Functional Map and Bases Design via ADMM

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Goal

Compute dense **correspondences** between shapes
Functional Maps

Linear alignment between functional spaces

Ovsjanikov et al. ‘12
Functional Maps

Find a matrix $C_{21}$ such that

$$C_{21}F_1 = F_2$$

where we assume

$$C_{21}(f_{1j}) \approx f_{1j} \circ \varphi \approx f_{2j}$$
Dimensionality Reduction

Find a matrix $C_{21}$ such that

$$C_{21} B_1^T F_1 \approx B_2^T F_2$$
Choice of basis

Eigenfunctions of the Laplace—Beltrami operator

$C_2 B_1^T F_1 = B_2^T F_2$
Basis Design

Find bases $B_1$ and $B_2$ such that

$$B_1^T F_1 \approx B_2^T F_2$$
Basis Design

Find bases $B_1$ and $B_2$ such that

$$\min_{B_1,B_2} \left| B_1^T F_1 - B_2^T F_2 \right|_F^2$$

s.t. $B_j^T G_j B_j = I$

$k \times m_1$

$k \times m_2$
Dimensionality Reduction

Find coefficient matrices $Q_1$ and $Q_2$ such that

$$\min_{Q_1, Q_2} \left| Q_1^T B_1^T F_1 - Q_2^T B_2^T F_2 \right|_F^2$$

s. t. $Q_j^T Q_j = I$
Coupled Quasi-Harmonic Bases

Find coefficient $Q_1$ and $Q_2$ such that

$$
\min_{Q_1, Q_2} \left| Q_1^T B_1^T F_1 - Q_2^T B_2^T F_2 \right|_F^2
$$

s.t. $Q_j^T Q_j = I$
Relation to Matrix Completion

Find a $k$–rank decomposition of $C_{21}$ such that

\[
\min_{Q_1, Q_2} \left| C_{21} B_1^T F_1 - B_2^T F_2 \right|_F^2
\]

s. t. $C_{21} = Q_2 Q_1^T$

Kovnatsky et al. ‘15
**FM vs. JD**

\[ C_{21} B_1^T F_1 \approx B_2^T F_2 \]

- Use first \( k \) basis elements
- metric/area consistency
- orientation...

\[ Q_1^T B_1^T F_1 \approx Q_2^T B_2^T F_2 \]

- Combines best \( k \) basis elements
- diagonalize \( L \)
- sparse p2p...

...
Should we use LB?
Should we use LB?

- Invariant to isometries
- Natural ordering
- Related to Fourier Analysis
- Related to a well studied discrete operator
- Aflalo et al. ‘15, ‘16
- …
Should we use LB?

Proper Orthogonal Decomposition (POD) modes:
Proper Orthogonal Decomposition

Based on the SVD of a matrix

\[ F = USV^* \]
LB vs. POD

\[
\sum_j (C_{21} B_1^T f_{1j} - B_2^T f_{2j})^2
\]
LB vs. POD

\[ \sum_j \left( C_{21} B_1^T f_{1j} - B_2^T f_{2j} \right)^2 \]

\[ \sum_j \left( B_2 C_{21} B_1^T f_{1j} - f_{2j} \right)^2 \]
Our Approach

We propose to

1. Solve for $C_{21}$ AND $Q_1$, $Q_2$

2. Use POD modes instead of LB

3. Regularize with consistency, smoothness and metric preservation
Our Approach

Find matrices $C_{21}, Q_1, Q_2 \ (C_{12})$ that minimize

$$\min \left| C_{21} Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2 \right|_F^2 +$$

$$\left| Q_1^T U_1^T F_1 - C_{12} Q_2^T U_2^T F_2 \right|_F^2 + \varepsilon_{iso} + \varepsilon_{dir}$$

s.t. $Q_j^T G_j Q_j = I, C_{pq} C_{qp} = I$
Fixed vs. Designed

\[ \sum_{j} (C_{21}B_{1}^{T}f_{1j} - B_{2}^{T}f_{2j})^2 \]

\[ \sum_{j} (B_{2}C_{21}B_{1}^{T}f_{1j} - f_{2j})^2 \]
Functional Map and Bases Design
Numerical Method

Our problem:

$$\min \left| C_{21} Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2 \right|_F^2 \quad \text{s.t.} \quad Q_j^T G_j Q_j = I$$
Numerical Method

Our problem:

\[
\min \left| C_{21} Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2 \right|_F^2 \quad \text{s.t.} \quad Q_j^T G_j Q_j = I
\]

An Alternating Direction Method of Multipliers (ADMM) version:

\[
\min \left| C_{21} Q_1^T U_1^T F_1 - Q_2^T U_2^T F_2 \right|_F^2 \\
\text{s.t.} \quad Q_j^T G_j Q_j = I, \quad Q_j = Q_j'
\]
An ADMM Approach

Algorithm:

For $k = 0, 1, 2, \ldots$ do

1. Solve a Sylvester-type equation $x = 2$
2. Solve a linear equation $x = 3$
3. Update the dual variables
Empirical Convergence
Provably Convergent Scheme

We consider the minimization

$$\min \mathcal{G}(X) + \mathcal{H}(Z)$$
$$s.t. \ P(X) + Q(Z) = 0$$

where

$$\mathcal{G}(X) = \left| C_{21} \tilde{Q}_1^T U_1^T F_1 - \tilde{Q}_2^T U_2^T F_2 \right|^2_F$$
$$\mathcal{H}(Z) = \left| Z - I \right|^2_F + \left| Q_j'' \right|^2_F + \left| \tilde{Q}_j'' \right|^2_F$$
$$P(X) = \begin{pmatrix} Q_j^T G_j Q_j' \\ Q_j - Q_j' \\ Q_j - \tilde{Q}_j' \end{pmatrix}, Q(Z) = \begin{pmatrix} -Z \\ -Q_j'' \\ -\tilde{Q}_j'' \end{pmatrix}$$

Gao et al. ‘18
Results
Shape Correspondences
Shape Correspondences

GT
FMAPS
AJD
CFM
DPC
Ours
Comparison to AJD

AJD

Ours
Comparison to AJD

GT  AJD  Ours
Joint Quadrangulation

Fixed LB

Designed POD
Function Transfer

Standard Transfer  Product Transfer

Nogneng et al. ‘18
Future Work

• Dependencies between constraints and bases

• Design a basis on a single shape
Self Functional Maps
Non Linear Dynamics
Gang Reduction in Youth

Personality questionnaires taken every 6 months

Q: I get very angry and “lose my temper”

Q: It is okay to beat people up if they hit me first

Q: Attacked someone with a weapon?
Gang Reduction in Youth
Transient Growth

$E^k$

$k$

$A_{12}$
$A_{23}$
$A_{13}$
Conclusions

• Design a basis based on POD modes

• Exploit established regularizers on fmaps

• Solve efficiently via ADMM

• Similar provably convergent scheme
Thank you!