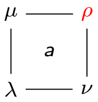
$$(\emptyset, \emptyset) \leftarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \left(\begin{array}{c} 3 \end{array}, \begin{array}{c} 1 \end{array} \right) \leftarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \left(\begin{array}{cc} 3 & 5 \end{array}, \begin{array}{cc} 1 & 1 \end{array} \right) \leftarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc} 2 & 5 \\ 3 & \end{array}, \begin{array}{cc} 1 & 1 \\ 2 & \end{array} \right) \leftarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} \qquad \left(\begin{array}{cc} 2 & 4 \\ 3 & 5 \end{array}, \begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \right) \leftarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} \qquad \left(\begin{array}{ccc} 2 & 4 & 4 \\ 3 & 5 & \end{array}, \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & \end{array} \right) \leftarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc} 1 & 4 & 4 \\ 2 & 5 & \\ 3 & & \end{array}, \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \right) \qquad \left(\begin{array}{cccc} 1 & 4 & 4 & 4 \\ 2 & 5 & & \\ 3 & & & \end{array}, \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 2 & & \\ 3 & & & \end{array} \right) \qquad \left(\begin{array}{ccccc} 1 & 4 & 4 & 4 & 5 \\ 2 & 5 & & & \\ 3 & & & & \end{array}, \begin{array}{ccccc} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & & & \\ 3 & & & & \end{array} \right)$$

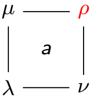
$$\left(\begin{array}{ccccc} 1 & 2 & 4 & 4 & 5 \\ 2 & 4 & & & \\ 3 & 5 & & & \end{array}, \begin{array}{ccccc} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & & & \\ 3 & 4 & & & \end{array} \right) \left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 4 & & \\ 3 & 5 & & & \end{array}, \begin{array}{ccccc} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 4 & & \\ 3 & 4 & & & \end{array} \right) = (P, Q)$$

Growth diagrams



$\lambda \prec \nu, \lambda \prec \mu$ ($\alpha \prec \beta$: $\beta_i \geq \alpha_i \geq \beta_{i+1}$ $\forall i$)
$\rho_1 := \max(\nu_1, \mu_1) + a$;
For $i = 2 \to \min(\ell(\mu), \ell(\nu)) + 1$:
$\rho_i = \max(\mu_i, \nu_i) + \min(\mu_{i-1}, \nu_{i-1}) - \lambda_{i-1}$;
$|\rho| = |\mu| + |\nu| - |\lambda| + a$

Growth diagrams



$\lambda \prec \nu, \lambda \prec \mu$ ($\alpha \prec \beta$: $\beta_i \geq \alpha_i \geq \beta_{i+1}\ \forall i$)
$\rho_1 := \max(\nu_1, \mu_1) + a$;
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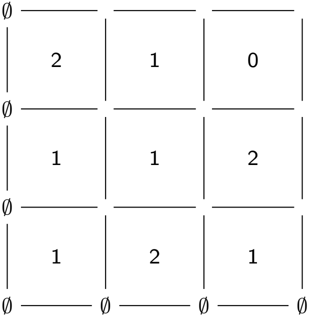
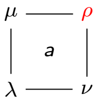
$|\rho| = |\mu| + |\nu| - |\lambda| + a$



Growth diagrams



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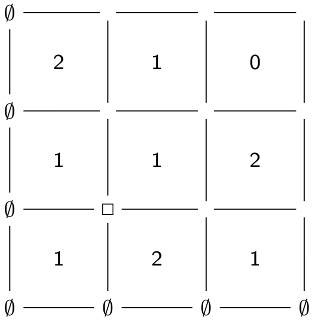
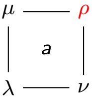


Growth diagrams



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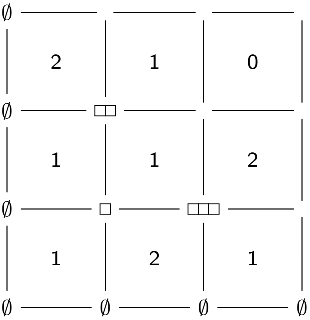
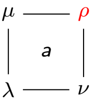


Growth diagrams



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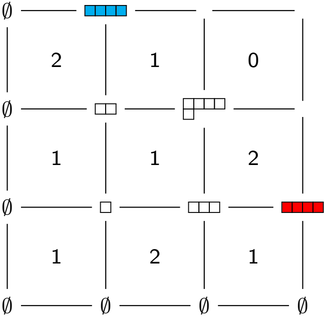
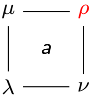


Growth diagrams



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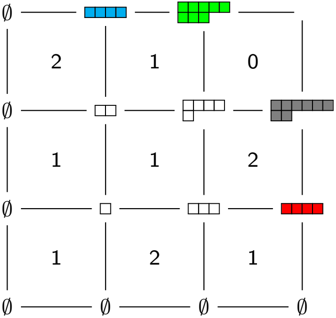
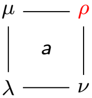


Growth diagrams



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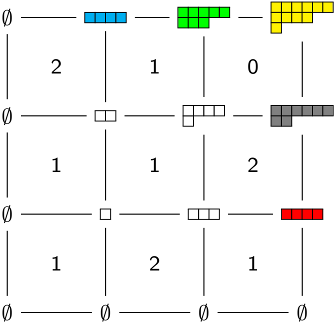
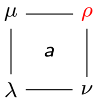


Growth diagrams



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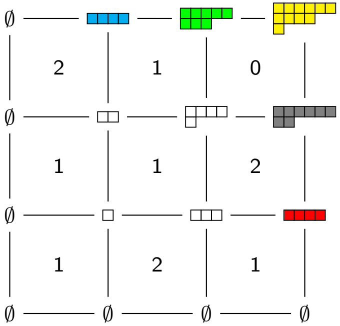
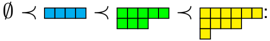
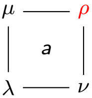


Insertion tableau: P

Top row:



Growth diagrams



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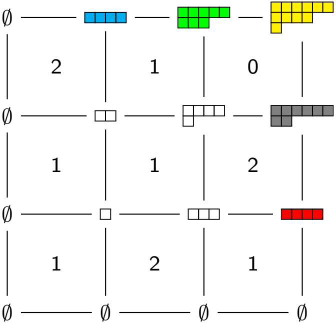
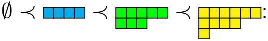
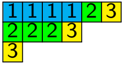


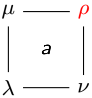
Insertion tableau: P

Top row:



$P =$ 

Growth diagrams



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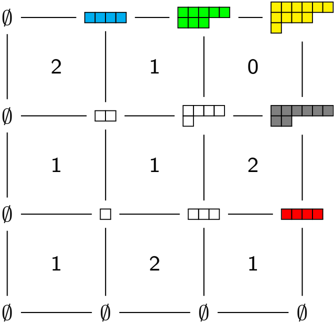
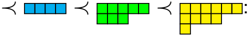


Insertion tableau: P

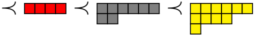
Top row:

$\emptyset \prec$ :

$P =$

1	1	1	1	2	3
2	2	2	3		
3					

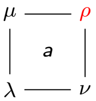
Recording tableau: Q

$\emptyset \prec$ :

$Q =$

1	1	1	1	2	2
2	2	2	3		
3					

Growth diagrams



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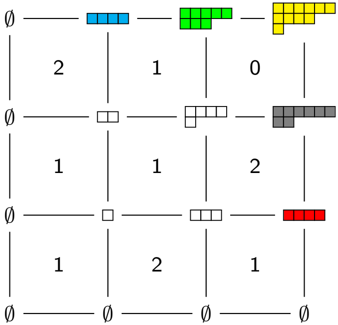
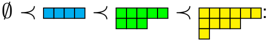
$\rho_1 := \max(\nu_1, \mu_1) + a$;
For $i = 2 \to \min(\ell(\mu), \ell(\nu)) + 1$:
$\rho_i = \max(\mu_i, \nu_i) + \min(\mu_{i-1}, \nu_{i-1}) - \lambda_{i-1}$;
$|\rho| = |\mu| + |\nu| - |\lambda| + a$



Insertion tableau: P

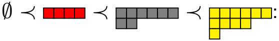
Top row:



$P =$

1	1	1	1	2	3
2	2	2	3		
3					

Recording tableau: Q



$Q =$

1	1	1	1	2	2
2	2	2	3		
3					

Corollary: Symmetry of the RSK:
$RSK(A) = (P, Q) \iff RSK(A^T) = (Q, P)$

Corollary: Cauchy identity

$$\prod_{i,j=1}^{n} \frac{1}{1-x_iy_j} = \sum_{A\in\mathbb{N}_0^{n\times n}} \prod_{i,j} (x_iy_j)^{A_{i,j}} = \sum_{A\in\mathbb{N}_0^{n\times n}} x^{row(A)}y^{col(A)}$$

$$RSK \; ||$$

$$\sum_{\lambda} s_\lambda(x)s_\lambda(y) = \sum_{\lambda} \sum_{P,Q,sh(P)=sh(Q)=\lambda} x^{type(P)}y^{type(Q)}$$

Corollary: Cauchy identity

$$\prod_{i,j=1}^{n} \frac{1}{1-x_iy_j} = \sum_{A\in\mathbb{N}_0^{n\times n}} \prod_{i,j} (x_iy_j)^{A_{i,j}} = \sum_{A\in\mathbb{N}_0^{n\times n}} x^{row(A)}y^{col(A)}$$

$$RSK \; ||$$

$$\sum_{\lambda} s_\lambda(x)s_\lambda(y) = \sum_{\lambda} \sum_{P,Q,sh(P)=sh(Q)=\lambda} x^{type(P)}y^{type(Q)}$$

Growth diagrams:
skew Cauchy:

$$\sum_{\rho} s_{\rho/\mu}(x)s_{\rho/\nu}(y) = \prod \frac{1}{1-x_iy_j}\left(\sum_{\lambda} s_{\mu/\lambda}(y)s_{\nu/\lambda}(x)\right)$$

Corollary: Cauchy identity

$$\prod_{i,j=1}^{n} \frac{1}{1-x_iy_j} = \sum_{A\in\mathbb{N}_0^{n\times n}} \prod_{i,j} (x_iy_j)^{A_{i,j}} = \sum_{A\in\mathbb{N}_0^{n\times n}} x^{row(A)}y^{col(A)}$$

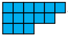
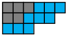
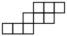
$$RSK \;||$$

$$\sum_{\lambda} s_\lambda(x)s_\lambda(y) = \sum_{\lambda} \sum_{P,Q,sh(P)=sh(Q)=\lambda} x^{type(P)}y^{type(Q)}$$

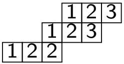
Growth diagrams:
skew Cauchy:

$$\sum_{\rho} s_{\rho/\mu}(x)s_{\rho/\nu}(y) = \prod \frac{1}{1-x_iy_j}\left(\sum_{\lambda} s_{\mu/\lambda}(y)s_{\nu/\lambda}(x)\right)$$

Skew Schur functions:

$\alpha =$ , $\beta =$  $\Longrightarrow \alpha/\beta =$  $=$ 

$$s_{\alpha/\beta}(x) = \sum_{T\in SSYT(\alpha/\beta)} x^{type(T)} = \quad x_1^3x_2^4x_3^2 \quad + \cdots$$



Greene's theorem

Let $w = w_1 \dots w_N$ be a word.
Increasing subsequence of w:
$w_{i_1} \leq w_{i_2} \leq \cdots \leq w_{i_k}$ for $i_1 < i_2 < \cdots < i_k$

[1] Connection: Last Passage Percolation

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E.g. $w = 637548192$, is longest, so $is(w) = 4$.

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Theorem (Greene)

If $rsk(w) = (P, Q)$ *and* $sh(P) = sh(Q) = \lambda$, *then* $is_j(w) = \lambda_1 + \dots + \lambda_j$.

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Theorem (Greene)

If $rsk(w) = (P, Q)$ and $sh(P) = sh(Q) = \lambda$, then $is_j(w) = \lambda_1 + \cdots + \lambda_j$.

$$rsk(236145) = \left(\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2&6 \\ \cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 4&5 \\ \cline{1-2}\end{array} \right)$$

$236145 \to is_1(w) = 4 \quad \to \quad \lambda_1 = 4$
$236145 \to is_2(w) = 3 + 3 \quad \to \quad \lambda_1 + \lambda_2 = 6$

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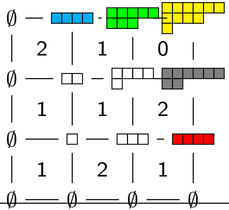
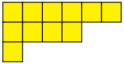


shape = 

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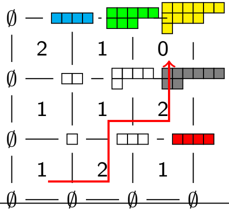
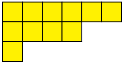


shape = 

$\lambda_1 = 1 + 2 + 1 + 2 + 0 = 6$

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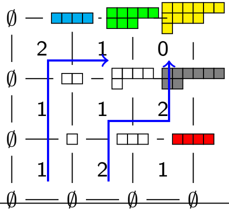
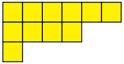


shape = 

$\lambda_1 = 1 + 2 + 1 + 2 + 0 = 6$

$\lambda_2 + \lambda_1 = (2 + 1 + 2 + 0) + (1 + 1 + 2 + 1) = 10$

[1] Connection: Last Passage Percolation

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Theorem (Greene)

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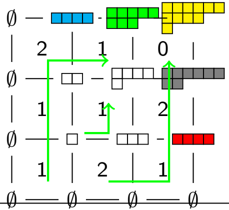
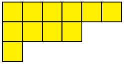


shape = 

$\lambda_1 = 1 + 2 + 1 + 2 + 0 = 6$

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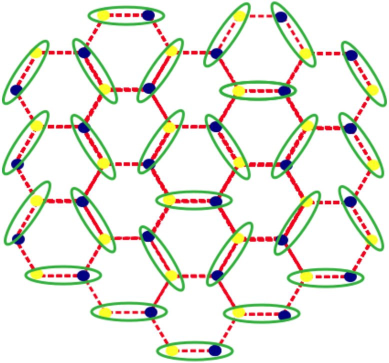
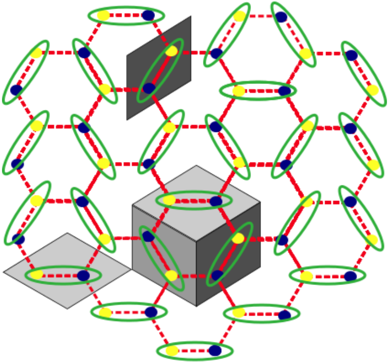
$\lambda_1 + \lambda_2 + \lambda_3 = (2 + 1 + 2 + 0) + (1 + 1 + 2 + 1) + (1)$

[1] Connection: Last Passage Percolation

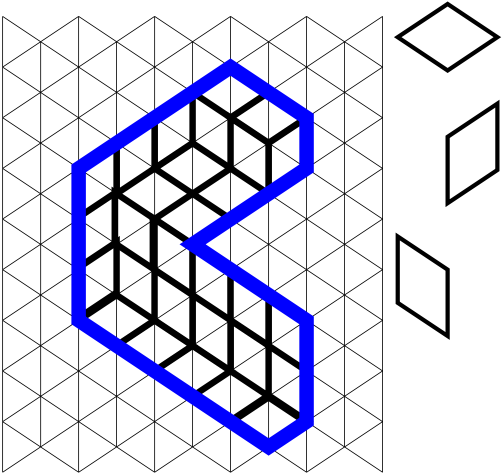
Dimer models and lattice paths



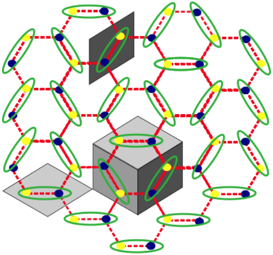
Dimer models and lattice paths



Dimer models and lattice paths



Dimer models and lattice paths



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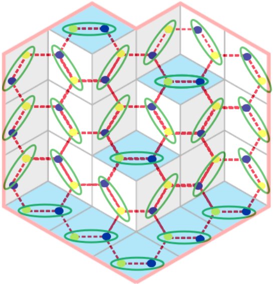

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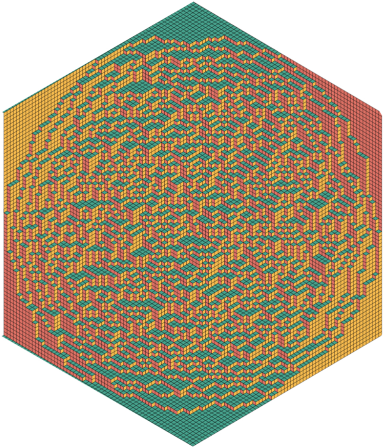

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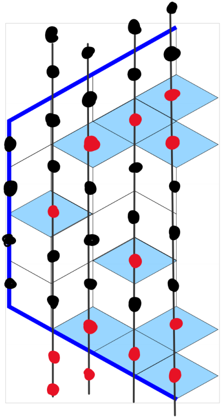
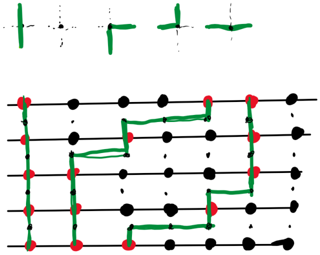
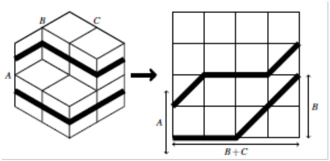
Dimer models and lattice paths



5 vertex model
<-> non-intersecting lattice paths

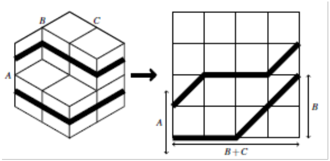


Dimer models and lattice paths



ⒸVadim Gorin, Lectures on Random Tilings

Dimer models and lattice paths



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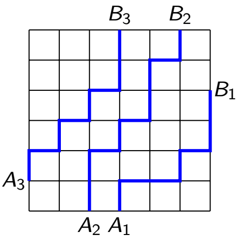


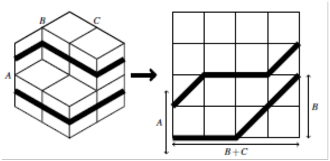
Non-Intersecting Lattice Paths (NILP):

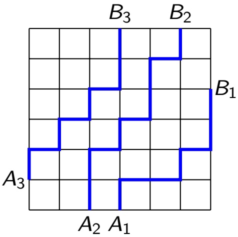
$(P_1, P_2, \ldots)$
$P_1 : A_1 \to B_1;\ P_2 : A_2 \to B_2;\ \ldots$

Theorem[Karlin–McGregor–Lindström–Gessel–Viennot]
(Number of) Nonintersecting Lattice Paths:

$$NILP(A_i \to B_i; i = 1..\ell) = \det[(A_i \to B_j)]_{i,j=1}^{\ell}$$

Dimer models and lattice paths





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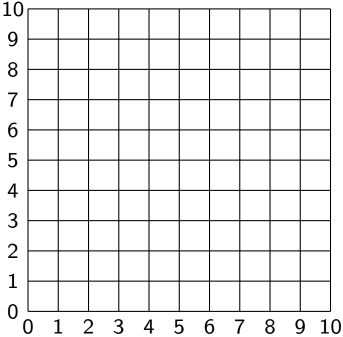
(Number of) Nonintersecting Lattice Paths:

$$NILP(A_i \to B_i; i = 1..\ell) = \det[(A_i \to B_j)]_{i,j=1}^{\ell}$$

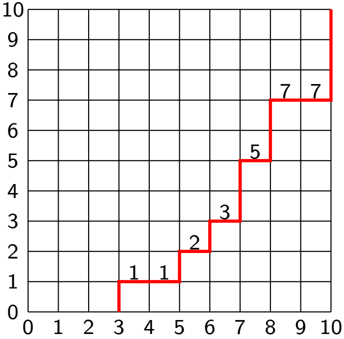
Proof: Sign reversing involution on intersecting pairs
$(A_{i_1} \to B_{j_1}, A_{i_2} \to B_{j_2}) \leftrightarrow (A_{i_1} \to B_{j_2}, A_{i_2} \to B_{j_1})$

Non-intersecting lattice paths



1	1	2	3	5	7	7
2	3	6	6	8		
4	4	7	9			
5	7	9				

Non-intersecting lattice paths



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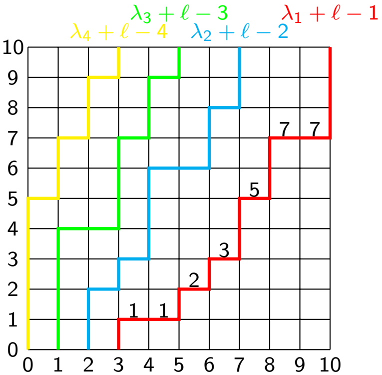

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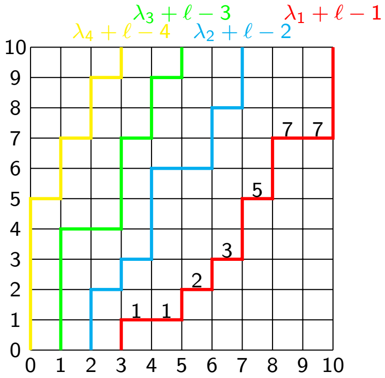


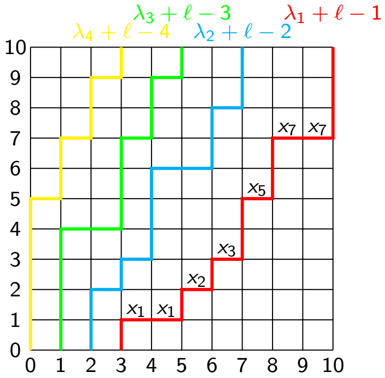
$NILP((\ell-i,1)\to(\lambda_i+\ell-i,N)\ ,i=1\to\ell)$

1	1	2	3	5	7	7
2	3	6	6	8		
4	4	7	9			
5	7	9				

$\ell := \ell(\lambda)$

$SSYT(\lambda;N)$

Non-intersecting lattice paths



$NILP((\ell - i, 1) \to (\lambda_i + \ell - i, N)\ , i = 1 \to \ell)$

1	1	2	3	5	7	7
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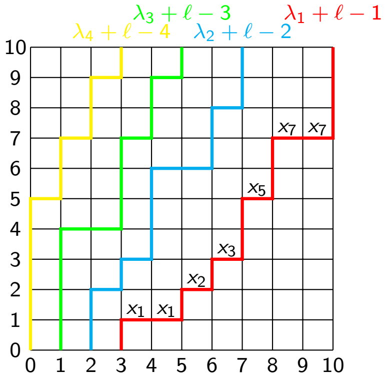
$SSYT(\lambda; N)$

Weighting

$$s_\lambda = \sum_{T \in SSYT(\lambda, N)} x^{type(T)}$$

Non-intersecting lattice paths



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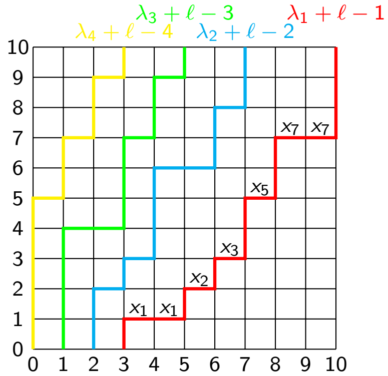
$$W(P : (a,b) \to (c,d)) = \prod_{(i,j)-(i+1,j) \in Path} x_j$$

$$\sum_{P:(a,b)\to(c,d)} W(P) = \sum_{b \le j_1 \le \cdots j_{c-a} \le d} x_{j_1} \cdots x_{j_{c-a}} = h_{c-a}(x_b, \ldots, x_d)$$

Non-intersecting lattice paths



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1	1	2	3	5	7	7
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$SSYT(\lambda; N)$

Theorem[KMLGV] Nonintersecting Lattice Paths:

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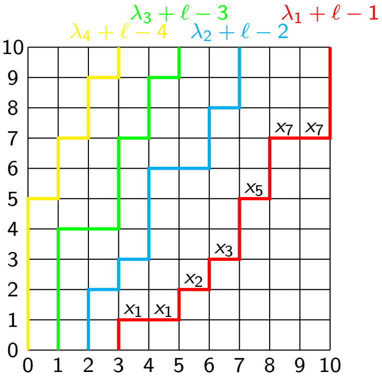
Non-intersecting lattice paths



$NILP((\ell - i, 1) \to (\lambda_i + \ell - i, N)\ , i = 1 \to \ell)$

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$$A_i = (\ell - i, 1), B_i = (\lambda + \ell - i, N)$$

Jacobi-Trudi identity:

$$s_\lambda(x) = \sum_{P_1,\ldots,P_\ell : NILP(\mathbf{A}\to\mathbf{B})} \prod_i W(P_i) = \det[\sum_{P:A_i\to B_j} W(P)]_{i,j=1}^{\ell} = \det[h_{\lambda_i - i + j}]_{i,j=1}^{\ell}$$

Thank you