An introduction to the Gaussian deformation

GOALS 2024 - Expository talk

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The intention of these notes is to serve as a detailed extension to the expository talks for GOALS 2024. The talks themselves will be more brief, focusing on Sections 4 and 5 and culminating in the role of the Gaussian deformation in the proof of Theorem 6.8. The material is borrowed from various sources as [AP; CS11; Va10b], with no claim of originality (except for any mistakes).

1 Introduction

Throughout this notes G will always denote a countable discrete group. Recall that we have a favourite representation, the **left regular representation**:

$$\lambda : G \to \mathscr{U}(\ell^2(G)) \text{ where } \lambda_g(\delta_h) = \delta_{gh} \text{ for any } g, h \in G.$$

For convenience (which later will become apparent) we write $u_g = \lambda_g$ and call these the **canonical group unitaries**. Then, the **group von Neumann algebra** is the von Neumann algebra generated by these unitaries $L(G) := \text{span}\{u_g : g \in G\}'' \subseteq \mathbb{B}(\ell^2(G))$.

The assignment $G \mapsto L(G)$ of a von Neumann algebra to every group has some nice behaviour:

Proposition 1.1. *Let $\theta : G \to H$ be a group isomorphism, then there exists a $*$-isomorphism $\pi_\theta : L(G) \to L(H)$ extending $\pi_\theta(u_g) = v_{\theta(g)}$ (where $\{u_g : g \in G\}$,$\{v_h : h \in H\}$ are the respective canonical group unitaries).*

Proof. We start by defining a unitary $W_\theta : \ell^2(G) \to \ell^2(H)$ by extending $W_\theta(\delta_g) = \delta_{\theta(g)}$ for any $g \in G$. Notice that $W_\theta^* = W_{\theta^{-1}}$ and for any $g \in G$, $h \in H$ we have $W_\theta u_g W_\theta^* \delta_h = \delta_{\theta(g)h} = v_{\theta(g)}\delta_h$. Therefore $W_\theta u_g W_\theta^* = v_{\theta(g)}$ for any $g \in G$ and $\pi_\theta = \text{Ad}_{W_\theta}$ gives the desired $*$-isomorphism. □

Proposition 1.2. *$L(G \times H) \cong L(G)\bar{\otimes}L(H)$ for any two groups G, H.*

Proof. First notice we have a unitary $W : \ell^2(G \times H) \to \ell^2(G) \otimes \ell^2(H)$ extending $W(\delta_{(g,h)}) = \delta_g \otimes \delta_h$ for any $g \in G$, $h \in H$. Furthermore, conjugating the respective canonical group unitaries we get $Wu_{(g,h)}W^* = u_g \otimes v_h$ for any $g \in G$, $h \in H$. Hence, $WL(G \times H)W^* = L(G)\bar{\otimes}L(H)$. □

Question: Is it true that $L(G) \cong L(H)$ implies $G \cong H$?
Answer: No! For instance $L(G) \cong L^\infty([0, 1])$ (with Lebesgue measure) for any infinite abelian group G, and $L(G) \cong \mathcal{R}$ (the hyperfinite II_1 factor) for any infinite i.c.c. amenable group.

Question: Are there cases where restrictions on G, H or the isomorphism $L(G) \cong L(H)$ will guarantee $G \cong H$?
Answer: Yes! First examples of ***W*-superrigid*** groups in [IPV30], (i.e. groups G such that $L(G) \cong L(H)$, for arbitrary H, implies $G \cong H$).

Many more questions: If not isomorphism class of the group, are there other properties from the group that can be recovered from the group von Neumann algebra?
Many answers: (we can talk about some of these later ...)

Aim of these notes: Finite rank free group factors $L(\mathbb{F}_n)$ are **prime** (i.e. any tensor decomposition $L(\mathbb{F}_n) \cong \mathcal{M}\bar{\otimes}\mathcal{N}$ implies one of $\mathcal{M}$ or $\mathcal{N}$ is finite dimensional).

2 Revisiting the group von Neumann algebra

The inclusion $L(G) \subseteq \mathbb{B}(\ell^2(G))$ has a lot of structure. Recall we had a description of $L(G)$ as the algebra of left convolvers of $\ell^2(G)$ (so with our above notation we have $u_g = L(\delta_g)$). Moreover, the commutant $L(G)' = R(G)$ corresponds to the algebra of right convolvers and is generated by the **right regular representation**:

$$\rho : G \to \mathscr{U}(\ell^2(G)) \quad \text{where} \quad \rho_g(\delta_h) = \delta_{hg^{-1}} \quad \text{for any } g, h \in G.$$

Hence $R(G) = \text{span}\{\rho_g : g \in G\}'' \subseteq \mathbb{B}(\ell^2(G))$, but we could also think of the operators ρ_g in an opposite way. By definition, we have an embedding $\pi_l : L(G) \to \mathbb{B}(\ell^2(G))$, but we can also extend ρ (linearly and to the SOT closure) to get a normal **$*$-anti-homomorphism** (i.e. a linear, $*$-preserving map that reverses the order of the product):

$$\pi_r : L(G) \to R(G) \subseteq \mathbb{B}(\ell^2(G)) \quad \text{where} \quad \pi_r(u_g) = \rho_{g^{-1}} \quad \text{for any } g, h \in G.$$

Remark 2.1. Just as we leave the left representation implicit and write $x\xi$ (or $x\cdot\xi$) instead of $\pi_l(x)\xi$, we do the same for the right representation and write ξx (or $\xi \cdot x$) instead of $\pi_r(x)\xi$. This notation turns out to be more convenient, as it respects associativity and multiplication order, namely the expression

$$ab\xi xy = a(b((\xi x)y)) = (\pi_l(a) \circ \pi_l(b) \circ \pi_r(y) \circ \pi_r(x))(\xi)$$

is independent of any chosen order of parenthesis, for all $\xi \in \ell^2(G)$ and $a, b, x, y \in L(G)$.

We call $\ell^2(G)$ an $L(G)$**-bimodule** as it carries normal $*$-(anti-)homomorphisms π_l and π_r that commute. Moreover, the vector δ_e is cyclic for both $L(G)$ and $R(G)$ and defines a normal tracial state $\tau(x) = \langle x\delta_e, \delta_e\rangle$ on $L(G)$. Therefore, the vector δ_e is separating for $L(G)$ (i.e. if $x \in L(G)$ and $x\delta_e = 0$, then $x = 0$) and hence τ is faithful on $L(G)$.

Remark 2.2. From the uniqueness of the GNS representation we know that $L^2(L(G), \tau) \cong \ell^2(G)$. More precisely, there is a unitary $W : L^2(L(G), \tau) \to \ell^2(G)$ uniquely determined by the following: $W\widehat{1} = \delta_e$ and $W\pi_\tau(x)W^* = x$ for any $x \in L(G)$. Going forward we make the identification $L^2(L(G), \tau) = \ell^2(G)$ and treat them indistinctly.

Since τ is faithful, the linear map $L(G) \to \widehat{L(G)} \subset \ell^2(G)$ given by $x \mapsto \widehat{x} := x\widehat{1}$ is injective and gives rise to the $\|\cdot\|_2$**-norm**: $\|x\|_2 := \|\widehat{x}\|_2 = \tau(x^*x)^{1/2}$ for any $x \in L(G)$. Since $\{\widehat{u_g} = \delta_g : g \in G\}$ is an orthonormal basis for $\ell^2(G)$, then for each $x \in L(G)$ there is a **Fourier expansion** defined by $x = \sum_{g\in G} x_g u_g$ with convergence in $\|\cdot\|_2$ (but not necessarily in any of the operator topologies). Notice that $x_g = \tau(xu_g^*)$ gives the **Fourier coefficients**.

Exercise 2.3. Prove that the Fourier expansion is well behaved under, sum, adjoint and product.

3 Tracial von Neumann algebras

We can generalize most of Section 2 to arbitrary tracial von Neumann algebras (except for the Fourier expansion, this will show up again in Section 4 where there is an underlying copy of $L(G)$). Recall that a **tracial von Neumann algebra** consists of a pair $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a von Neumann algebra and $\tau \in \mathcal{M}_*$ is a normal faithful tracial state on $\mathcal{M}$. Given that τ is normal and faithful, we identify $\mathcal{M}$ with its image under the GNS representation, so $\mathcal{M} \subseteq \mathbb{B}(L^2(\mathcal{M}))$ as a unital von Neumann subalgebra (notice we drop τ in the notation $L^2(\mathcal{M}, \tau)$ for convenience, and for the significance of this so-called **standard representation**).

We have a linear embedding $\mathcal{M} \hookrightarrow \widehat{\mathcal{M}} \subset L^2(\mathcal{M})$ giving the $\|\cdot\|_2$-norm for $\mathcal{M}$, together with a normal $*$-homomorphism $\pi_l : \mathcal{M} \to \mathbb{B}(L^2(\mathcal{M}))$ and a normal $*$-anti-homomorphism $\pi_r : \mathcal{M} \to \mathbb{B}(L^2(\mathcal{M}))$ given by $\pi_l(a)\pi_r(b)\widehat{x} = a \cdot \widehat{x} \cdot b = \widehat{axb}$. Notice that the definition of π_r makes sense (i.e. the operators can be extended from acting on $\widehat{\mathcal{M}}$ to all of $L^2(\mathcal{M})$) precisely because τ is tracial. Indeed, for any $b, x \in \mathcal{M}$ we have

$$\|xb\|_2^2 = \tau(b^*x^*xb) = \tau(xbb^*x^*) \leq \|bb^*\|\tau(xx^*) = \|b\|^2\tau(x^*x) = \|b\|^2\|x\|_2^2.$$

In an analogous way as the study of left and right convolvers on $\ell^2(G)$ allows to prove $L(G)' = R(G)$ (inside $\mathbb{B}(\ell^2(G))$), the study of multiplication by left and right bounded vectors of $L^2(\mathcal{M})$ allows to show $\mathcal{M}' = \pi_r(\mathcal{M}) \subseteq \mathbb{B}(L^2(\mathcal{M}))$. Moreover, there is an **anti-unitary** $J : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ (i.e. a conjugate-linear isometric isomorphism) extending $J\widehat{x} = \widehat{x^*}$ that satisfies $JxJ = \pi_r(x^*)$ and $J\mathcal{M}J = \mathcal{M}' = \pi_r(\mathcal{M})$. The anti-unitary J is called the **canonical** (or **Tomita**) **conjugation**.

The following Proposition 3.1 and its Corollary 3.2 are very useful properties of tracial von Neumann algebras.

Proposition 3.1. *Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra. Then the unit ball $(\mathcal{M})_1$ is complete with respect to the $\|\cdot\|_2$-norm. Moreover, the topology induced by $\|\cdot\|_2$-norm on $(\mathcal{M})_1$ is the restriction of the SOT topology.*

Proof. Suppose $(x_n)_n \subset (\mathcal{M})_1$ is a Cauchy sequence with respect to the $\|\cdot\|_2$-norm. Then for any $y \in \mathcal{M}$ we have

$$\|x_n\widehat{y} - x_m\widehat{y}\|_2 = \|\widehat{x_n y} - \widehat{x_m y}\|_2 \leq \|y\|_\infty \|x_n - x_m\|_2$$

implying that there is a $\|\cdot\|_2$-limit $x \cdot \widehat{y} = \lim_n x_n\widehat{y}$. The map $\widehat{y} \mapsto x \cdot \widehat{y}$ is linear and bounded, so it extends to $x \in \mathbb{B}(L^2(\mathcal{M}))$. Since the sequence $(x_n)_n$ is uniformly bounded in $\|\cdot\|_\infty$, we can prove $x_n \to x$ in SOT and consequently $x \in (\mathcal{M})_1$.
For the second assertion, notice we have just shown a sequence $(x_n)_n \subset (\mathcal{M})_1$ converging in $\|\cdot\|_2$ converges in SOT. The converse is immediate by applying the operators to the vector $\widehat{1}$. □

Corollary 3.2. *Let $(\mathcal{B}, \tau_\mathcal{B})$, $(\mathcal{M}, \tau_\mathcal{M})$ be tracial von Neumann algebras and $\mathcal{B}_0 \subset \mathcal{B}$ an SOT-dense $*$-subalgebra. If $\Psi_0 : \mathcal{B}_0 \to \mathcal{M}$ is a trace preserving $*$-homomorphism (i.e. $\tau_\mathcal{M} \circ \Psi_0 = \tau_\mathcal{B}|_{\mathcal{B}_0}$), then Ψ_0 extends to a normal $*$-embedding (i.e. injective $*$-homomorphism) $\Psi : \mathcal{B} \to \mathcal{M}$.*

Suppose $\mathcal{B} \subseteq \mathcal{M}$ is a (unital) von Neumann subalgebra, we give $\mathcal{B}$ the trace $\tau|_\mathcal{B}$, so $(\mathcal{B}, \tau)$ is a tracial von Neumann algebra (we drop the $|_\mathcal{B}$ in front of τ for convenience). Since the embedding $\mathcal{B} \hookrightarrow \mathcal{M}$ preserves the trace, then the embedding $\widehat{\mathcal{B}} \hookrightarrow \widehat{\mathcal{M}}$ is isometric and we can identify $L^2(\mathcal{B})$ as the Hilbert subspace $\overline{\mathcal{B}\widehat{1}} \subseteq L^2(\mathcal{M})$. We let $e_\mathcal{B} : L^2(\mathcal{M}) \to L^2(\mathcal{B})$ be the orthogonal projection (sometimes called **Jones's projection**).

Proposition 3.3. *Given a (unital) von Neumann subalgebra $\mathcal{B}$ of a tracial von Neumann algebra $(\mathcal{M}, \tau)$ the projection $e_\mathcal{B}$ satisfies:*

1. *$e_{\mathcal{B}}b = be_{\mathcal{B}}$ for any $b \in \mathcal{B}$.*

2. *$Je_{\mathcal{B}} = e_{\mathcal{B}}J$, notice the Tomita conjugation $J_{\mathcal{B}}$ on $L^2(\mathcal{B})$ is precisely the restriction $J|_{L^2(\mathcal{B})}$ of the Tomita conjugation on $L^2(\mathcal{M})$.*

3. *$(JbJ)(e_{\mathcal{B}}xe_{\mathcal{B}}) = (e_{\mathcal{B}}xe_{\mathcal{B}})(JbJ)$ for any $x \in \mathcal{M}$ and $b \in \mathcal{B}$.*

4. *$e_{\mathcal{B}}\mathcal{M}e_{\mathcal{B}} = \mathcal{B}e_{\mathcal{B}}$ and this induces a map $E_{\mathcal{B}} : \mathcal{M} \to \mathcal{B}$ uniquely determined by $e_{\mathcal{B}}xe_{\mathcal{B}} = E_{\mathcal{B}}(x)e_{\mathcal{B}}$ for any $x \in \mathcal{M}$.*

5. *The map $E_{\mathcal{B}} : \mathcal{M} \to \mathcal{B}$ is unital, normal, faithful, completely positive, trace preserving and $\mathcal{B}$-linear ($\mathcal{B}$-linearity means $E_{\mathcal{B}}(axb) = aE_{\mathcal{B}}(x)b$ for any $a, b \in \mathcal{B}$ and $x \in \mathcal{M}$).*

6. *If $E : \mathcal{M} \to \mathcal{B}$ is any trace preserving conditional expectation, then $E = E_{\mathcal{B}}$.*

7. *$E_{\mathcal{B}}$ is faithful and normal.*

Items 5. and 6. make $E_{\mathcal{B}}$ the **(unique) trace preserving conditional expectation**.

The proof of the existence and uniqueness of the trace preserving conditional expectation is layed out in Proposition 3.3 and left as an exercise. (Hint: To prove 6 consider how $E : \widehat{\mathcal{M}} \to \widehat{\mathcal{B}}$ extends to the (unique) orthogonal projection).

Example 3.4. Let G be a countable discrete group and $H \leq G$ be a subgroup. Thus we have a trace preserving inclusion of von Neumann algebras $L(H) \subseteq L(G)$ and an inclusion of Hilbert spaces $\ell^2(H) \subseteq \ell^2(G)$. Using the Fourier expansion, we see that for any $x = \sum_{g \in G} x_g u_g \in L(G)$ we must have $E_{L(H)}(x) = \sum_{h \in H} x_h u_h$.

Remark 3.5. Notice that Corollary 3.2 implies any injective group homomorphism $\theta : H \to G$ induces a normal $*$-embedding $L(H) \to L(G)$. Moreover if $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra containing a countable group of unitaries $G \subset \mathscr{U}(\mathcal{M})$ such that $\tau(u) = \delta_{1,u}$ for all $u \in G$, then $L(G) \cong \overline{\text{span}\{u \in G\}}^{SOT} \subseteq \mathcal{M}$ (this is a reason to use the notation u_g, instead of λ_g, for canonical group unitaries, since they are determined by the trace and not by how they act on a particular Hilbert space).

4 Tracial crossed products

If $\varphi : \mathcal{M} \to \mathcal{M}$ is a $*$-automorphism such that $\tau \circ \varphi = \tau$, then $\widetilde{\varphi}(\widehat{x}) = \widehat{\varphi(x)}$ extends to a unitary operator $\widetilde{\varphi} \in \mathscr{U}(L^2(\mathcal{M}))$ (as $\|\varphi(x)\|_2 = \|x\|_2$ for any $x \in \mathcal{M}$). Moreover, we have $\widetilde{\varphi}x\widetilde{\varphi} = \varphi(x)$ since

$$\widetilde{\varphi}x\widetilde{\varphi}^*\widehat{y} = \widetilde{\varphi}(x\varphi^{-1}(y)\widehat{1}) = \varphi(x\varphi^{-1}(y))\widehat{1} = \varphi(x)\widehat{y} \quad \text{for any } x, y \in \mathcal{M}.$$

We denote the group of all trace preserving $*$-automorphisms by $\text{Aut}(\mathcal{M}, \tau)$. We say G **acts by trace preserving $*$-automorphisms**, and denote it by $G \curvearrowright^{\sigma} (\mathcal{M}, \tau)$, if there is a group homomorphism $\sigma : G \to \text{Aut}(\mathcal{M}, \tau)$. Notice that in this situation we obtain a unitary representation $\widetilde{\sigma}$ of G such that $\widetilde{\sigma_g}x\widetilde{\sigma_g}^* = \sigma_g(x)$ for any $x \in \mathcal{M}$ and $g \in G$.

We want to construct a tracial von Neumann algebra $\mathcal{M} \rtimes^{\sigma} G$, called the **(tracial) crossed product of $\mathcal{M}$ by G**, that encodes the action $G \curvearrowright^{\sigma} (\mathcal{M}, \tau)$. It should contain a copy of both $\mathcal{M}$, and $L(G)$ in such a way that $u_g x u_g^* = \sigma_g(x)$ for any $x \in \mathcal{M}$ and $g \in G$. Let's start by defining a $*$-homomorphism $\Psi : \mathcal{M} \to \mathbb{B}(L^2(\mathcal{M}) \otimes \ell^2(G))$ and a unitary representation $\Lambda : G \to \mathscr{U}(L^2(\mathcal{M}) \otimes \ell^2(G))$ as follows:

$$\Psi(x) = x \otimes 1 \quad \text{for all } x \in \mathcal{M} \quad \text{and} \quad \Lambda_g = \widetilde{\sigma_g} \otimes \lambda_g \quad \text{for all } g \in G.$$

From the definition it follows that Ψ is injective and normal, so $\Psi(\mathcal{M}) \subseteq \mathbb{B}(L^2(\mathcal{M}) \otimes \ell^2(G))$ is a von Neumann subalgebra isomorphic to $\mathcal{M}$. For convenience we identify $\mathcal{M}$ and $\Psi(\mathcal{M})$ and write x instead of $\Psi(x)$. Further we will give $\mathcal{M} \rtimes G$ a normal faithful trace such that $\tau(\Lambda_g) = \delta_{e,g}$, so Remark 3.5 shows $L(G) \cong \text{span}\{\Lambda_g : g \in G\}'' \subseteq \mathbb{B}(L^2(\mathcal{M}) \otimes \ell^2(G))$ and hence we will write u_g instead of Λ_g (and call these again the **canonical group unitaries**).

With the above notations we have a $*$-algebra $\text{span}\{xu_g : x \in \mathcal{M}, g \in G\}$ that satisfies

$$xu_g yu_h = x\sigma_g(y)u_{gh} \quad \text{and} \quad (xu_g)^* = u_g^* x^* = \sigma_{g^{-1}}(x)u_{g^{-1}} \quad \text{for all } x \in \mathcal{M},\ g \in G.$$

We let $\mathcal{M} \rtimes^\sigma G := \text{span}\{xu_g : x \in \mathcal{M}, g \in G\}'' \subseteq \mathbb{B}(L^2(\mathcal{M}) \otimes \ell^2(G))$. Notice that the vector $\xi_1 = \widehat{1} \otimes \delta_e$ is cyclic for $\mathcal{M} \rtimes^\sigma G$ and the corresponding vector state ω_{ξ_1} is tracial on $\mathcal{M} \rtimes^\sigma G$. Indeed, it suffices to check the traciality on $\mathcal{M} \rtimes_{\text{alg}} G = \text{span}\{xu_g : x \in \mathcal{M}, g \in G\}$ which is an SOT-dense $*$-subalgebra, so we only need to check

$$\omega_{\xi_1}(xu_g yu_h) = \left\langle \widehat{x\sigma_g(y)} \otimes \delta_{gh}, \widehat{1} \otimes \delta_e \right\rangle = \tau(x\sigma_g(y))\delta_{gh,e} = \tau(y\sigma_{g^{-1}}(x))\delta_{hg,e} = \omega_{\xi_1}(yu_h xu_g)$$

for any $x, y \in \mathcal{M}$ and $g, h \in G$. As with the standard representation of a tracial von Neumann algebra, we also define a "right acting" algebra $R(\mathcal{M} \rtimes^\sigma G) \subseteq \mathbb{B}(L^2(\mathcal{M}) \otimes \ell^2(G))$ to be the von Neumann algebra generated by operators $\{R(xu_g) : x \in \mathcal{M}, g \in G\}$ where

$$R(xu_g)\xi \otimes \delta_h = \xi \otimes \delta_h \cdot (xu_g) = (\xi \cdot \sigma_h(x)) \otimes \delta_{hg} \quad \text{for all } \xi \in L^2(\mathcal{M}),\ h \in G.$$

We can prove that $R(\mathcal{M} \rtimes^\sigma G) \subseteq (\mathcal{M} \rtimes^\sigma G)'$ and $\widehat{1} \otimes \delta_e$ is cyclic for $R(\mathcal{M} \rtimes^\sigma G)$. Therefore, $\widehat{1} \otimes \delta_e$ is separating for $\mathcal{M} \rtimes^\sigma G$ and we obtain a normal faithful trace $\tau = \omega_{\xi_1}$ on $\mathcal{M} \rtimes^\sigma G$.

We are now with $\mathcal{M} \rtimes^\sigma G$ in the setting of tracial von Neumann algebras, and so we can make use of the tools ($\|\cdot\|_2$, standard representation, Tomita conjugation, trace preserving conditional expectations) of Section 3. Notice that $L^2(\mathcal{M} \rtimes^\sigma G) = L^2(M) \otimes \ell^2(G)$ with the left and right multiplication as described above. Moreover, we do have a **Fourier expansion** defined on $\mathcal{M} \rtimes^\sigma G$.

Proposition 4.1. *For every $x \in \mathcal{M} \rtimes^\sigma G$ the series $x = \sum_{g \in G} x_g u_g$ given by $x_g = E_{\mathcal{M}}(xu_g^*)$ converges in $\|\cdot\|_2$-norm and uniquely determines x.*

Proof. The uniqueness follows from the equality $\widehat{x} = \sum_{g \in G} x_g \widehat{u_g}$ in $L^2(\mathcal{M} \rtimes^\sigma G)$ and the faithfulness of τ. Recall $e_{\mathcal{M}} : L^2(\mathcal{M} \rtimes^\sigma G) \to L^2(\mathcal{M})$ is the orthogonal projection, where $\mathcal{M} = \mathcal{M}1 = \mathcal{M}u_e$ and so we identify $L^2(\mathcal{M}) = L^2(\mathcal{M}) \otimes \delta_e \subseteq L^2(\mathcal{M} \rtimes^\sigma G)$. Since $\{\delta_g : g \in G\}$ is an orthonormal basis for $\ell^2(G)$, we obtain a direct sum decomposition $L^2(\mathcal{M} \rtimes^\sigma G) = \oplus_{g \in G} L^2(\mathcal{M}) \otimes \delta_g$ and thus a unique expansion $\widehat{x} = \oplus_{g \in G} \xi_g \otimes \delta_g$ for some $\xi_g \in L^2(\mathcal{M})$. Finally, by applying Proposition 3.3 we obtain $\xi_g \otimes \delta_g = e_{\mathcal{M}}(\widehat{xu_g^*}) = E_{\mathcal{M}}(xu_g^*)\widehat{1} \otimes \delta_e$. □

5 The Gaussian construction

We finally turn to the idea of a deformation. Roughly speaking, we want to use some group data to construct a tracial von Neumann algebra $(\widetilde{\mathcal{M}}, \tau)$ containing $\mathcal{M} := L(G)$ in a trace preserving manner (i.e. $\tau|_{\mathcal{M}} = \tau_{\mathcal{M}}$), and construct a one parameter group of automorphisms $\alpha : \mathbb{R} \to \text{Aut}(\widetilde{\mathcal{M}})$ from which we can extract information of $L(G)$ (and G). There are multiple variations and generalizations of this idea, but we will focus on the following construction.

Let $\mathcal{H}_{\mathbb{R}}$ be a real (separable) Hilbert space, we define a $*$-algebra $\mathcal{D}_0 := \text{span}\{w(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}$ where product and adjoint are given by $w(\xi)w(\eta) = w(\xi + \eta)$ and $w(\xi)^* = w(-\xi)$. Then, $\mathcal{D}_0$ is abelian, and we equip it with a (tracial, by default) linear functional $\tau(w(\xi)) = e^{-\|\xi\|^2}$ for any $\xi \in \mathcal{H}_{\mathbb{R}}$. We then apply the GNS construction to the pair $(\mathcal{D}_0, \tau)$ (careful, the Hilbert space construction makes sense, for any $*$-algebra and positive functional, but it need not give a bounded representation! This case works fine, though). We let $\mathcal{D}$ be the (tracial) von Neumann algebra generated by the image under the GNS representation of $\mathcal{D}_0$.

Exercise 5.1. Show that τ is positive (i.e. $\tau(x^*x) \geq 0$ for all $x \in \mathcal{D}_0$) and faithful.

Exercise 5.2. Show that we can apply the GNS construction to the pair $(\mathcal{D}_0, \tau)$. Namely, show that each $w(\xi)$ induces a bounded operator (in fact, a unitary) on $L^2(\mathcal{D}_0, \tau)$.

Now suppose $\pi : G \to \mathcal{O}(\mathcal{H}_\mathbb{R})$ is an orthogonal representation. We obtain a trace preserving action by $*$-automorphisms $G \curvearrowright^{\sigma^\pi} \mathcal{D}_0$ given by $\sigma^\pi_g(w(\xi)) = w(\pi_g(\xi))$ which then extends to a trace preserving action $G \curvearrowright^{\sigma^\pi} (\mathcal{D}, \tau)$. At this point we can define $\widetilde{\mathcal{M}} = \mathcal{D} \rtimes^{\sigma^\pi} G$ and as discussed in Section 4 we have trace preserving embeddings $\mathcal{D}, L(G) \hookrightarrow \mathcal{D} \rtimes G$. It remains to find a one parameter group of automorphisms.

Given an orthogonal representation $\pi : G \to \mathcal{O}(\mathcal{H}_\mathbb{R})$, we say that a map $b : G \to \mathcal{H}_\mathbb{R}$ is a **1-cocycle** for π if it satisfies $b(gh) = \pi_g b(h) + b(g)$ for any $g, h \in G$. Notice that a 1-cocycle always satisfies $b(e) = 0$. For each $t \in \mathbb{R}$ we will define an automorphism $\alpha_t \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ so it satisfies that $\alpha_t(w(\xi)u_g) = w(\xi + tb(g))u_g$ for all $\xi \in \mathcal{H}_\mathbb{R}$ and $g \in G$. Let $V_t : L^2(\mathcal{D}) \otimes \ell^2(G) \to L^2(\mathcal{D}) \otimes \ell^2(G)$ be given by $V_t(\zeta \otimes \delta_h) = w(tb(h))\zeta \otimes \delta_h$, for any $\zeta \in L^2(\mathcal{D})$ and $h \in G$.

Exercise 5.3. Prove that $V_t \in \mathscr{U}(L^2(\mathcal{D}) \otimes \ell^2(G))$ and it satisfies $V_t w(\xi) V_t^* = w(\xi)$, $V_t u_g V_t^* = w(tb(g))u_g$ for all $t \in \mathbb{R}$, $\xi \in \mathcal{H}_\mathbb{R}$ and $g \in G$. (Further, notice that for all $\xi \in \mathcal{H}_\mathbb{R}$ and $g \in G$ we have $\|V_t w(\xi) V_t^* \widehat{1}\|_2 = \|w(\xi)\|_2$ and $\|V_t u_g V_t^* \widehat{1}\|_2 = \|u_g\|_2$.)

Exercise 5.4. Show that Ad_{V_t} restricts to an automorphism α_t of $\mathcal{D} \rtimes G$ (the results of the above exercise can help) and that the group homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{D} \rtimes G)$ is continuous in the point-$\|\cdot\|_2$ topology (i.e. for any $x \in \mathcal{D} \rtimes G$ the map $t \mapsto \alpha_t(x)$ is continuous with respect to the $\|\cdot\|_2$-norm).

The following is a very useful inequality for a deformation to satisfy. We will not need it in the rest of these notes, though it will be mentioned again in the next section.

Proposition 5.5 (Popa's transversality inequality)**.** *For any $x \in L(G)$ and $t \in \mathbb{R}$ we have*

$$\|\alpha_t(x) - E_{L(G)}(\alpha_t(x))\|_2 \leq \|x - \alpha_t(x)\|_2 \leq \sqrt{2}\|\alpha_t(x) - E_{L(G)}(\alpha_t(x))\|_2 \,.$$

6 Solidity of free group factors

As mentioned in the introduction, our aim will be to sketch the proof that the finite rank free group factors $L(\mathbb{F}_n)$ are prime. The existence of prime II_1 factors was a long standing problem, first tackled in the non-separable case by [Po83]. The existence of a prime separable II_1 factor was first shown by [Ge96], who used Voiculescu's free probability theory to show that all free group factors are prime. Later on, [Oza03] generalized Ge's primeness result by tackling an open problem from the same paper [Ge96]. In [Oza03], it is proved that for any hyperbolic group (e.g. $\mathbb{F}_n$ for $n \in \mathbb{N}$) the group von Neumann algebra $L(G)$ is **solid**, i.e. if $\mathcal{A} \subseteq L(G)$ is a (unital) diffuse von Neumann subalgebra, then the relative commutant $\mathcal{A}' \cap L(G)$ is amenable.

Corollary 6.1 (Primeness from solidity)**.** *If G is an i.c.c. non-amenable hyperbolic group, then $L(G)$ is prime.*

Proof. Suppose we have a tensor product decomposition $L(G) = \mathcal{P}\bar{\otimes}\mathcal{Q}$ for some von Neumann subfactors $\mathcal{P}, \mathcal{Q} \subseteq L(G)$ (both $\mathcal{P}$ and $\mathcal{Q}$ must be factors so their tensor product $L(G)$ is again a factor). Since $L(G)$ is a II_1 factor, then at least one of $\mathcal{P}$ or $\mathcal{Q}$ must be diffuse (otherwise $L(G)$ would contain a minimal projection). Without loss of generality, let's say $\mathcal{P}$ is diffuse. Since $L(G)$ is solid, then $\mathcal{Q} \subseteq \mathcal{P}' \cap L(G)$ is amenable. Now, we have that $\mathcal{Q}$ is a separable tracial amenable factor. Assume for a contradiction that $\mathcal{Q}$ is not finite dimensional. As an infinite dimensional tracial factor we would get that $\mathcal{Q}$ is diffuse. However, a new application of solidity would yield that $\mathcal{P}$ is also amenable, thus making $L(G) = \mathcal{P}\bar{\otimes}\mathcal{Q}$ amenable in contradiction to the non-amenability of the group G. □

Ozawa's approach is based on C^*-algebra theory. Later on, [Po06] provided a new proof in the free group case based on his deformation/rigidity theory and [Pe06] provided a new approach based on closable derivations. The work of many others led to generalizations, strengthenings and corollaries of Ozawa's solidity result. Here we will take an approach based on [CS11] (but, in the restricted setting presented here, it is fundamentally [Va10b, Theorem 3.6]). In order to fit the main proof in this lecture, we will need to state a few results without a proof.

Definition 6.2. Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra. Recall that an **$\mathcal{M}$-bimodule** is a Hilbert space $\mathcal{H}$ together with a pair of a normal $*$-homomorphism $\pi_l : \mathcal{M} \to \mathbb{B}(\mathcal{H})$ and a normal $*$-anti-homomorphism $\pi_r : \mathcal{M} \to \mathbb{B}(\mathcal{H})$ such that the images of $\pi_l(\mathcal{M})$ and $\pi_r(\mathcal{M})$ commute. An $\mathcal{M}$-bimodule $\mathcal{H}$ induces a norm on the algebraic tensor product $\mathcal{M} \otimes_{\text{alg}} \mathcal{M}^{\text{op}}$ given by

$$\|\sum_{i=1}^{n} x_i \otimes y_i^{\text{op}}\|_{\infty,\mathcal{H}} = \|\sum_{i+1}^{n} \pi_l(x_i)\pi_r(y_i)\|_{\mathbb{B}(\mathcal{H})}$$

for any $\sum_{i+1}^{n} x_i \otimes y_i^{\text{op}} \in \mathcal{M} \otimes_{\text{alg}} \mathcal{M}^{\text{op}} = \text{span}\{x \otimes y^{\text{op}} : x, y \in \mathcal{M}\}$. Given two $\mathcal{M}$-bimodules $\mathcal{H}$ and $\mathcal{K}$ we say $\mathcal{H}$ is **weakly contained** in $\mathcal{K}$ (and denote it by $\mathcal{H} \prec \mathcal{K}$) if $\|T\|_{\infty,\mathcal{H}} \leq \|T\|_{\infty,\mathcal{K}}$ for all $T \in \mathcal{M} \otimes_{\text{alg}} \mathcal{M}^{\text{op}}$. Notice that weak containment is a transitive relation.

Definition 6.3. The following are the two most common $\mathcal{M}$-bimodules:

- $L^2(\mathcal{M})$, called the standard or **trivial bimodule** (with the left and right multiplication as discussed in Section 3).
- $L^2(\mathcal{M}) \otimes L^2(\mathcal{M})$, called the **coarse bimodule**. Here left and right multiplication act on each of the tensor factors separately, i.e. $x \cdot (\xi \otimes \eta) \cdot y = (x \cdot \xi) \otimes (\eta \cdot y)$.

Proposition 6.4 (Theorem 13.4.1 in[AP])**.** *A tracial von Neumann algebra $\mathcal{M}$ is amenable if and only if the trivial bimodule is weakly contained in the coarse bimodule, i.e. $L^2(\mathcal{M}) \prec L^2(\mathcal{M})\otimes L^2(\mathcal{M})$.*

Definition 6.5. An orthogonal representation $\pi : G \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ is called **weakly-ℓ^2** if it can be identified with a subrepresentation of $\oplus_{\mathbb{N}}\lambda_{\mathbb{R}}$ (the infinite direct sum of copies of the (real) left regular representation). This definition comes from the notion of weak containment of representations.

The following proposition can be proved using Problem 6 (at the end of these notes) and noticing an $L(G)$-bimodule $\mathcal{H}$ is completely determined by the left and right G action on $\mathcal{H}$ induced by left and right multiplication by u_g.

Proposition 6.6. *Let $\pi : G \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ be an orthogonal representation and $\mathcal{D} \rtimes G$ be the algebra of the Gaussian deformation as in Section 5. If π is weakly-ℓ^2 (i.e. π is weakly contained in $\lambda_{\mathbb{R}}$), then the $L(G)$-bimodule $L^2(\mathcal{D} \rtimes G) \ominus L^2(L(G)) := (1 - e_{L(G)})L^2(\mathcal{D} \rtimes G)$ (i.e. the orthogonal complement of $L^2(L(G))$ inside $L^2(\mathcal{D} \rtimes G)$) is weakly contained in the coarse $L(G)$-bimodule.*

Proposition 6.7 (Corollary to Haagerup's amenability criterion [Ha83, Lemma 2.2])**.** *Suppose $(\mathcal{M},\tau)$ is a tracial von Neumann algebra and $\mathcal{N}\subseteq\mathcal{M}$ is any (unital) von Neumann subalgebra. Then, $\mathcal{N}$ is amenable if and only if for any finite set of unitaries $F\Subset\mathscr{U}(\mathcal{N})$ and any non-zero central projection $p\in\mathcal{Z}(\mathcal{N})$ we have $\|\sum_{u\in F}\pi_l(pu)\pi_r(pu^*)\|_{\infty,L^2(\mathcal{M})\otimes L^2(\mathcal{M})}=|F|$.*

We are now almost ready to put all the ingredients together and prove that free group factors are prime. In fact, we can prove solidity for a more general class of group von Neumann algebras. The final ingredient is the existence of a "nice" 1-cocycle. We say that a map $b:G\to\mathcal{H}_{\mathbb{R}}$ is **proper** if the set $\{g\in G:\|b(g)\|<n\}$ is finite for every $n\in\mathbb{N}$. As it turns out, there is a way to induce a proper cocycle for $\mathbb{F}_n=\mathbb{Z}*\cdots*\mathbb{Z}$ into a weakly-ℓ^2 representation from proper cocycles $b:\mathbb{Z}\to\ell^2(\mathbb{Z})$ for each of the free factors. The proof of this is omitted for brevity of the exposition.

Theorem 6.8. *Suppose G admits a proper cocycle $b:G\to\mathcal{H}_{\mathbb{R}}$ into a weakly-ℓ^2 representation $\pi:G\to\mathcal{O}(\mathcal{H}_{\mathbb{R}})$. Then $L(G)$ is solid.*

Proof. Let $\mathcal{A}\subseteq L(G)$ be a diffuse von Neumann subalgebra. To simplify notation, let $\mathcal{M}:=L(G)$, $\widetilde{\mathcal{M}}:=\mathcal{D}\rtimes^{\sigma^\pi}G$ and $\mathcal{N}:=\mathcal{A}'\cap\mathcal{M}$. Recall we form $\widetilde{\mathcal{M}}=\mathcal{D}\rtimes G$ from the abelian tracial von Neumann algebra $\mathcal{D}=\operatorname{span}\{w(\xi):\xi\in\mathcal{H}_{\mathbb{R}}\}''$ and the action $\sigma^\pi_g(w(\xi))=w(\pi_g(\xi))$. Since $\mathcal{A}$ is diffuse, then there is a sequence $(v_k)_{k\in\mathbb{N}}\subset\mathscr{U}(\mathcal{A})$ converging in WOT to 0 (in particular $\tau(xv_ky)\to0$ for any $x,y\in\widetilde{\mathcal{M}}$).

Our aim will be to verify $\mathcal{N}$ satisfies Haagerup's amenability criterion. Thus, fix a non-zero projection $p\in\mathcal{Z}(\mathcal{N})$, so $v_kp\in\mathscr{U}(pL(G)p)$ and we let $v_kp=\sum_{g\in G}v_g^{(k)}u_g$ be the Fourier expansion. For any fixed $t\in\mathbb{R}$ we have

$$\begin{aligned}\|E_{\mathcal{M}}(\alpha_t(v_kp))\|_2^2&=\|E_{\mathcal{M}}(\sum_{g\in G}v_g^{(k)}w(tb(g))u_g)\|_2^2=\|\sum_{g\in G}E_{\mathcal{M}}(w(tb(g)))v_g^{(k)}u_g\|_2^2\\&=\sum_{g\in G}|v_g^{(k)}|^2\|E_{\mathcal{M}}(w(tb(g)))\|_2^2=\sum_{g\in G}|v_g^{(k)}|^2e^{-2t^2\|b(g)\|^2}.\end{aligned}$$

In particular, since $v_g^{(k)}=\langle v_kp,u_g\rangle$ given a finite subset $F\Subset G$ we can write

$$\|E_{\mathcal{M}}(\alpha_t(v_kp))\|_2^2=\sum_{g\in F}|\langle v_kp,u_g\rangle|^2e^{-2t^2\|b(g)\|^2}+\sum_{g\in G\setminus F}|v_g^{(k)}|^2e^{-2t^2\|b(g)\|^2}$$

and notice that the first sum goes to 0 since $v_k\to0$ in WOT and F is finite. Let $0<\varepsilon<1$, then there is a finite subset $F\Subset G$ such that for all $g\notin F$ we have $\|b(g)\|^2>-\frac{\log(\varepsilon)}{2t^2}$. Hence,

$$\sum_{g\in G\setminus F}|v_g^{(k)}|^2e^{-2t^2\|b(g)\|^2}\le\sum_{g\in G\setminus F}|v_g^{(k)}|^2\varepsilon\le\sum_{g\in G}|v_g^{(k)}|^2\varepsilon=\|v_kp\|_2^2\varepsilon\le\varepsilon$$

for all k. Altogether, it follows that for any fixed $t\in\mathbb{R}$ we have $\|E_{\mathcal{M}}(\alpha_t(v_kp))\|_2^2\xrightarrow{k\to\infty}0$. Hence, we have $\|\alpha_t(v_kp)-E_{\mathcal{M}}(\alpha_t(v_kp))\|_2^2\xrightarrow{k\to\infty}\|\alpha_t(v_kp)\|_2^2=\|\alpha_t(p)\|_2^2=\|p\|_2^2>0$. Notice we have that the map $x\mapsto\alpha_t(x)-E_{\mathcal{M}}(\alpha_t(x))$ converges to 0 as $t\to0$ in $\|\cdot\|_2$ pointwise but not uniformly on the unit ball $(\mathcal{A}p)_1$, hence Popa's transversality inequality implies that the convergence $\alpha_t\to\mathrm{id}$ (in $\|\cdot\|_2$-norm) is not uniform on $(\mathcal{A}p)_1$.

The non-uniform convergence of $x\mapsto\alpha_t(x)-E_{\mathcal{M}}(\alpha_t(x))$ to 0 on $(\mathcal{A}p)_1$ obtained above implies there is $\delta>0$ such that for any $t\in\mathbb{R}$ there exists $x_t\in(\mathcal{A}p)_1$ satisfying $\|\alpha_t(x_t)-E_{\mathcal{M}}(\alpha_t(x_t))\|_2>\delta$. Define $\zeta_t:=\widehat{\alpha_t(x_t)}$ and $\xi_t:=\zeta_t-e_{\mathcal{M}}(\zeta_t)=(\alpha_t(x_t)-E_{\mathcal{M}}(\alpha_t(x_t)))\widehat{1}$, and notice $\|\xi_t\|_2>\delta>0$. Now, given any $u \in \mathscr{U}(\mathcal{N}p)$ we have that

$$\begin{aligned}\|u\xi_t u^* - \xi_t\|_2 &= \|u\alpha_t(x_t)u^* - \alpha_t(x_t) + E_{\mathcal{M}}(u\alpha_t(x_t)u^*) - E_{\mathcal{M}}(\alpha_t(x_t))\|_2 \\ &\leq \|u\alpha_t(x_t)u^* - \alpha_t(x_t)\|_2 = \|u\alpha_t(x_t) - \alpha_t(x_t)u\|_2 = \|\alpha_{-t}(u)x_t - x_t\alpha_{-t}(u)\|_2 \\ &\leq \|(\alpha_{-t}(u) - u)x_t\|_2 + \|ux_t - x_tu\|_2 + \|x_t(\alpha_{-t}(u) - u)\|_2 \\ &\leq 2\|\alpha_{-t}(u) - u\|_2 \xrightarrow{t\to 0} 0\end{aligned}$$

(since $u \in \mathcal{N}p$ commutes with $x_t \in \mathcal{A}p$). Moreover, for any finite subset $F \Subset \mathscr{U}(\mathcal{N}p)$ we get

$$\frac{\|\sum_{u\in F}(u\xi_t u^* - \xi_t)\|_2}{\|\xi_t\|_2} \xrightarrow{t\to 0} 0$$

(since $\|\xi_t\|_2 > \delta > 0$ for all t) and so, for any $\varepsilon > 0$ there is t such that

$$\frac{\|\sum_{u\in F} u\xi_t u^*\|_2}{\|\xi_t\|_2} \geq \frac{\|\sum_{u\in F} \xi_t\|_2}{\|\xi_t\|_2} - \varepsilon = |F| - \varepsilon.$$

Hence, as an operator on $L^2(\widetilde{\mathcal{M}}) \ominus L^2(\mathcal{M}) := (1 - e_{\mathcal{M}})L^2(\widetilde{\mathcal{M}})$ we have

$$|F| \leq \|\sum_{u\in F} \pi_l(u)\pi_r(u^*)\|_{\mathbb{B}(L^2(\widetilde{\mathcal{M}})\ominus L^2(\mathcal{M}))} \leq \sum_{u\in F} \|\pi_l(u)\pi_r(u^*)\|_{\mathbb{B}(L^2(\widetilde{\mathcal{M}})\ominus L^2(\mathcal{M}))} \leq |F|.$$

Therefore, from Proposition 6.6 we obtain the bound

$$|F| = \|\sum_{u\in F} \pi_l(u)\pi_r(u^*)\|_{\mathbb{B}(L^2(\widetilde{\mathcal{M}})\ominus L^2(\mathcal{M}))} \leq \|\sum_{u\in F} \pi_l(u)\pi_r(u^*)\|_{\mathbb{B}(L^2(\mathcal{M})\otimes L^2(\mathcal{M}))} \leq |F|$$

and by Haagerup's criterion (Proposition 6.7) we conclude that $\mathcal{N}$ is amenable. □

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