What is a groupoid?

A topological groupoid is a joint generalization of a topological space and a group, that includes equivalence relations and group actions on sets. This flexible structure allows us to unify many examples of C^*-algebras under one umbrella.

This first lecture is based heavily on Aiden Sim's notes "Hausdorff étale groupoids & their C^*-algebras." For general theory of gpds I also recommend Ronald Brown's book "Topology and Groupoids".

A quick way to define a groupoid is "a small category in which every element has an inverse." More explicitly,

Defn: A groupoid is a tuple $(G, G^{(0)}, r, s)$ where:

- $G^{(0)}$ is a set, called the <u>unitspace</u>, which is a subset of G.

- $r, s : G \to G^{(0)}$ are functions called the range / source maps

- $r(x) = x = s(x)$ for all $x \in G^{(0)}$
- There is a multiplication defined on a <u>subset</u> $G^{(2)} \subseteq G \times G$ (called the set of <u>composable pairs</u>) where $(\alpha, \beta) \in G^{(2)}$ iff

$$s(\alpha) = r(\beta)$$

  Mult. is associative in the sense $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ whenever $s(\alpha) = r(\beta)$ & $s(\beta) = r(\gamma)$
- $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$ for all $\gamma \in G$
- $r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$ for all $(\alpha, \beta) \in G^{(2)}$
- For every $\gamma \in G$, $\exists\, \gamma^{-1} \in G$ satisfying

$$r(\gamma^{-1}) = s(\gamma) \text{ and } s(\gamma^{-1}) = r(\gamma)$$

and

$$\gamma\gamma^{-1} = r(\gamma) \text{ and } \gamma^{-1}\gamma = s(\gamma)$$

(These axioms can be minimized more)

Thus we can think of elements of G as arrows

$$r(\gamma) \bullet \xleftarrow{\ \gamma\ } \bullet\, s(\gamma)$$

with γ^{-1} travelling backwards along γ; and $(\alpha,\beta) \in G^{(2)}$ means you have

$$\xleftarrow{\ \alpha\ } \xleftarrow{\ \beta\ }$$

(Hence the small category w/ inverses)

The category language is why the most dominant (Australian?) convention is to have source on right, range on left, and mult. in order of composition.

<u>Exercise 1</u> [Pick one to check using gpd axioms]

- $(\gamma^{-1})^{-1} = \gamma$
- If $\alpha = \gamma^{-1}\gamma$, then $\alpha \in G^{(0)}$

  If $\alpha^2 = \alpha$, then $\alpha \in G^{(0)}$
- $\alpha\gamma = \beta\gamma \Rightarrow \alpha = \beta$

Note: A groupoid morphism is just a morphism of categories where $\mathcal{G}^{(0)}$ is the objects of category and $\mathcal{G}$ is the morphisms.

Examples of Groupoids

- Topological spaces

- Groups (& group bundles)

- Equivalence relations (incl. matrix gpds)

$\sim$ on $X \to$ subset of $X \times X$

$\mathcal{G}^{(0)} = \{(x,x) : x \in X\}$

$\mathcal{G}^{(2)} = \{((x,y),(y,z)) : x,y,z \in X\}$

- Deaconu-Renault groupoids

X-set, Γ-abelian gp $(+)$, $S \subseteq \Gamma$ subsemigp w/ 0, $S \curvearrowright X$

$$\mathcal{G} = \{(x, s-t, y) : s\cdot x = t \cdot y\}$$

$$\mathcal{G}^{(0)} = \{(x, 0, x) : x \in X\} \sim X$$

$$r(x, s-t, y) = x, \quad s(x, s-t, y) = y,$$

$$(x,g,y)^{-1} = (y, g^{-1}, x)$$

$$(x,g,y)(y,h,z) = (x, gh, z)$$ —when does this make sense?

<u>Exercise 2:</u> The <u>transformation gpd</u>

X-set, Γ group, $\Gamma \curvearrowright X$ via bijections, i.e. $g : X \to X$, $x \mapsto g\cdot x$ is a bij. of X

$$\mathcal{G} := \Gamma \times X$$

$$\mathcal{G}^{(0)} := \{e\} \times X \text{ (identified with } X)$$

$$r((g,x)) := g\cdot x, \quad s((g,x)) := x$$

$$(g, h\cdot x)(h, x) := (gh, x), \quad (g,x)^{-1} := (g^{-1}, g\cdot x)$$

<u>Check some axioms</u>:
- $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$
- multiplication is associative

Defn: A <u>topological groupoid</u> $\mathcal{G}$ is a groupoid with a loc. compact topology [not auto Hausdorff] for which:

- $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ is Hausdorff (rel. top)
- $r, s, {}^{-1}$ are continuous
- $(g,h) \mapsto gh$ is cont. w/rt <u>rel. top</u> on $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$

(Maybe include ex)

Prop: $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ iff $\mathcal{G}$ is Hausdorff.

Pf: $\boxed{\Leftarrow}$ Take net $(x_i) \subseteq \mathcal{G}^{(0)}$, $x_i \to \gamma \in \mathcal{G}$.

r cont. $\Rightarrow$ $x_i \to r(\gamma)$

Hausdorff $\Rightarrow$ limits unique

$\Rightarrow \gamma \in \mathcal{G}^{(0)}$

$\boxed{\Rightarrow}$ Since it suffices to show convgt nets have unique limits, basically backwards argument

Notes on top. for previous ex's:

- top sp. & top. gp obvious

- Top. equivalence rel. $\to$ X Hausdorff

  R is top gpd in <u>relative</u> top.
  from $X \times X$

  If topology is finer, principal.

- D-R groupoids

  X - loc cpt T2, Γ discrete, $\overset{\circ}{S} \subseteq \Gamma$ subsemi

  $S \curvearrowright X$ by local homeo.

  $\mathcal{G} \rightsquigarrow$ loc cpct T2 gpd, basic open sets

  $$Z(U,p,q,V) := \left\{(x, p-q, y) : \begin{matrix} x \in U, y \in V, \\ p \cdot x = q \cdot y \end{matrix}\right\}$$

In top spaces, we localize by restricting to an open set. In groupoids, it's very helpful to be able to restrict to open sets that also are simple w/r/t mult. structure:

Defn: A <u>bisection</u> $B \subseteq \mathcal{G}$ is a subset s.t.

$$BB^{-1} \subseteq \mathcal{G}^{(0)} \text{ and } B^{-1}B \subseteq \mathcal{G}^{(0)}.$$

Defn: A groupoid is called <u>étale</u> when

the source map is a local homeomorphism,

or equivalently,

it carries a topology with a basis of <u>open</u> <u>bisections</u> & which is closed under ptwise products & inverses.

If you don't req. gpd to be étale, you might instead be in the setting of a Lie gpd. where things are more "manifold-y".

Some facts (won't prove)

- $\mathcal{G}$ is étale $\Rightarrow$ $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$

Exercise 3:

- If $\mathcal{G}$ is étale, $x\mathcal{G}$ & $\mathcal{G}x$ are <u>discrete</u> $\forall x \in G^{(0)}$ in relative top.

Will use in conv. alg. section

Note this implies if G is a gp, then $\Rightarrow G$ discrete

- A D-R gpd w/ action given by local homeo is always étale

Note bonus, don't include

$$\boxed{\begin{array}{l}\bullet\ \text{If } G \text{ is étale, multiplication is an open map.}\\ \bullet\ \text{transf. gpd is étale} \Leftrightarrow \text{gp acting is discrete}\end{array}}$$

The convolution Algebra

Let G be a (2^{nd} count?) loc. cpct. Hausdorff étale gpd.

We will build our C^*-algebra on G in a similar way to how the group C^*-alg. is built.

Prop: Since G is étale,

For $f, g \in C_c(G)$, the set

$$\{(\alpha, \beta) \in G^{(2)} : \alpha\beta = \gamma \text{ and } f(\alpha) g(\beta) \neq 0\}$$

is finite.

Pf: (disrete sets intersected w/ compact sets)

Prop: $C_c(\mathcal{G})$ is a $*$-alg under ptwise add, involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$, and mult.

$$(f*g)(\gamma) = \sum_{\gamma=\alpha\beta} f(\alpha)\, g(\beta)$$

note if $\mathcal{G}$ is a gp, is gp alg mult.

$$= \sum_{\alpha \in r(\gamma)\mathcal{G}} f(\alpha)\, g(\alpha^{-1}\gamma)$$

Lemma: $C_c(\mathcal{G}) = \operatorname{span}\{ f \in C_c(\mathcal{G}) : \overline{\operatorname{supp}}(f) \text{ is a bisection} \}$

Pf: [Uses Hausdorff + loc. cpct. $\Rightarrow \exists$ finite partitions of unity]

Given $f \in C_c(\mathcal{G})$, cover $\overline{\operatorname{supp} f}$ by open bisections and pass to finitely many.

Choose partition of unity h_i subordinate

to open sets, $f \cdot h_i \in C_c(\mathcal{G})$ and
$f = \sum_{i=1}^{n} f h_i$ pointwise

This allows us to prove

Lemmas on mult. in $C_c(\mathcal{G})$: Let $f, g \in C_c(\mathcal{G})$

(1) If $\mathcal{U}, V$ are open bisections with $\overline{\text{supp}}\, f \subseteq \mathcal{U}$, $\overline{\text{supp}}\, g \subseteq V$, then $\overline{\text{supp}}\,(f * g) \subseteq \mathcal{U} \cdot V$ and moreover for $\gamma = \alpha\beta$, $\alpha \in \mathcal{U}, \beta \in V$ we get $(f*g)(\gamma) = f(\alpha) g(\beta)$.

(2) $C_c(\mathcal{G}^{(0)}) \subseteq C_c(\mathcal{G})$ Uses étale $\Rightarrow \mathcal{G}^{(0)} \subseteq \mathcal{G}$ open

(3) If f is supported on a bisection, $f^* * f$ is supported on $s(\overline{\text{supp}}\, f)$

Exercise 4: Check (3) and calculate $(f^* * f)(s(\gamma))$ for $\gamma \in \text{supp} f$

Also calculate $f * g$ when $g \in C_c(\mathcal{G}^{(0)})$.

Defn: The <u>reduced gpd C^*-alg</u> $C^*_r(\mathcal{G})$ is the completion of

$$\Big(\bigoplus_{x\in G^{(0)}} \pi_x\Big)(C_c(\mathcal{G})) \subseteq \bigoplus_{x\in G^{(0)}} B(\ell^2(G_x))$$

where $\pi_x : C_c(\mathcal{G}) \to B(\ell^2(\mathcal{G}_x))$ is the <u>regular $*$-repn associated to x</u>

$$\pi_x(f)\,\delta_\gamma = \sum_{\alpha\in \mathcal{G}_{r(\gamma)}} f(\alpha)\,\delta_{\alpha\gamma} \quad (\text{for } \gamma\in \mathcal{G}_x)$$

extended to rest of $\ell^2(\mathcal{G}x)$

<u>Note</u>: Can also define via Hilbert modules.

To be continued...

<u>Ex 5</u>: Check that for each $\eta\in\mathcal{G}$,

$$U_\eta : \ell^2(\mathcal{G}_{s(\eta)}) \to \ell^2(\mathcal{G}_{r(\eta)})$$
$$\delta_\gamma \longmapsto \delta_{\gamma\eta^{-1}}$$

is a unitary operator and that

$$\pi_{r(\eta)} = U_\eta\, \pi_{s(\eta)}\, U_\eta^*$$