Data Analysis with the Riemannian Geometry of SPD Matrices

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Domain Adaptation

- **P:** Datasets often comprise multiple domains:
  - Sessions
  - Subjects
  - Batches
- **Q:** How to adapt a given model that is well performing on a particular domain to a different yet related domain?
- **Q:** How to train a classifier based on data from one domain and apply it to data from another domain?

Domain Adaptation

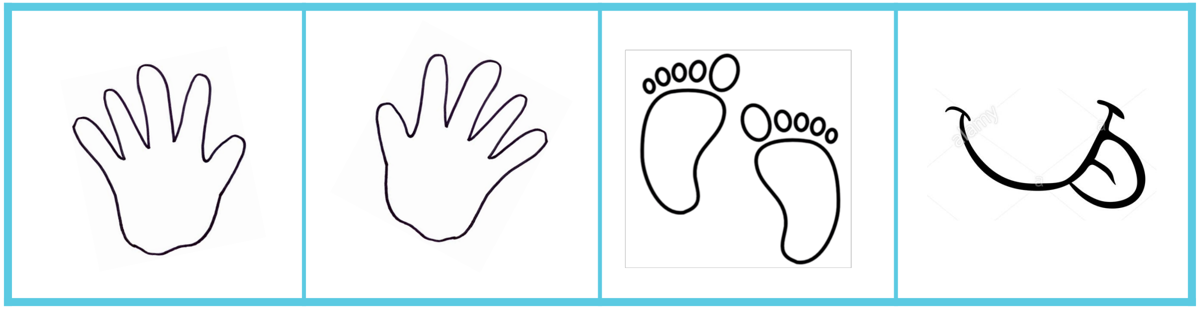
- Highly researched subject
- Many previous studies, e.g.:
  - [Ben-David et al, 07]
  - [Raina et al, 07]
  - [Dai et al, 09]
  - [Pan et al, 10]
  - …

Data Analysis in High Dimension

- **P:** The data do not live in a Euclidean space
  - Multiple modalities
  - Dependence between coordinates
- **Q:** What is the proper non-Euclidean metric?
- **Q:** How to find an embedding into a Euclidean space?
- **P:** Unsupervised

Illustrative Application

- Brain Computer Interface (BCI) [Barachant et al, 13]
- Data: [recent BCI competition]
  - EEG from 9 subjects
  - 22 electrodes
  - 2 days of experiments
  - 288 trials per subject
- In each trial, imagine performing 1 of 4 motor tasks:
  - Left hand
  - Right hand
  - Both feet
  - Tongue



Illustrative Application

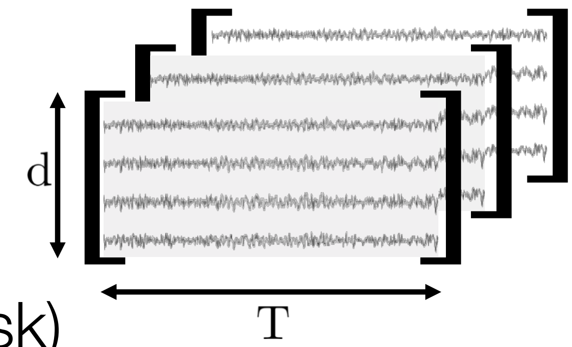
- Consider two datasets from two subjects:

$$\left\{ \boldsymbol{X}_i^{(1)}, y_i^{(1)} \right\}_{i=1}^{N_1} \qquad \left\{ \boldsymbol{X}_i^{(2)}, y_i^{(2)} \right\}_{i=1}^{N_2}$$

(target) (source)

  - Each set contains N_k matrices of observations

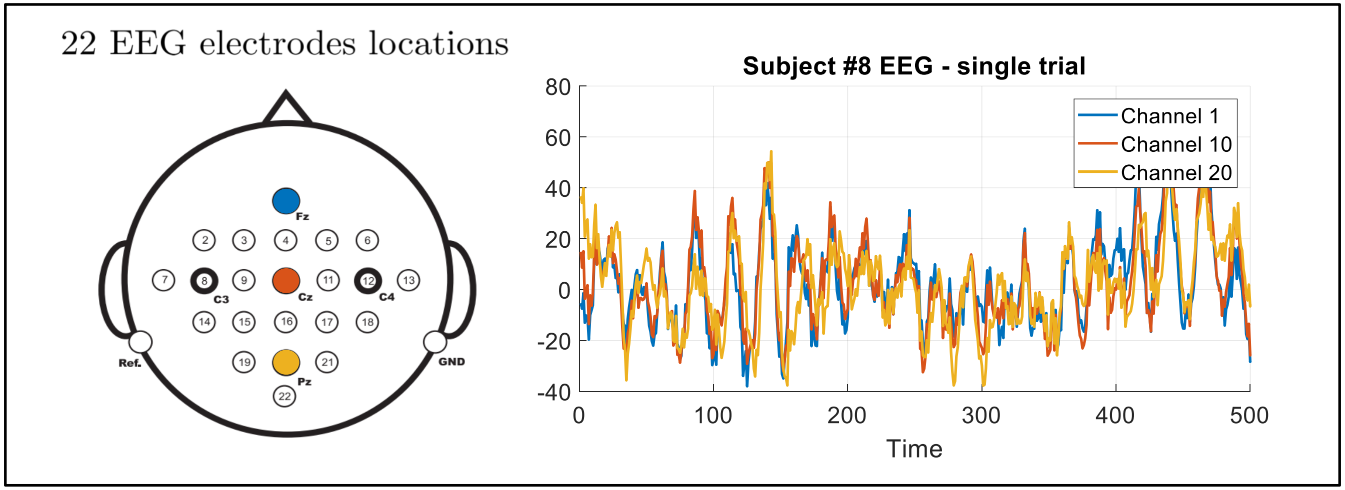
$$\boldsymbol{X}_i^{(k)} \in \mathbb{R}^{d \times T_i^{(k)}}$$



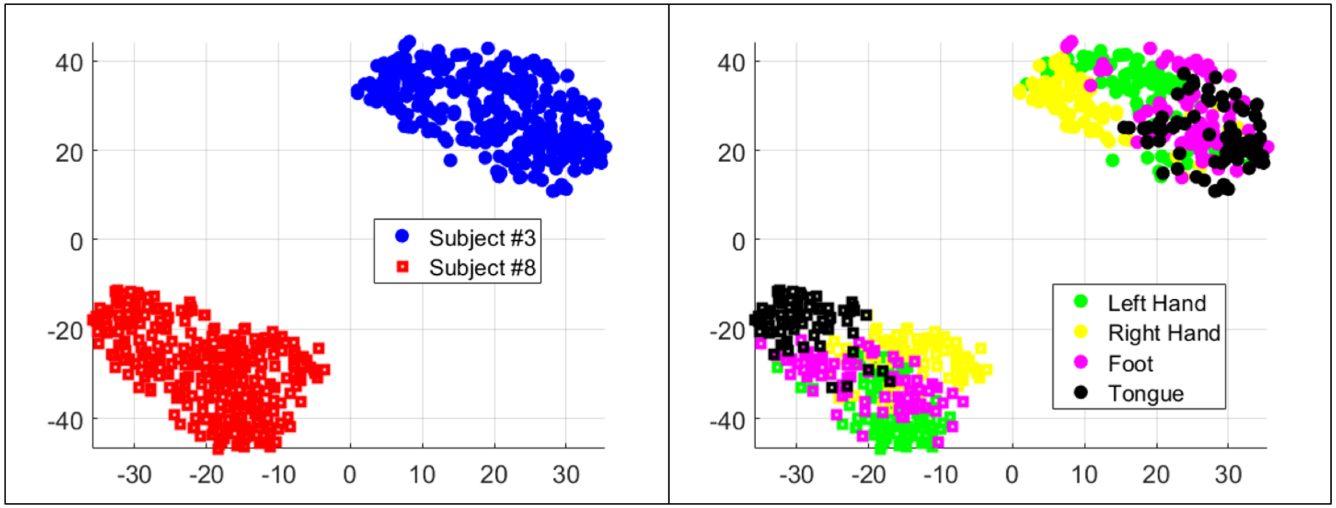
  - d - dimension (# of EEG electrodes)
  - $T_i^{(k)}$ - observation length
  - $y_i^{(k)}$ - hidden label (imagined motor task)

- Let $\boldsymbol{P}_i^{(k)} \in \mathbb{R}^{d \times d}$ be the (sample) covariance of $\boldsymbol{X}_i^{(k)}$

Illustrative Application



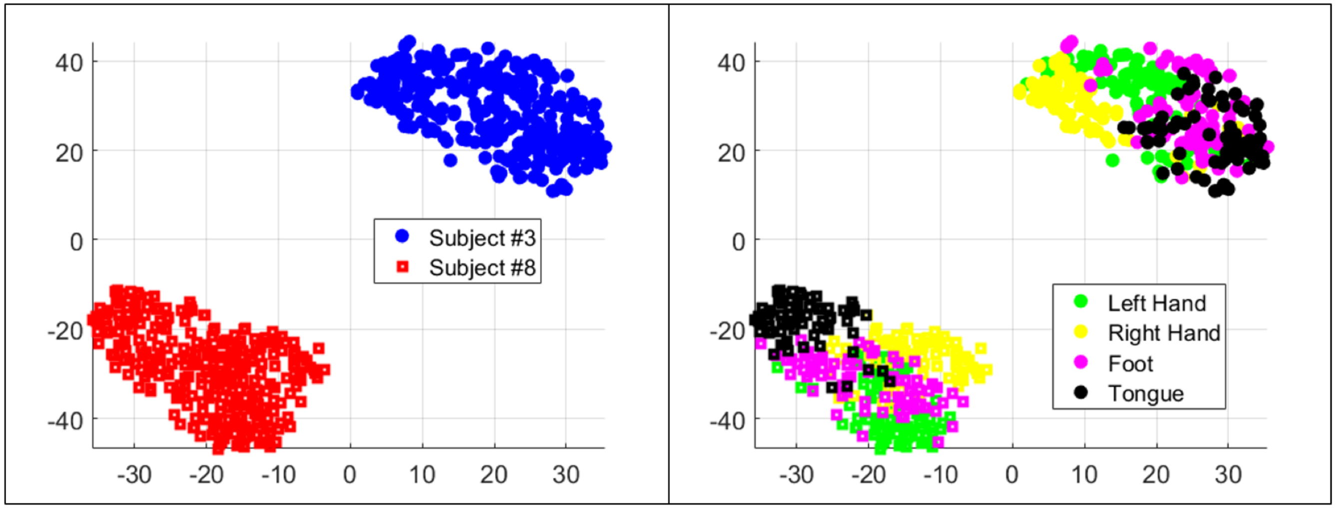
Low dimensional representation of the covariance matrices from **two** subjects



Illustrative Application

Identify the imagined motor activity per trial *from multiple subjects*

- → Training a classifier from one subject and testing on another subject
- → Unsupervised



Our Solution

Riemannian Geometry of SPD matrices

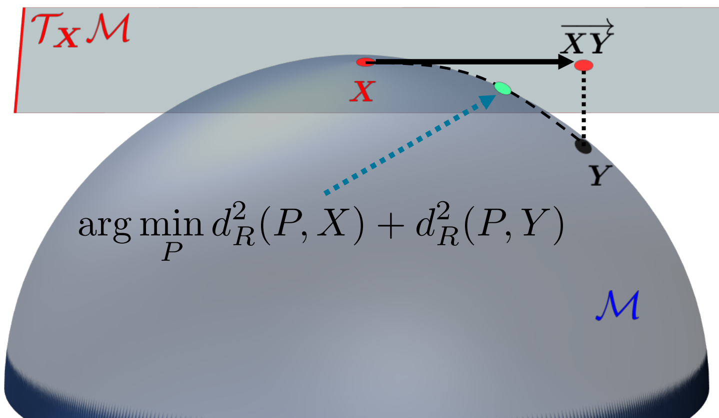
Benefits

- *Known* non-Euclidean space
  facilitating comparisons, additions, subtractions
- Joint representation from multiple domains
- Following recent work:
  - Theory [Pennec et al, 06], [Sra & Hosseini, 15]
  - Applications in BCI [Barachant et al, 13]
  - Applications in computer vision [Tuzel el al, 08], [Freifeld et al, 14], [Bergman et al, 17]

Preliminaries on Riemannian Geometry

- Let $x, y \in \mathcal{M}$ be two points on a Riemannian manifold
- Let $\mathcal{T}_x\mathcal{M}$ be the tangent plane at the point x
- Define the following operations:

	Vector Space	Riemannian Manifold
Subtraction	$\overrightarrow{\boldsymbol{x}\boldsymbol{y}} = \boldsymbol{y} - \boldsymbol{x}$	$\overrightarrow{\boldsymbol{x}\boldsymbol{y}} = \mathrm{Log}_{\boldsymbol{x}}(\boldsymbol{y})$
Addition	$\boldsymbol{y} = \boldsymbol{x} + \overrightarrow{\boldsymbol{x}\boldsymbol{y}}$	$\boldsymbol{y} = \mathrm{Exp}_{\boldsymbol{x}}(\overrightarrow{\boldsymbol{x}\boldsymbol{y}})$
Mean	$\arg\min_{\boldsymbol{x}} \sum_i \lVert \boldsymbol{x} - \boldsymbol{x}_i \rVert_i^2$	$\arg\min_{\boldsymbol{x}} \sum_i d_R^2(\boldsymbol{x}, \boldsymbol{x}_i)$

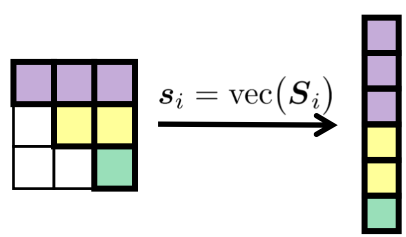


The SPD Cone

- The SPD matrices constitute a convex half-cone in the space of real symmetric matrices
- This cone forms a differentiable Riemannian manifold $\mathcal{M}$ equipped with the inner product

$$\langle S_1, S_2 \rangle_{\mathcal{T}_P\mathcal{M}} = \langle P^{-\frac{1}{2}} S_1 P^{-\frac{1}{2}}, P^{-\frac{1}{2}} S_2 P^{-\frac{1}{2}} \rangle$$

  - $\mathcal{T}_P\mathcal{M}$ is the tangent plane at $P \in \mathcal{M}$
  - $S_1, S_2 \in \mathcal{T}_P\mathcal{M}$
  - $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product
- The symmetric matrices $S \in \mathcal{T}_P\mathcal{M}$ live in a linear space
  - We can view them as vectors



With $\sqrt{2}$ scaling on the off-diagonal elements

The SPD Cone – Properties

- There exists a **unique geodesic curve** between any two SPD matrices $P_1, P_2 \in \mathcal{M}$:

$$\varphi(t) = P_1^{\frac{1}{2}} \left(P_1^{-\frac{1}{2}} P_2 P_1^{-\frac{1}{2}}\right)^t P_1^{\frac{1}{2}}, \qquad 0 \leq t \leq 1$$

- Define a **Riemannian distance** on the manifold as the arc-length of the geodesic curve:

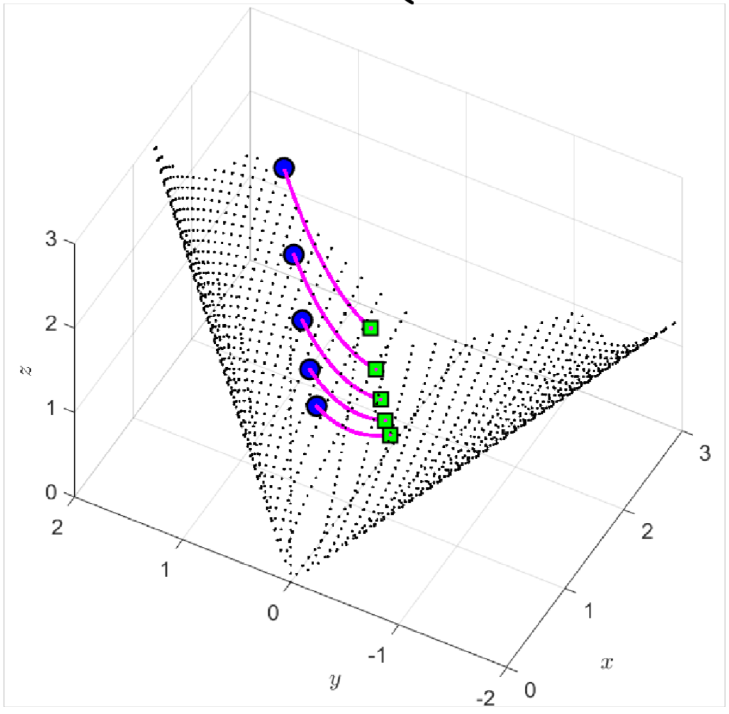
$$\begin{aligned} d_R^2(P_1, P_2) &= \left\| \log \left(P_2^{-\frac{1}{2}} P_1 P_2^{-\frac{1}{2}}\right) \right\|_F^2 \\ &= \sum_{i=1}^n \log^2 \left(\lambda_i \left(P_2^{-\frac{1}{2}} P_1 P_2^{-\frac{1}{2}}\right)\right) \end{aligned}$$

  - $\lambda_i(P)$ is the i-th eigenvalue of P
  - Scale-invariant

The SPD Cone – Properties

$$M = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} \mathrm{Tr}\,(M) = x + z > 0 \\ |(M)| = xz - y^2 > 0 \end{cases} \Rightarrow \begin{cases} x > 0 \\ z > 0 \end{cases} \Rightarrow \boxed{|y| < \sqrt{xz}}$$



The SPD Cone – Properties

- Logarithm map:

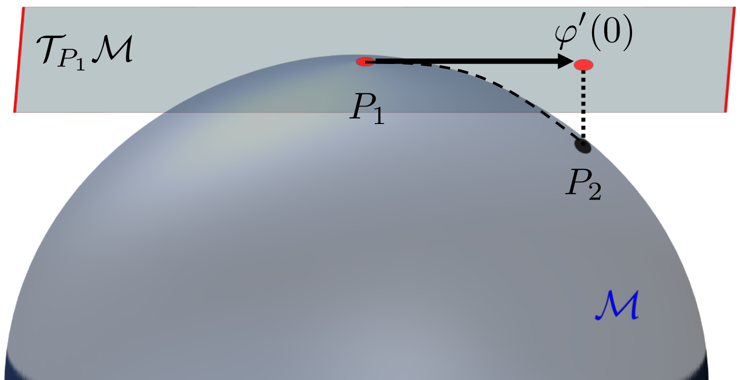
$$S_1 = \mathrm{Log}_P(P_1) = P^{\frac{1}{2}} \log\left(P^{-\frac{1}{2}} P_1 P^{-\frac{1}{2}}\right) P^{\frac{1}{2}} \in \mathcal{T}_P \mathcal{M}$$

- Exponential map:

$$P_1 = \mathrm{Exp}_P(S_1) = P^{\frac{1}{2}} \exp\left(P^{-\frac{1}{2}} S_1 P^{-\frac{1}{2}}\right) P^{\frac{1}{2}} \in \mathcal{M}$$

- Relation to the (unique) geodesic curve $\varphi(t)$ from P_1 to P_2 is given by the initial velocity

$$\varphi'(0) = \mathrm{Log}_{P_1}(P_2) \in \mathcal{T}_{P_1}\mathcal{M}$$



Domain Adaptation

Formulation

- Consider two subsets $\mathcal{P}^{(1)}$ (target) and $\mathcal{P}^{(2)}$ (source) of SPD matrices from *two different domains*
  - $\overline{P}^{(1)}$ and $\overline{P}^{(2)}$- Riemannian means
  - $\varphi(t)$ - the unique geodesic from $\overline{P}^{(2)}$ to $\overline{P}^{(1)}$
  - $S_i^{(k)}$ - the symmetric matrix (or vector) in $\mathcal{T}_{\overline{P}^{(k)}}\mathcal{M}$

$$S_i^{(k)} = \mathrm{Log}_{\overline{P}^{(k)}}\left(P_i^{(k)}\right)$$

Domain Adaptation

Formulation

- **Goal:**
  Derive a new representation $\Gamma\left(S_i^{(2)}\right)$:
  $$\Gamma : \mathcal{T}_{\overline{P}^{(2)}}\mathcal{M} \to \mathcal{T}_{\overline{P}^{(1)}}\mathcal{M}$$
  so that $\left\{S_i^{(1)}\right\}$ and $\left\{\Gamma\left(S_i^{(2)}\right)\right\}$ live in the same linear space
- **Benefit:**
  Relate samples from the two subsets
  - Compute quantities such as $\left\langle S_i^{(1)}, \Gamma\left(S_j^{(2)}\right)\right\rangle_{\overline{P}^{(1)}}$

Domain Adaptation

Formulation

- **Constraints:**
  - Zero mean:

$$\frac{1}{N_2}\sum_{i=1}^{N_2}\Gamma\left(S_i^{(2)}\right) = \frac{1}{N_1}\sum_{i=1}^{N_1} S_i^{(1)} = 0$$

  - Inner product preservation:

$$\left\langle \Gamma\left(S_i^{(2)}\right), \Gamma\left(S_j^{(2)}\right)\right\rangle_{\overline{P}^{(1)}} = \left\langle S_i^{(2)}, S_j^{(2)}\right\rangle_{\overline{P}^{(2)}}$$

  - Geodesic velocity preservation:

$$\Gamma\left(\varphi'\left(0\right)\right) = \varphi'\left(1\right)$$

Domain Adaptation

Formulation

- **Constraints:**
  - Zero mean:

$$\frac{1}{N_2}\sum_{i=1}^{N_2}\Gamma\left(S_i^{(2)}\right) = \frac{1}{N_1}\sum_{i=1}^{N_1} S_i^{(1)} = 0$$

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Imply that the map preserves inter-sample relations

Domain Adaptation

Formulation

- **Constraints:**
  - Zero mean:

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Not unique:
If Γ admits to these properties, then also $R \circ \Gamma$
where R is an arbitrary rotation

Domain Adaptation

Formulation

- **Constraints:**
  - Zero mean:

$$\frac{1}{N_2}\sum_{i=1}^{N_2}\Gamma\left(S_i^{(2)}\right)=\frac{1}{N_1}\sum_{i=1}^{N_1}S_i^{(1)}=0$$

  - Inner product preservation:

$$\left\langle\Gamma\left(S_i^{(2)}\right),\Gamma\left(S_j^{(2)}\right)\right\rangle_{\overline{P}^{(1)}}=\left\langle S_i^{(2)},S_j^{(2)}\right\rangle_{\overline{P}^{(2)}}$$

Simple implementation by mean subtraction

Domain Adaptation

Formulation

- **Constraints:**
  - Geodesic velocity preservation:

$$\Gamma\left(\varphi'\left(0\right)\right) = \varphi'\left(1\right)$$

Use the *unique* geodesic to resolve the arbitrary degree of freedom

The two intrinsic symmetric matrices (vectors) $\varphi'(0) \in \mathcal{T}_{\overline{P}^{(2)}}\mathcal{M}$ and $\varphi'(1) \in \mathcal{T}_{\overline{P}^{(1)}}\mathcal{M}$ are used to fix the rotation

Domain Adaptation

Formulation

- **Constraints:**
  - Geodesic velocity preservation:

$$\Gamma\left(\varphi'\left(0\right)\right) = \varphi'\left(1\right)$$

Unsupervised – no labels are used

Present a closed-form expression (no optimization)

Parallel Transport

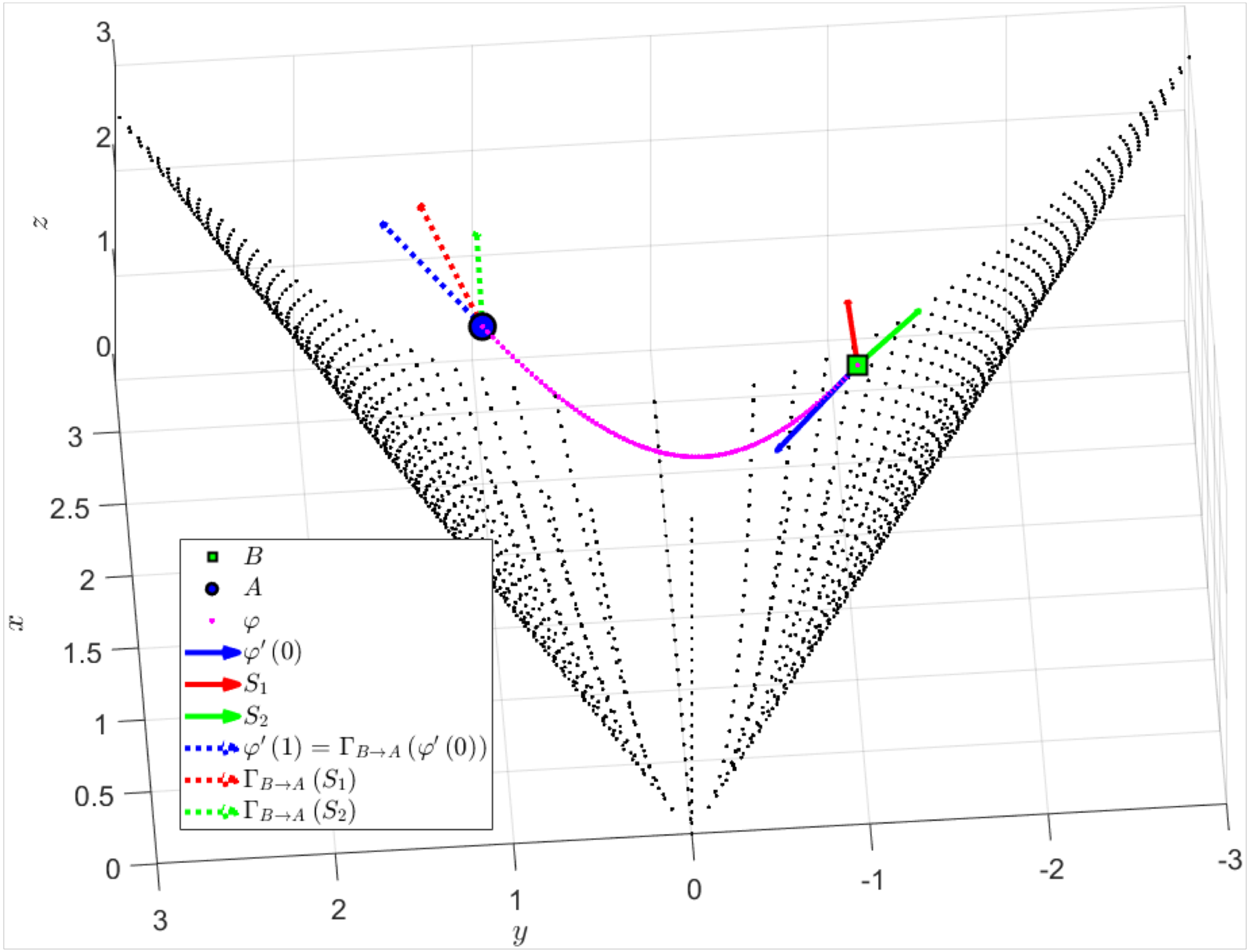
Lemma (Parallel Transport)
Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}$.
The PT from $\boldsymbol{B}$ to $\boldsymbol{A}$ of any $\boldsymbol{S} \in \mathcal{T}_{\boldsymbol{B}}\mathcal{M}$ is:

$$\Gamma_{\boldsymbol{B} \to \boldsymbol{A}}\left(\boldsymbol{S}\right) \triangleq \boldsymbol{E}\boldsymbol{S}\boldsymbol{E}^T$$

where $\boldsymbol{E} = \left(\boldsymbol{A}\boldsymbol{B}^{-1}\right)^{\frac{1}{2}}$.

Theorem.
The representation $\Gamma_{\overline{\boldsymbol{P}}^{(2)} \to \overline{\boldsymbol{P}}^{(1)}}\left(\boldsymbol{S}_i^{(2)}\right)$,
i.e., the unique PT of $\boldsymbol{S}_i^{(2)}$ from $\overline{\boldsymbol{P}}^{(2)}$ to $\overline{\boldsymbol{P}}^{(1)}$,
is well defined and satisfies properties $(1) - (3)$.

Parallel Transport



Domain Adaptation with PT

Algorithm:

1. Project the SPD matrix $\boldsymbol{P}_i^{(2)}$ to the tangent plane $\mathcal{T}_{\overline{\boldsymbol{P}}^{(2)}}\mathcal{M}$

$$\boldsymbol{S}_i^{(2)} = \mathrm{Log}_{\overline{\boldsymbol{P}}^{(2)}}\left(\boldsymbol{P}_i^{(2)}\right)$$

2. Parallel transport $\boldsymbol{S}_i^{(2)}$ from $\overline{\boldsymbol{P}}^{(2)}$ to $\overline{\boldsymbol{P}}^{(1)}$ by computing

$$\boldsymbol{S}_i^{(2)\to(1)} = \Gamma_{\overline{\boldsymbol{P}}^{(2)}\to\overline{\boldsymbol{P}}^{(1)}}\left(\boldsymbol{S}_i^{(2)}\right)$$

3. Project the symmetric matrix $\boldsymbol{S}_i^{(2)\to(1)} \in \mathcal{T}_{\overline{\boldsymbol{P}}^{(1)}}\mathcal{M}$ back to the manifold using $\mathrm{Exp}_{\overline{\boldsymbol{P}}^{(1)}}\left(\boldsymbol{S}_i^{(2)\to(1)}\right)$.

Domain Adaptation with PT

Define the map $\Psi : \mathcal{M} \to \mathcal{M}$ that adapts the domain of $\mathcal{P}^{(2)}$ to the domain of $\mathcal{P}^{(1)}$

$$\Psi\big(P_i^{(2)}\big) = \mathrm{Exp}_{\overline{P}^{(1)}}\left(\Gamma_{\overline{P}^{(2)} \to \overline{P}^{(1)}}\left(\mathrm{Log}_{\overline{P}^{(2)}}\big(P_i^{(2)}\big)\right)\right)$$

for any $P_i^{(2)} \in \mathcal{P}^{(2)}$

Implementation

Theorem.
Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{P} \in \mathcal{M}$ and let $\boldsymbol{S} = \mathrm{Log}_{\boldsymbol{B}}(\boldsymbol{P}) \in \mathcal{T}_{\boldsymbol{B}}\mathcal{M}$.
Then,

$$\mathrm{Exp}_{\boldsymbol{A}}\left(\Gamma_{\boldsymbol{B}\to\boldsymbol{A}}\left(\boldsymbol{S}\right)\right) = \boldsymbol{E}\boldsymbol{P}\boldsymbol{E}^T,$$

where $\boldsymbol{E} = \left(\boldsymbol{A}\boldsymbol{B}^{-1}\right)^{\frac{1}{2}}$.

- Ψ can be efficiently implemented

$$\Psi\left(P_i^{(2)}\right) = \Gamma_{\overline{P}^{(2)}\to\overline{P}^{(1)}}\left(P_i^{(2)}\right) = EP_i^{(2)}E^T$$

$$E \triangleq \left(\overline{P}^{(1)}\left(\overline{P}^{(2)}\right)^{-1}\right)^{\frac{1}{2}}$$

Implementation

- **Important consequence:**
  - The covariance adaptation:

$$E P_i^{(2)} E^T$$

    where

$$E \triangleq \left(\overline{P}^{(1)} \left(\overline{P}^{(2)} \right)^{-1} \right)^{\frac{1}{2}}$$

  - Can be applied directly to data by:

$$E X_i^{(2)}$$

Toy Problem

- Consider the set of hidden multi-dimensional times series $\{s_i[n]\}_{i=1}^{100}$:

$$\boldsymbol{s}_i\left[n\right]=\begin{bmatrix}\sin\left(2\pi f_0 n/T\right)\\ \cos\left(2\pi f_0 n/T+\phi_i\right)\end{bmatrix}, n=0,\ldots,T-1$$

where $f_0=10$, $T=500$, and $\phi_i \sim U\left[-\pi/2,0\right]$

- Short segments of two oscillatory signals
- Governed by a 1-dimensional hidden variable ϕ_i (the initial phase of the oscillations)

Toy Problem

- The population covariance of $\boldsymbol{s}_i[n]$ is

$$\frac{1}{2}\begin{bmatrix} 1 & -\sin\left(\phi_i\right) \\ -\sin\left(\phi_i\right) & 1 \end{bmatrix}$$

  which depends only on ϕ_i

- <u>Note</u>:
  when presenting the population covariances as vectors in $\mathbb{R}^3$, two coordinates are fixed and only one varies

Toy Problem

- We generate two observable subsets

$$\mathcal{X}^{(1)} = \{\boldsymbol{x}_i^{(1)}[n]\}_{i=1}^{100}, \mathcal{X}^{(2)} = \{\boldsymbol{x}_i^{(2)}[n]\}_{i=1}^{100}$$

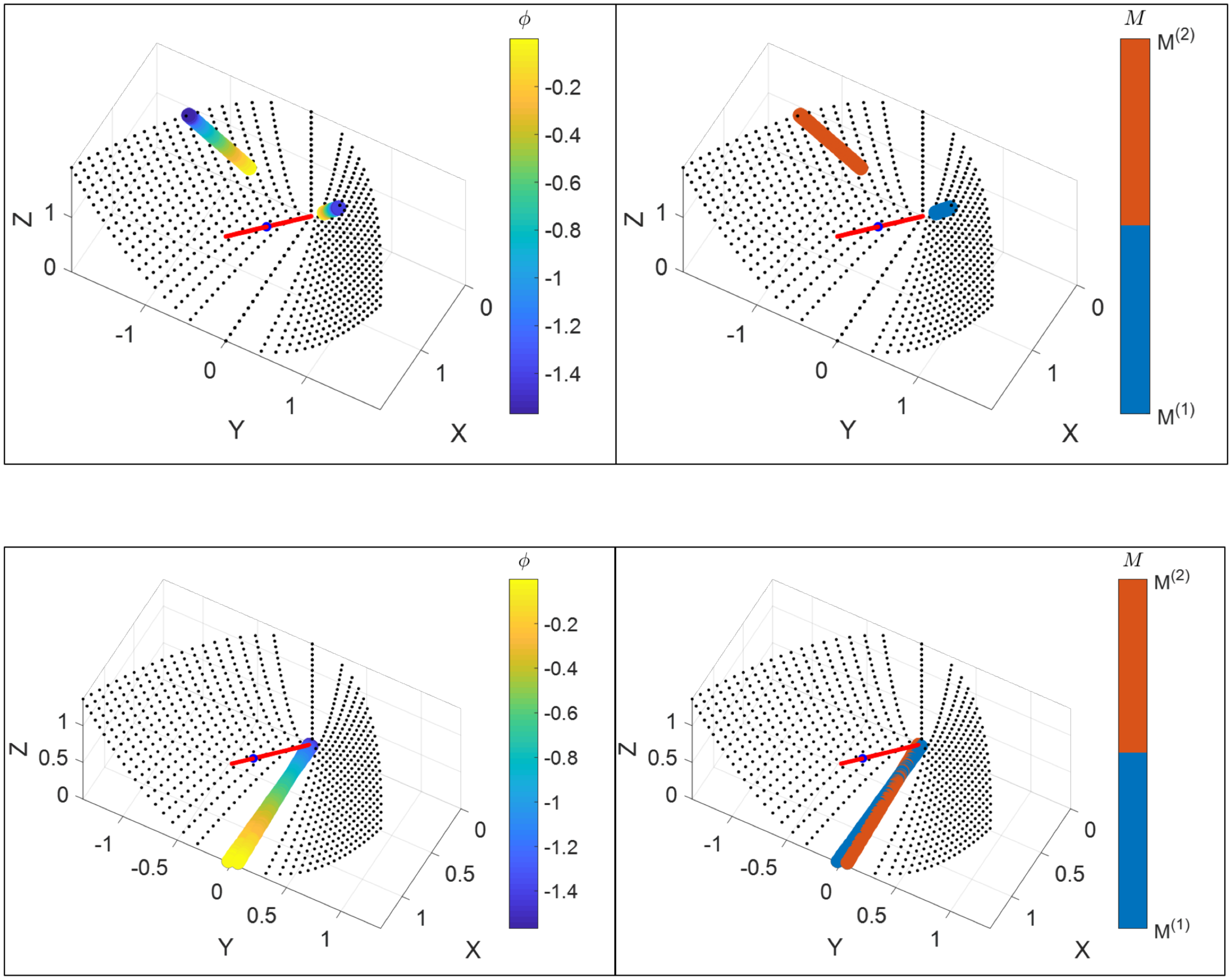
such that:

$$x_i^{(k)}[n] = M^{(k)} s_i[n]$$

where

- $M^{(1)}$ is randomly chosen
- $M^{(2)} = 1.5 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M^{(1)}$

Toy Problem



After PT*

* PT of the source and target sets to the mid-point

Brain Computer Interface

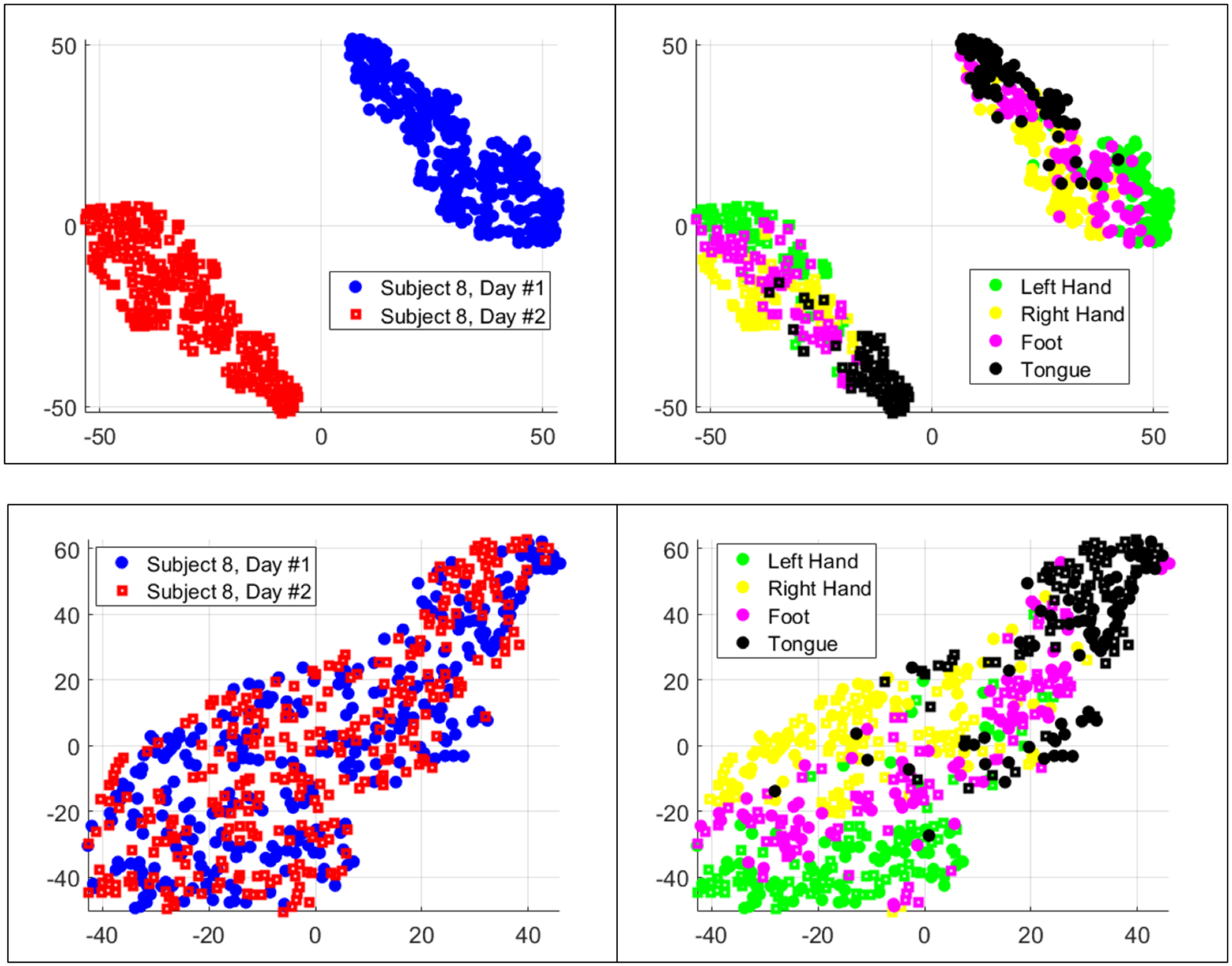
- Data: [BCI competition]
  - 22 EEG electrodes
  - 9 subjects
  - 2 days of experiments
  - 288 trials per subject
  - In each trial the subject is asked to imagine performing 1 out of 4 motor tasks:
    left hand, right hand, both feet, tongue

Brain Computer Interface

tSNE [v. d. Maaen & Hinton, 08] representation of Covariance matrices

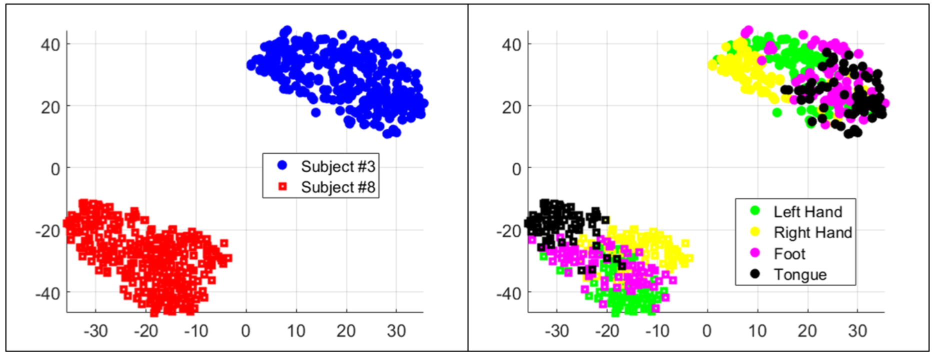
One subject
Two sessions

After PT

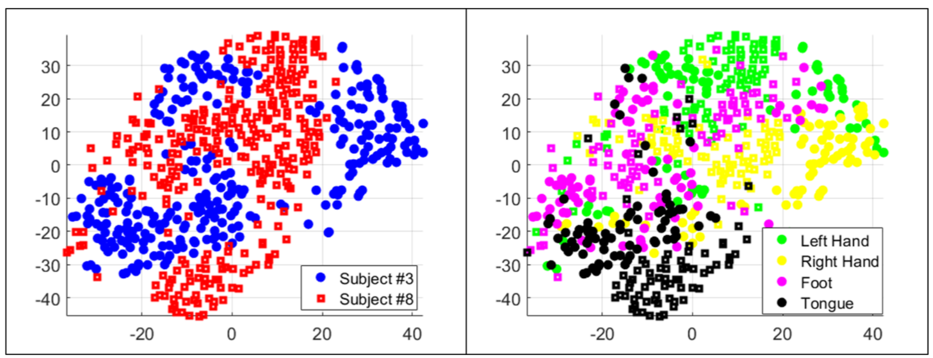


Brain Computer Interface

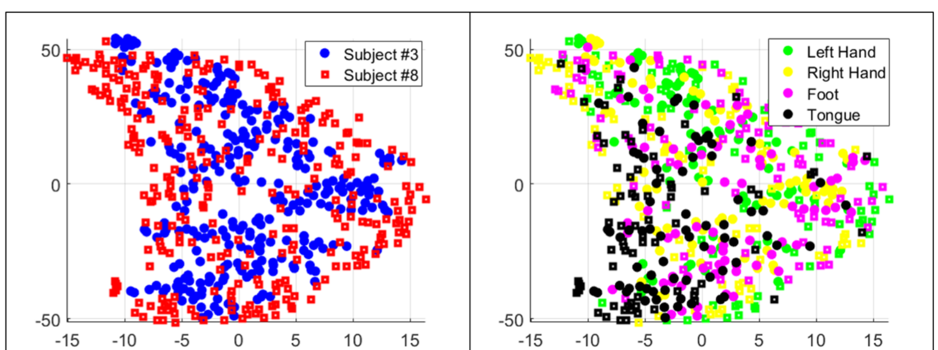
Two subjects



After PT



Mean subtraction
[Barachant et al, 13]



Intrinsicness

- Using *affine transformation* [Zanini et al, 18]:

$$\left(\overline{P}^{(k)}\right)^{-\frac{1}{2}} P_i^{(k)} \left((\overline{P}^{(k)})^{-\frac{1}{2}}\right)^T$$

  - Equivalent to parallel transport to the identity:

$$\Psi\left(P_i^{(k)}\right) = \Gamma_{\overline{P}^{(k)} \to I}\left(P_i^{(k)}\right) = E P_i^{(k)} E^T$$

$$E \triangleq \left(I(\overline{P}^{(k)})^{-1}\right)^{\frac{1}{2}}$$

  - When there are two sets, it is equivalent to

$$\Gamma_{I \to \overline{P}^{(1)}} \circ \Gamma_{\overline{P}^{(2)} \to I}$$

  - Coincides with our work when

$$\Gamma_{\overline{P}^{(2)} \to \overline{P}^{(1)}} = \Gamma_{I \to \overline{P}^{(1)}} \circ \Gamma_{\overline{P}^{(2)} \to I}$$

    - $\overline{P}^{(1)}$ and $\overline{P}^{(2)}$ commute and have the same eigenvectors (PC)

- **Q:** what is special about the proposed transport?

Intrinsicness

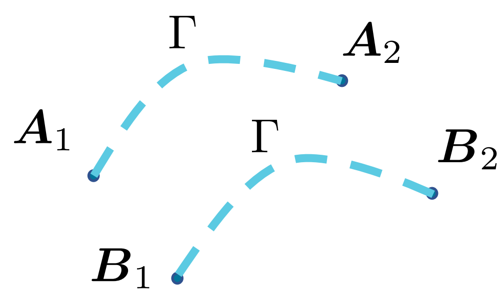
Definition (Equivalent Pairs)
Two pairs $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ and $(\boldsymbol{A}_2, \boldsymbol{B}_2)$, such that $\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{A}_2, \boldsymbol{B}_2 \in \mathcal{M}$, are *equivalent* if there exists an invertible matrix $\boldsymbol{E}$ such that

$$\boldsymbol{A}_2 = \Gamma(\boldsymbol{A}_1) = \boldsymbol{E}\boldsymbol{A}_1\boldsymbol{E}^T$$

$$\boldsymbol{B}_2 = \Gamma(\boldsymbol{B}_1) = \boldsymbol{E}\boldsymbol{B}_1\boldsymbol{E}^T$$

We denote this relation by

$$(\boldsymbol{A}_1, \boldsymbol{B}_1) \sim (\boldsymbol{A}_2, \boldsymbol{B}_2)$$



Intrinsicness

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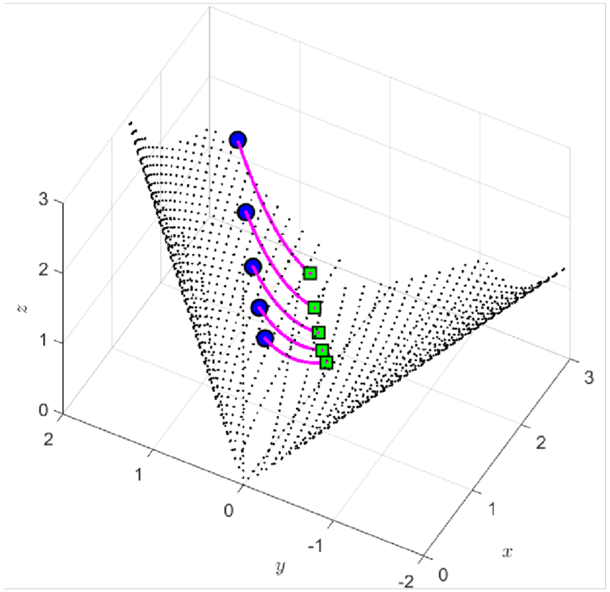
We denote this relation by

$$(\boldsymbol{A}_1, \boldsymbol{B}_1) \sim (\boldsymbol{A}_2, \boldsymbol{B}_2)$$

Lemma.
The relation $\sim$ is an equivalence relation, satisfying reflexivity, symmetry, and transitivity.

Intrinsicness

- Interpretation:
  - Equivalent pairs are matrices with equivalent intra-relations
  - E.g. if $(A_1, B_1) \sim (A_2, B_2)$ then
$$d_R(A_1, B_1) = d_R(A_2, B_2)$$
    but with a different global position on the manifold



Intrinsicness

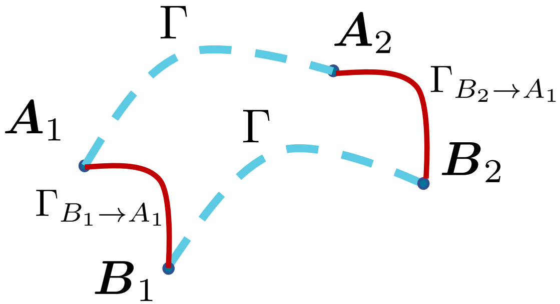
Proposition.
Let (A_1, B_1) be a pair of SPD matrices $A_1, B_1 \in \mathcal{M}$, and let $\big[(A_1, B_1)\big]$ denote the equivalence class

$$\big[(A_1, B_1)\big] = \big\{(A_2, B_2) \in \mathcal{M} \times \mathcal{M} \,|\, (A_2, B_2) \sim (A_1, B_1)\big\},$$

of all matrix pairs that are equivalent to (A_1, B_1).
Then, for any $(A_2, B_2) \in \big[(A_1, B_1)\big]$:

$$\Gamma \circ \Gamma_{B_1 \to A_1} = \Gamma_{B_2 \to A_2} \circ \Gamma\,,$$

where $\Gamma(P) = EPE^T$ and E satisfies the equivalence relation.



Intrinsicness

- Direct consequence:
  - Domain adaptation via Ψ is invariant to the relative position of $\overline{P}^{(1)}$ and $\overline{P}^{(2)}$ on the manifold
  - It is constructed equivalently for every pair in the equivalence class

$$[(\overline{P}^{(1)}, \overline{P}^{(2)})]$$

- Guarantees consistence
  - For example, two subjects in two sessions in the BCI problem

Extension to Multiple Subsets

Algorithm.
Input: $\{\boldsymbol{P}_i^{(1)}\}_{i=1}^{N_1}, \{\boldsymbol{P}_i^{(2)}\}_{i=1}^{N_2}, \dots, \{\boldsymbol{P}_i^{(K)}\}_{i=1}^{N_K}$
Output: $\{\tilde{\boldsymbol{S}}_i^{(1)}\}_{i=1}^{N_1}, \{\tilde{\boldsymbol{S}}_i^{(2)}\}_{i=1}^{N_2}, \dots, \{\tilde{\boldsymbol{S}}_i^{(K)}\}_{i=1}^{N_K}$

1. **For** each $k \in \{1, 2, \dots, K\}$, compute $\overline{\boldsymbol{P}}^{(k)}$ the Riemannian mean of the subset $\{\boldsymbol{P}_i^{(k)}\}$.

2. Compute $\hat{\boldsymbol{P}}$, the Riemannian mean of $\{\overline{\boldsymbol{P}}^{(k)}\}_{k=1}^{K}$.

3. **For** all k and all i, apply Parallel Transport using:

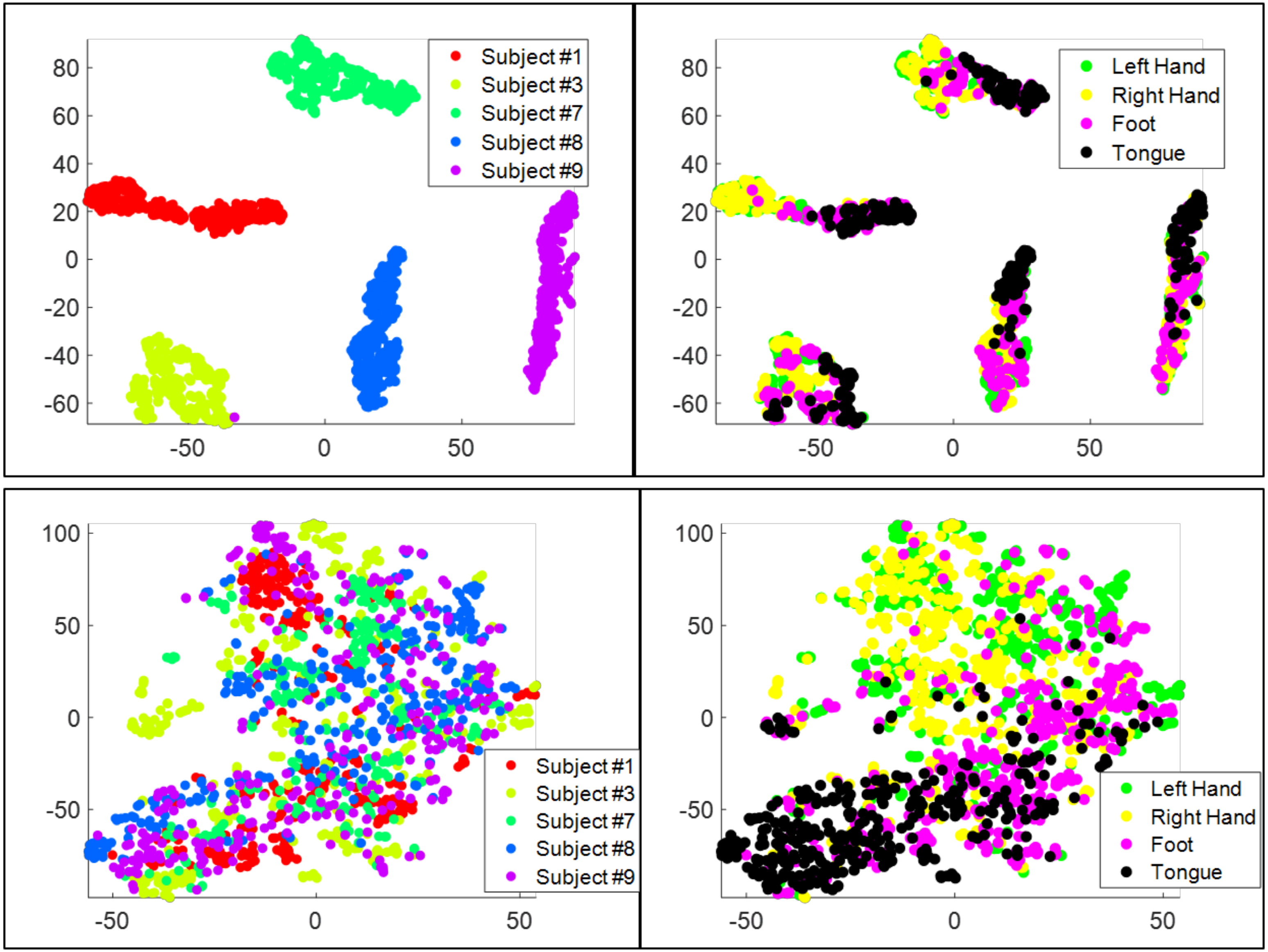
$$\boldsymbol{\Gamma}_i^{(k)} = \Gamma_{\overline{\boldsymbol{P}}^{(k)} \to \hat{\boldsymbol{P}}}\left(\boldsymbol{P}_i^{(k)}\right).$$

4. **For** all k and all i, project the transported matrix to the tangent space via:

$$\tilde{\boldsymbol{S}}_i^{(k)} = \log\left(\hat{\boldsymbol{P}}^{-\frac{1}{2}} \boldsymbol{\Gamma}_i^{(k)} \hat{\boldsymbol{P}}^{-\frac{1}{2}}\right).$$

Brain Computer Interface

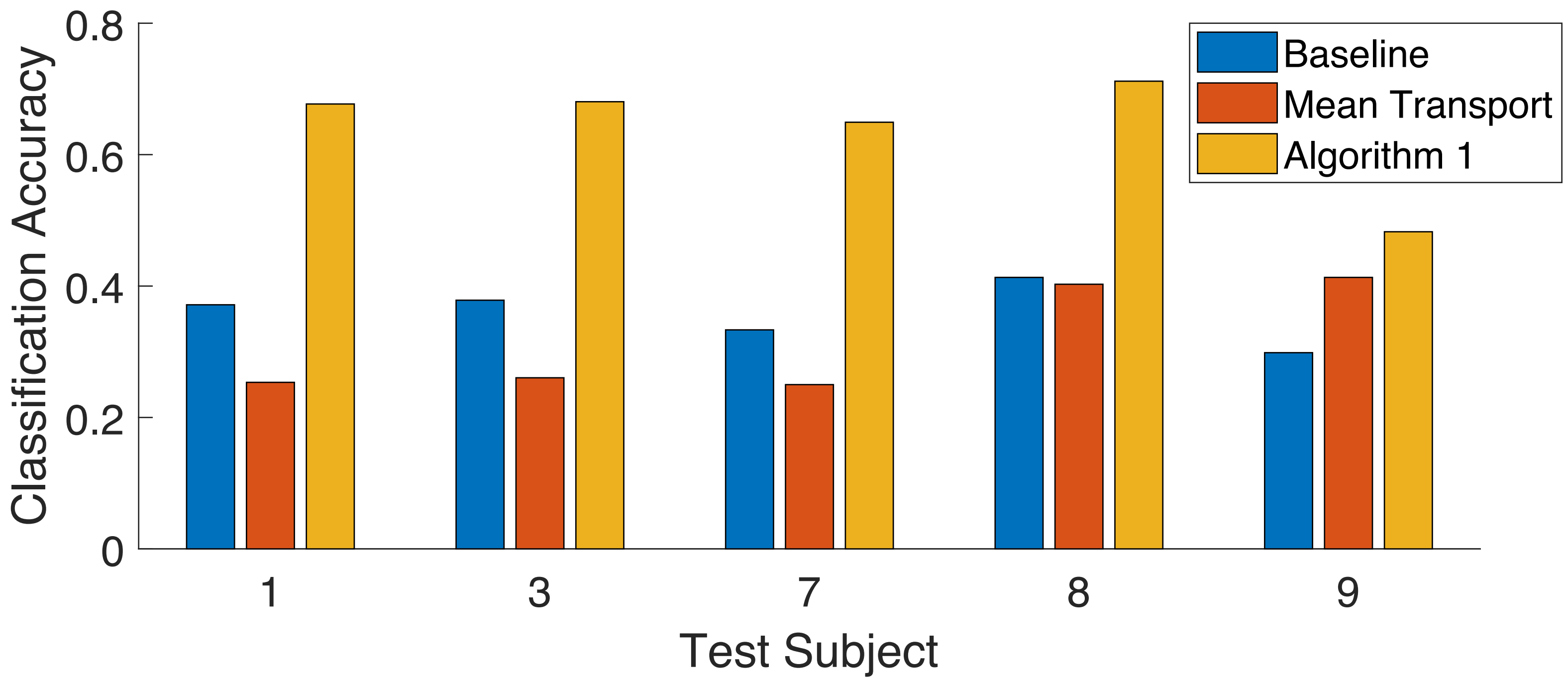
Five subjects



After PT

Brain Computer Interface

- Objective evaluation via classification
  - Leave-one-subject-out
  - Linear SVM



Sleep Stage Identification

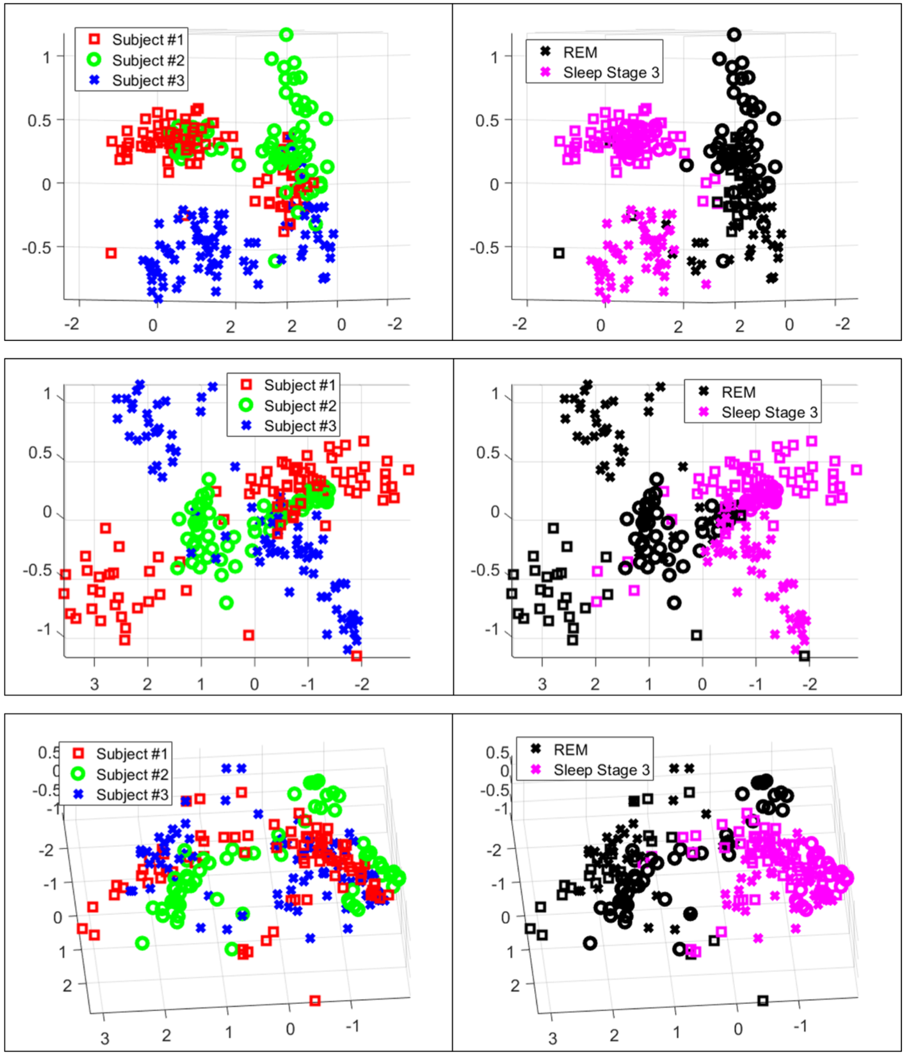
- Six different sleep stages:
  awake, REM, and sleep stages 1-4
- Recordings [PhysioNet.org]:
  - Two EEG channels
  - One electrooculography (EOG) channel
- Data from three subjects

Sleep Stage Identification

PCA of the covariance matrices[1]

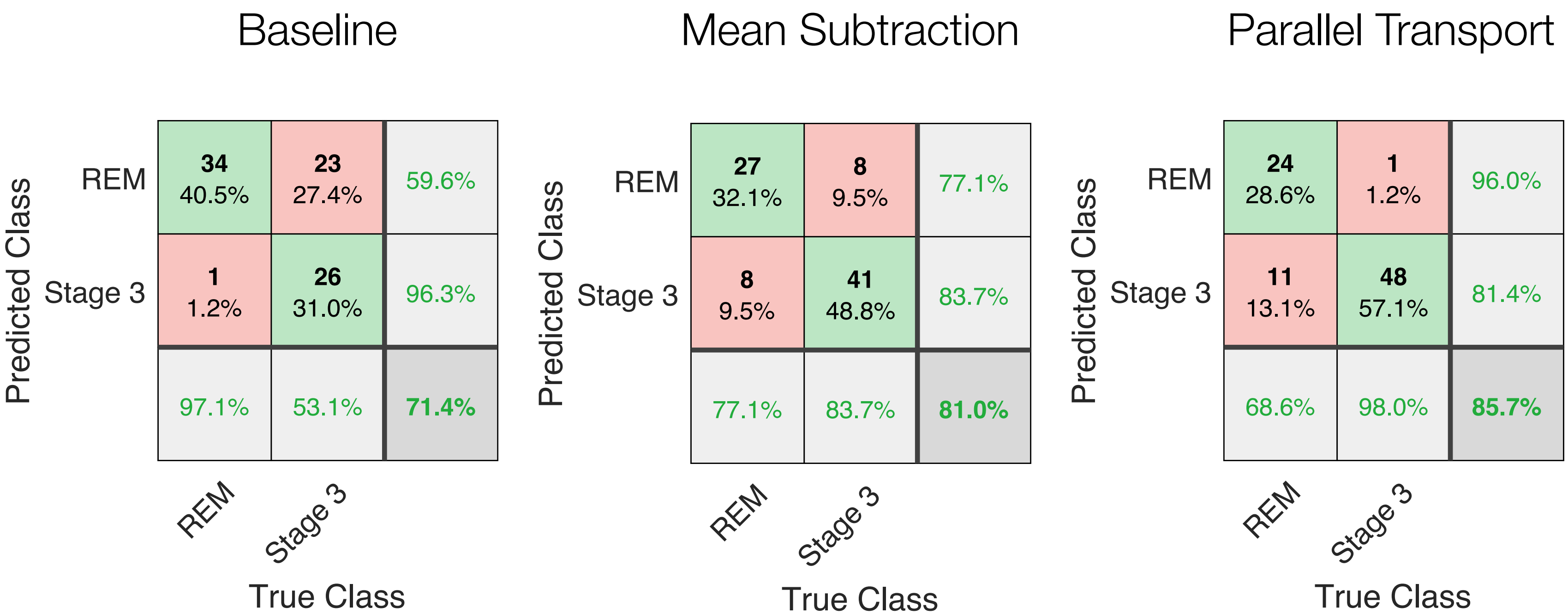
Mean subtraction

Parallel transport



[1]Since the covariance matrices are 3x3, dimension reduction using PCA was sufficient

Sleep Stage Identification

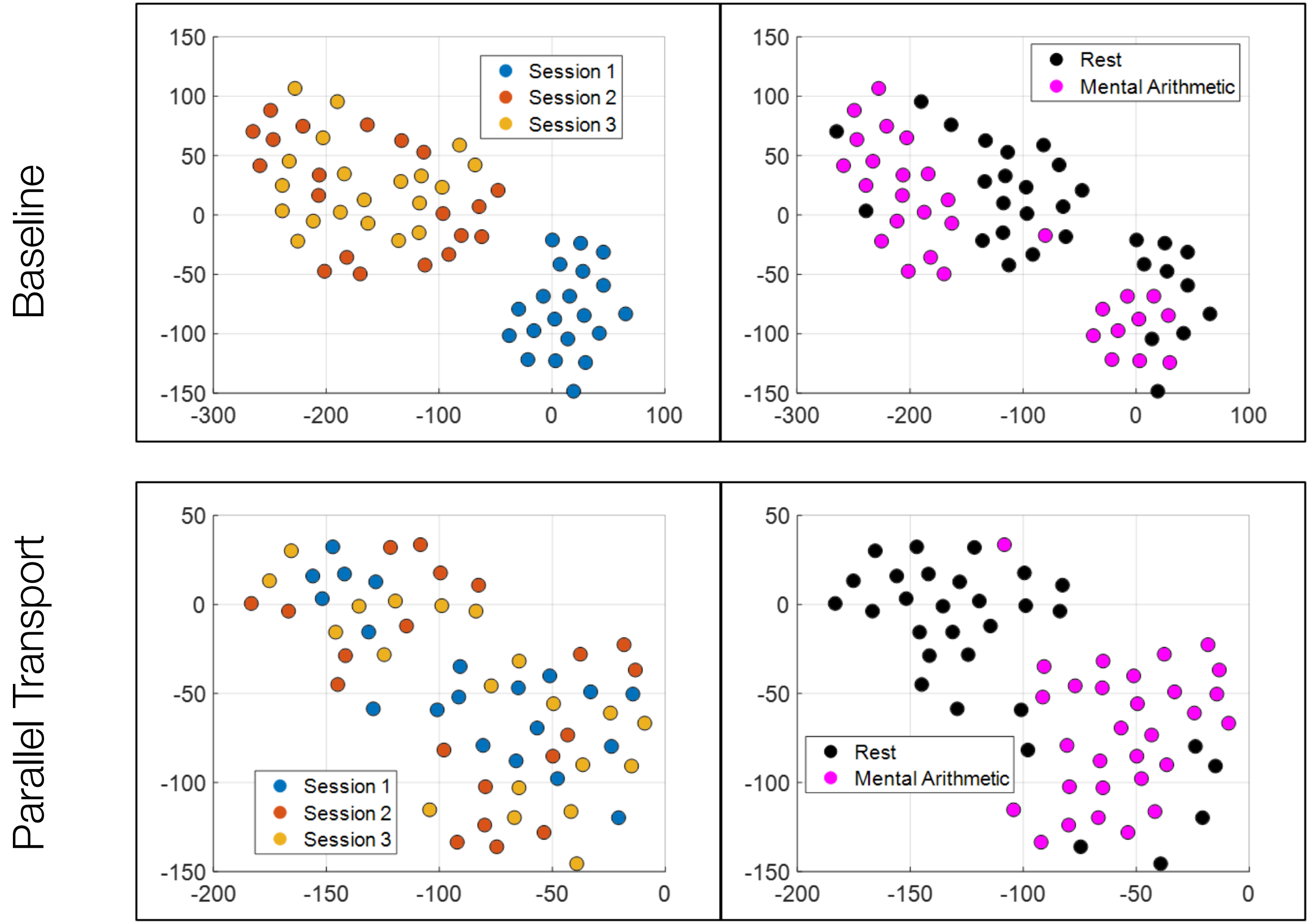


Mental Arithmetic Identification

- Recordings [Shin et al, 17]:
  - EEG from 29 subjects
  - 30 electrodes at 1000Hz
  - 3 sessions per subject
  - 20 repetitions/trials per session
- Two mental states:
  - Performing repeated simple arithmetic calculations
  - Baseline resting state

Mental Arithmetic Identification

tSNE representation of trials from subject #1



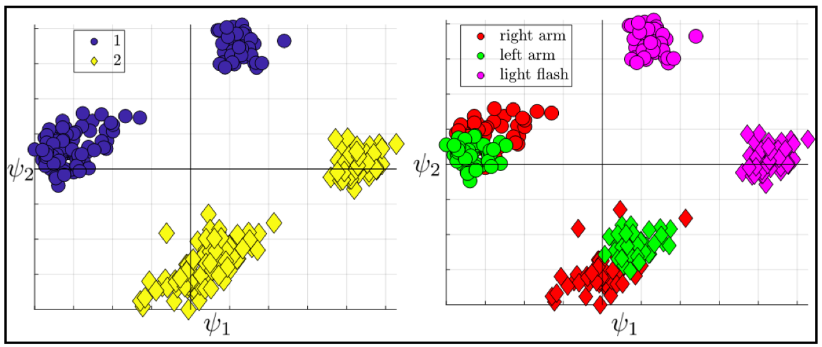
Mental Arithmetic Identification

- Average classification results over all 29 subjects
  - Leave-one-session-out cross-validation
  - Linear SVM

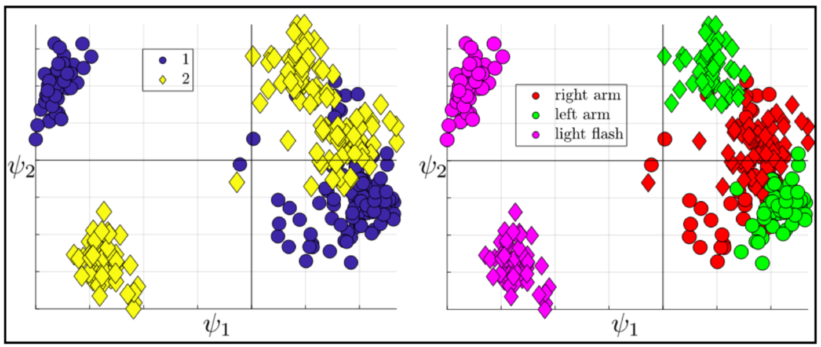
Baseline	Mean Subtraction	Parallel Transport
74%	73%	78%

Extensions and Outlook

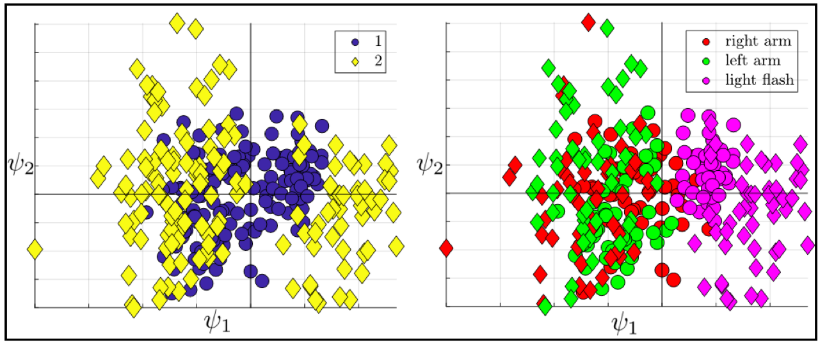
- To facilitate the internal structure of each subset:
  - Rotation following the PT
  - *Unsupervised moments alignment*



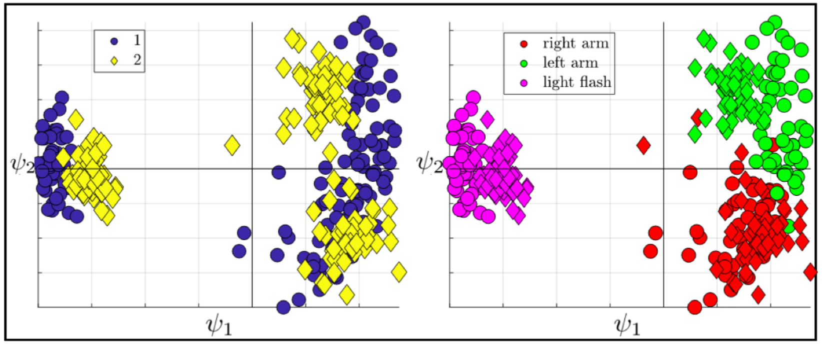
Baseline



Parallel Transport



Mean Subtraction



Parallel Transport & Moments Alignment

Extensions and Outlook

- Take home message
  - High-dimensional data live in a non-Euclidean space
  - Covariance matrices are informative features
  - They live in a non-Euclidean space with operations given in closed-form
- Covariance matrices might be insufficient features
- Instead, we could use:
  - Correlation and Partial Correlation matrices
  - Positive Kernels
  - Graph Laplacians
  - Transition probability matrices of random walks on graphs

Thank you

O. Yair, M. Ben-Chen and R. Talmon
"Parallel Transport on the Cone Manifold of SPD Matrices for Domain Adaptation"
IEEE Transactions on Signal Processing, 2019

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