Shape Analysis Based on Computational Conformal Geometry

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Geometry and Learning from Data in 3D and Beyond
Workshop II: Shape Analysis

Thanks

Thanks for the invitation.

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

Motivation

Conformal geometry lays down the theoretic foundation for

- Surface mapping
- Geometry classification
- Shape analysis

Applied in computer graphics, computer vision, geometric modeling, wireless sensor networking and medical imaging, and many other engineering, medical fields.

History

- In pure mathematics, conformal geometry is the intersection of complex analysis, algebraic topology, Riemann surface theory, algebraic curves, differential geometry, partial differential equation.
- In applied mathematics, computational complex function theory has been developed, which focuses on the conformal mapping between planar domains.
- Recently, computational conformal geometry has been developed, which focuses on the conformal mapping between surfaces.

History

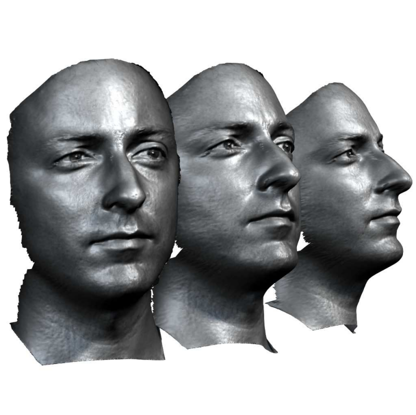
Conventional conformal geometric method can only handle the mappings among planar domains.

- Applied in thin plate deformation (biharmonic equation)
- Membrane vibration
- Electro-magnetic field design (Laplace equation)
- Fluid dynamics
- Aerospace design

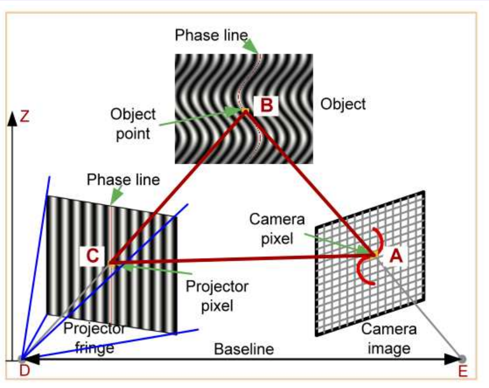
Reasons for Booming

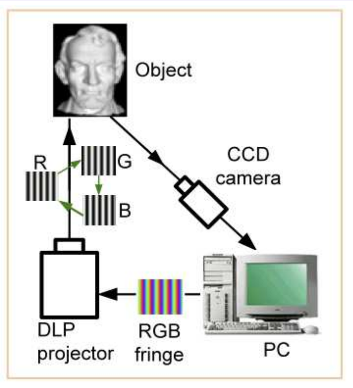
Data Acquisition

3D scanning technology becomes mature, it is easier to obtain surface data.

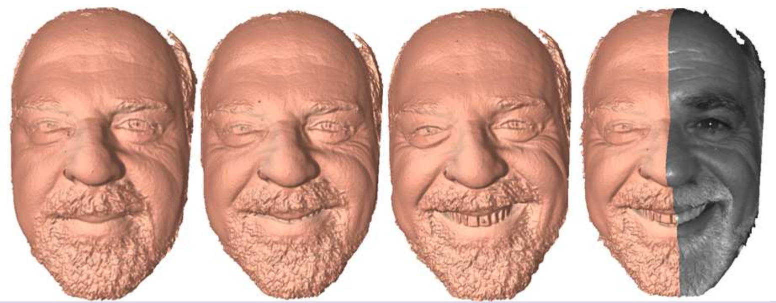


System Layout

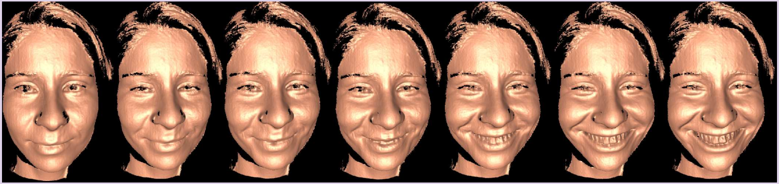




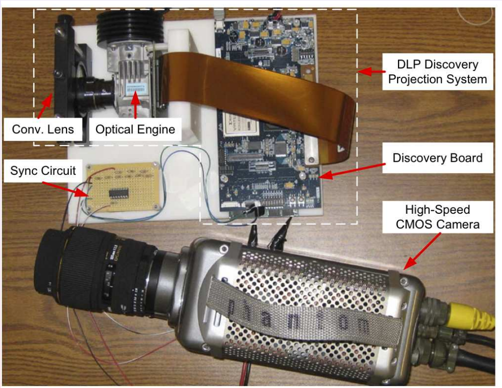
3D Scanning Results

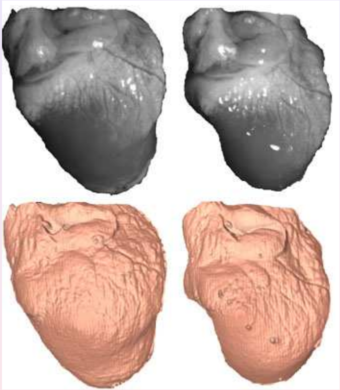


3D Scanning Results



System Layout





Reasons for Booming

Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

Computational Power

Computational power has been increased tremendously. With the incentive in graphics, GPU becomes mature, which makes numerical methods for solving PDE's much easier.

Fundamental Problems

1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.
2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmuller shape space.
3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics
4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.
5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.
6. Compute the extremal quasi-conformal maps.
7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.

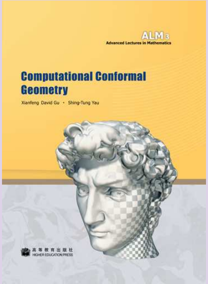
Complete Tools

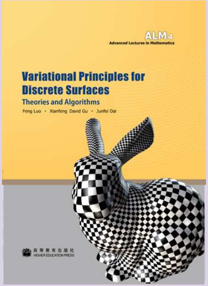
Computational Conformal Geometry Library

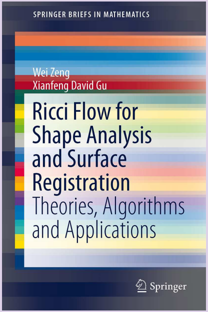
1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation

Books

The theory, algorithms and sample code can be found in the following books.







You can find them in the book store.

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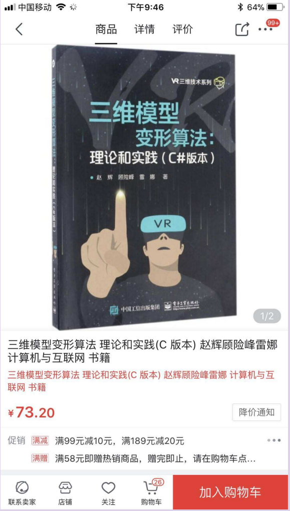
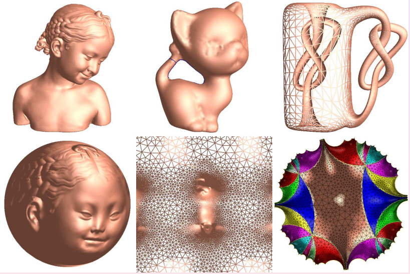
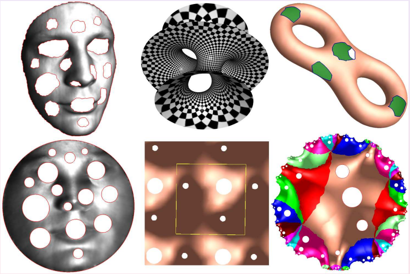


Source Code Library

Please email me gu@cs.sunysb.edu for updated code library on computational conformal geometry.





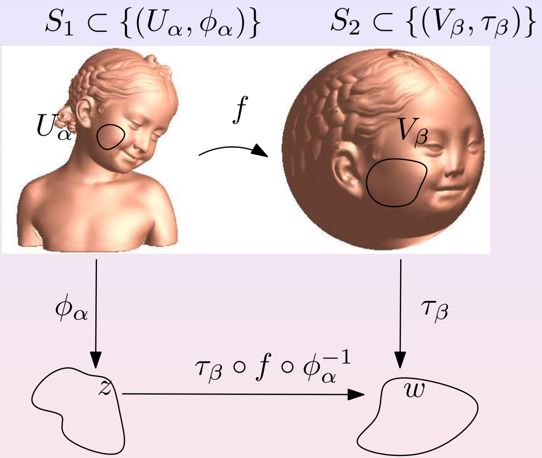
Conformal Mapping

biholomorphic Function

Definition (biholomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then then f is a biholomorphic function.

Conformal Map



The restriction of the mapping on each local chart is biholomorphic, then the mapping is conformal.

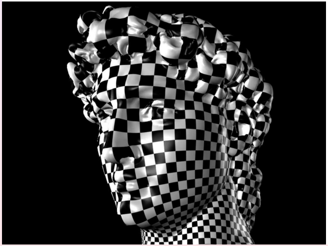
Conformal Mapping

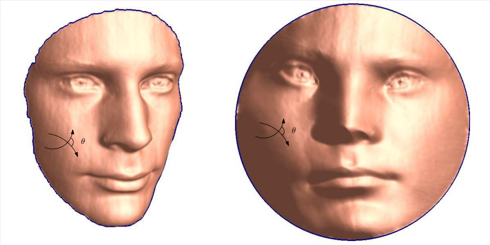
Definition (Conformal Map)

Let $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$ is a homeomorphism, ϕ is conformal if and only if

$$\phi^* \mathbf{g}_2 = e^{2u} \mathbf{g}_1.$$

Conformal Mapping preserves angles.





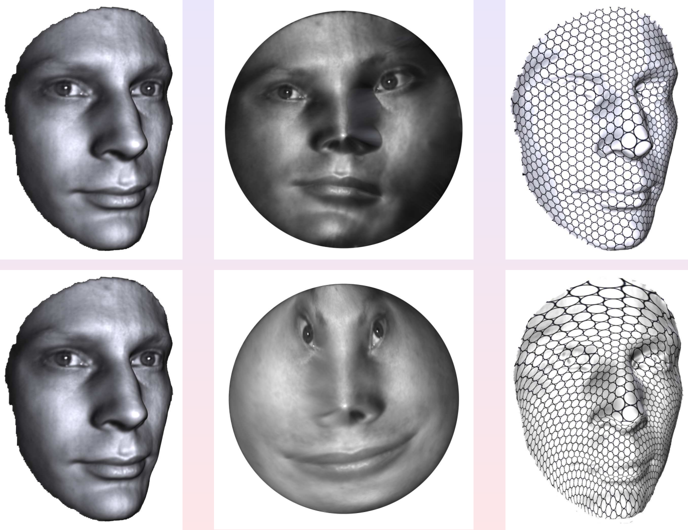
Conformal Mapping

Conformal maps Properties

Map a circle field on the surface to a circle field on the plane.

Quasi-Conformal Map

Diffeomorphisms: maps ellipse field to circle field.

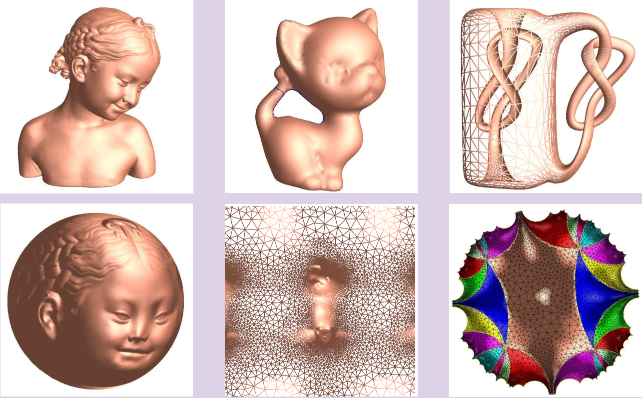


Uniformization

Conformal Canonical Representations

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ *be a compact 2-dimensional Riemannian manifold. Then there is a metric* $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ *conformal to* $\mathbf{g}$ *which has constant Gauss curvature.*



Spherical Euclidean Hyperbolic

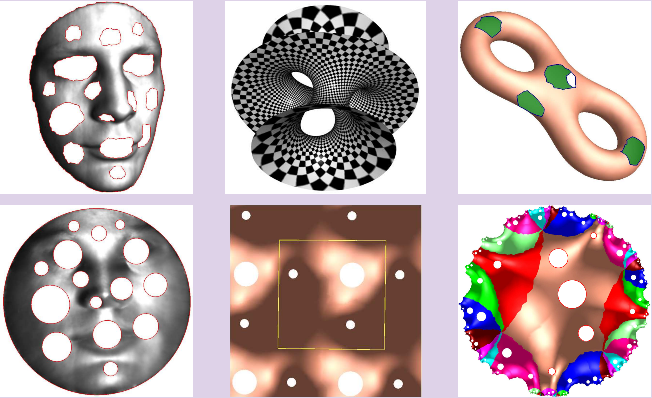
Definition (Circle Domain)

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain Ω in $\hat{\mathbb{C}}$, whose boundary $\partial\Omega$ has at most countably many components, is conformally homeomorphic to a circle domain Ω^* in $\hat{\mathbb{C}}$. Moreover Ω^* is unique upto Möbius transformations, and every conformal automorphism of Ω^* is the restriction of a Möbius transformation.

Uniformization of Open Surfaces

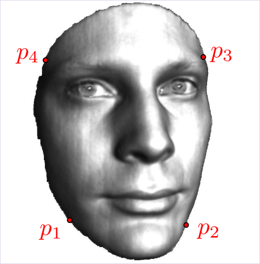


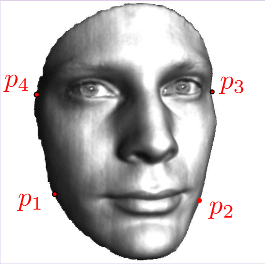
Spherical Euclidean Hyperbolic

Simply Connected Domains

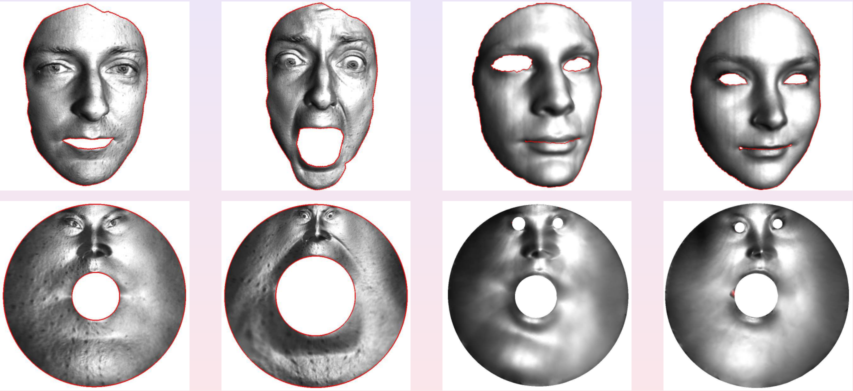
Conformal Canonical Forms

Topological Quadrilateral



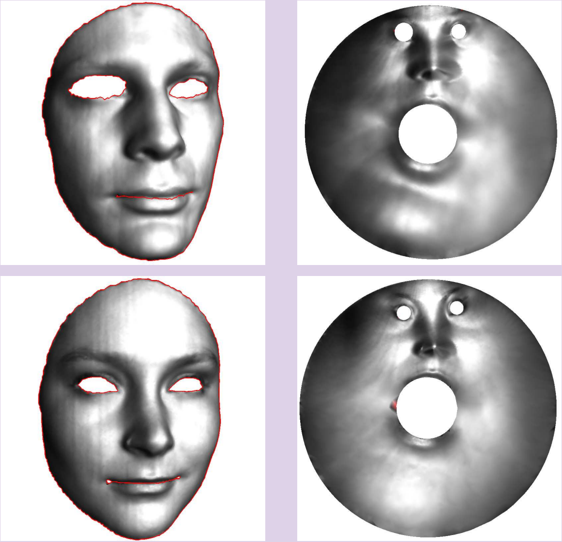


Multiply Connected Domains



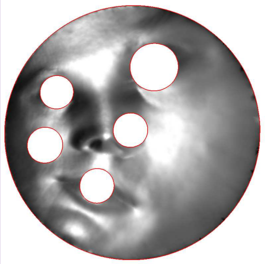
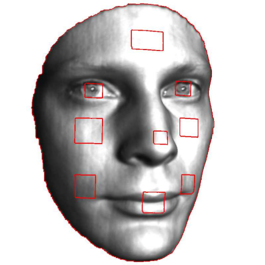
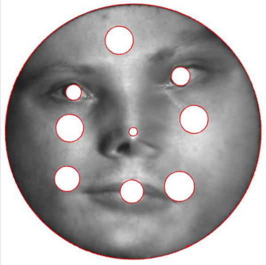
Conformal Canonical Forms

Multiply Connected Domains

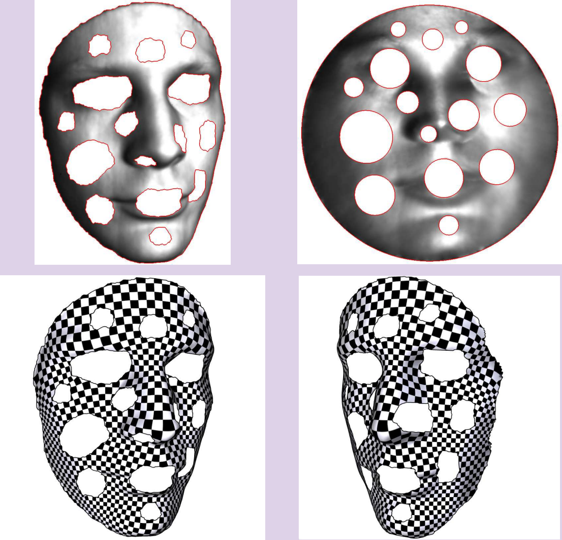


Conformal Canonical Forms

Multiply Connected Domains

Multiply Connected Domains



Conformal Canonical Representations

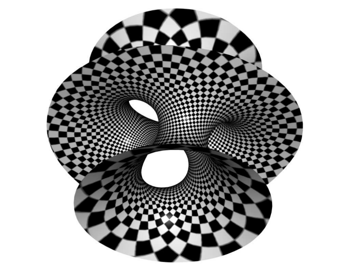
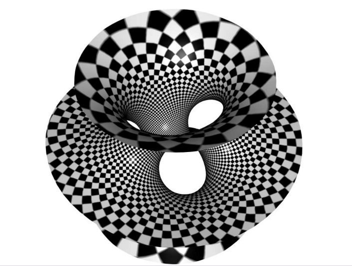
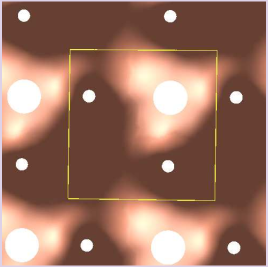
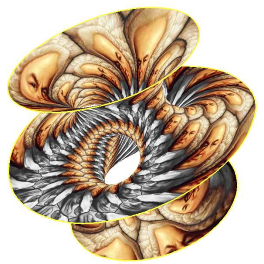
Definition (Circle Domain in a Riemann Surface)

A circle domain in a Riemann surface is a domain, whose complement's connected components are all closed geometric disks and points. Here a geometric disk means a topological disk, whose lifts in the universal cover or the Riemann surface (which is $\mathbb{H}^2$, $\mathbb{R}^2$ or $\mathbb{S}^2$ are round.

Theorem

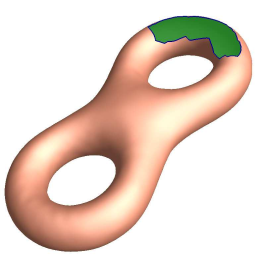
Let Ω be an open Riemann surface with finite genus and at most countably many ends. Then there is a closed Riemann surface R^* such that Ω is conformally homeomorphic to a circle domain Ω^* in R^*. More over, the pair (R^*, Ω^*) is unique up to conformal homeomorphism.

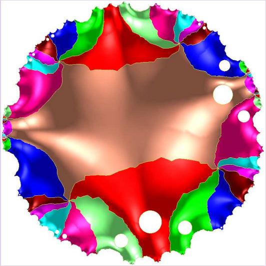
Tori with holes

Conformal Canonical Form

High Genus Surface with holes





Teichmüller Space

Definition (Conformal Mapping)

Suppose $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are two metric surfaces, $\phi : S_1 \to S_2$ is conformal, if on S_1

$$\mathbf{g}_1 = e^{2\lambda} \phi^* \mathbf{g}_2,$$

where $\phi^* \mathbf{g}_2$ is the pull-back metric induced by ϕ.

Definition (Conformal Equivalence)

Suppose two surfaces S_1, S_2 with marked homotopy group generators, $\{a_i, b_i\}$ and $\{\alpha_i, \beta_i\}$. If there exists a conformal map $\phi : S_1 \to S_2$, such that

$$\phi_*[a_i] = [\alpha_i], \phi_*[b_i] = [\beta_i],$$

then we say two marked surfaces are conformal equivalent.

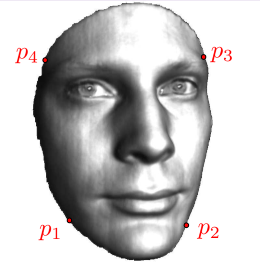
Definition (Teichmüller Space)

Fix the topology of a marked surface S, all conformal equivalence classes sharing the same topology of S, form a manifold, which is called the Teichmüller space of S. Denoted as T_S.

- Each point represents a class of surfaces.
- A path represents a deformation process from one shape to the other.
- The Riemannian metric of Teichmüller space is well defined.

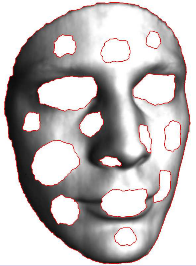
Teichmüller Space

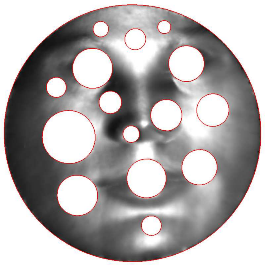
Topological Quadrilateral



Conformal module: $\frac{h}{w}$. The Teichmüller space is 1 dimensional.

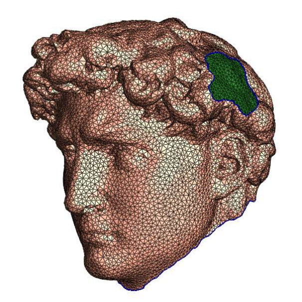
Multiply Connected Domains

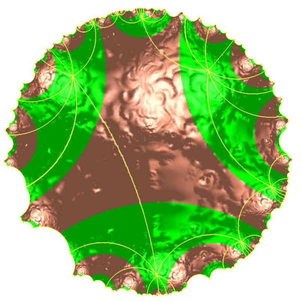




Conformal Module : centers and radii, with Möbius ambiguity. The Teichmüller space is $3n-3$ dimensional, n is the number of holes.

Topological Pants

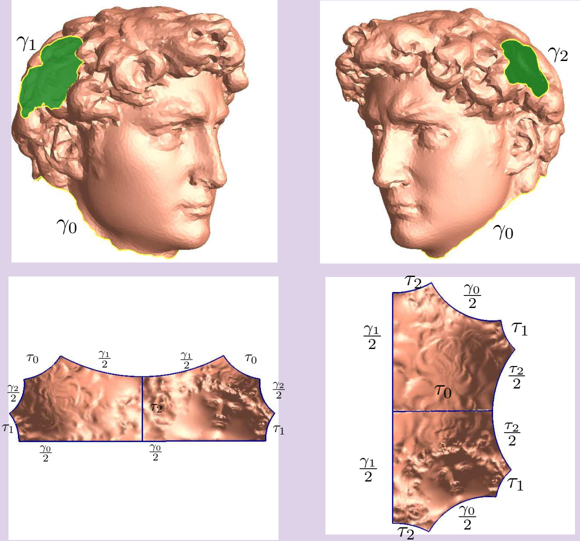




Genus 0 surface with 3 boundaries is conformally mapped to the hyperbolic plane, such that all boundaries become geodesics.

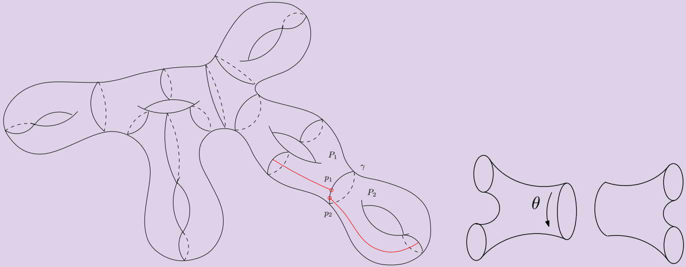
Teichmüller Space

Topological Pants



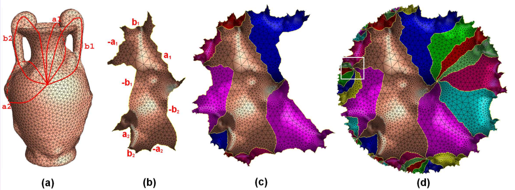
Teichmüller Space

Topological Pants Decomposition - $2g-2$ pairs of Pants

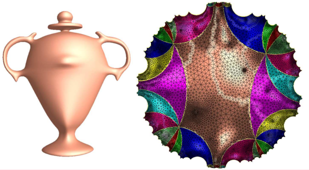


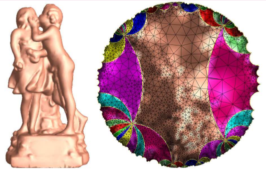
Compute Teichmüller coordinates

Step 1. Compute the hyperbolic uniformization metric.

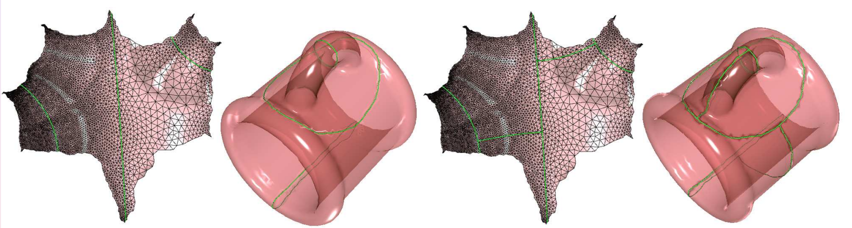


Step 2. Compute the Fuchsian group generators.





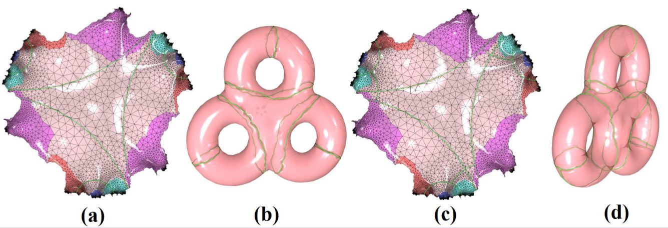
Step 3. Pants decomposition using geodesics and compute the twisting angle.



Teichmüller Coordinates

Compute Teichmüller coordinates

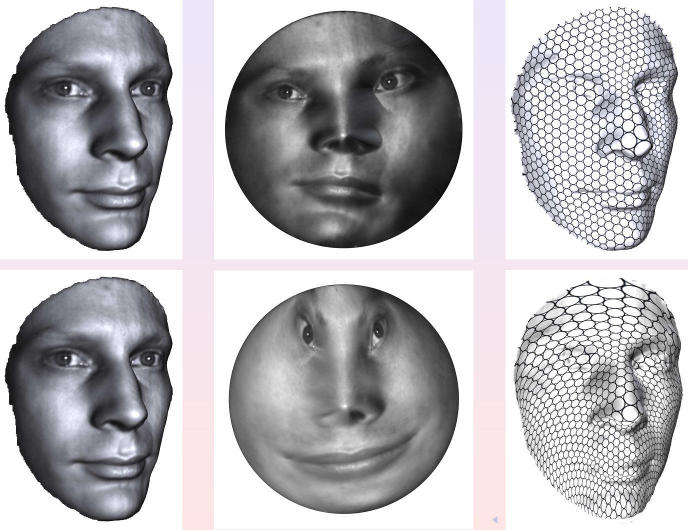
Compute the pants decomposition using geodesics and compute the twisting angle.



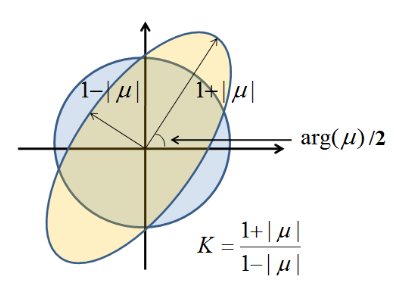
Quasi-Conformal Maps

Quasi-Conformal Map

Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.



Beltrami-Equation



Beltrami Coefficient

Let $\phi : S_1 \to S_2$ be the map, z, w are isothermal coordinates of S_1, S_2, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$

Diffeomorphism Space

Theorem

Given two genus zero metric surface with a single boundary,

$$\{Diffeomorphisms\} \cong \frac{\{Beltrami\ Coefficient\}}{\{Mobius\}}.$$

The problem of computing Quasi-conformal map is converted to compute a conformal map.

Solveing Beltrami Equation

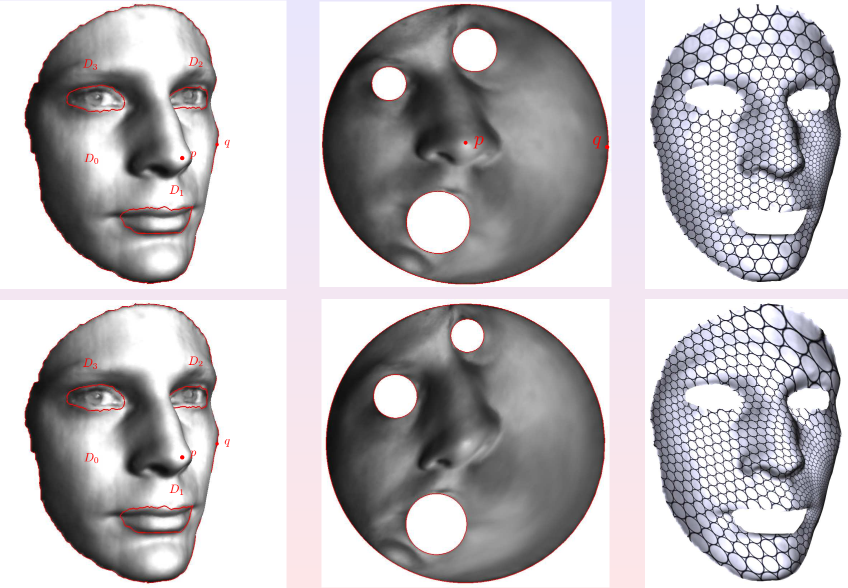
Given metric surfaces $(S_1,\mathbf{g}_1)$ and $(S_2,\mathbf{g}_2)$, let z,w be isothermal coordinates of S_1,S_2,$w=\phi(z)$.

$$\mathbf{g}_1 = e^{2u_1}dzd\bar{z} \tag{1}$$
$$\mathbf{g}_2 = e^{2u_2}dwd\bar{w}, \tag{2}$$

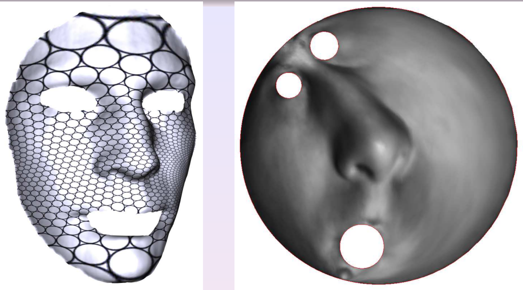
Then

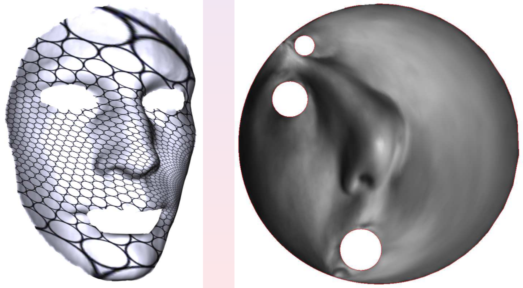
- $\phi:(S_1,\mathbf{g}_1)\to(S_2,\mathbf{g}_2)$, quasi-conformal with Beltrami coefficient μ.
- $\phi:(S_1,\phi^*\mathbf{g}_2)\to(S_2,\mathbf{g}_2)$ is isometric
- $\phi^*\mathbf{g}_2 = e^{u_2}|dw|^2 = e^{u_2}|dz+\mu d\bar{z}|^2$.
- $\phi:(S_1,|dz+\mu d\bar{z}|^2)\to(S_2,\mathbf{g}_2)$ is conformal.

Quasi-Conformal Map Examples



Quasi-Conformal Map Examples





Applications

Computer Graphics

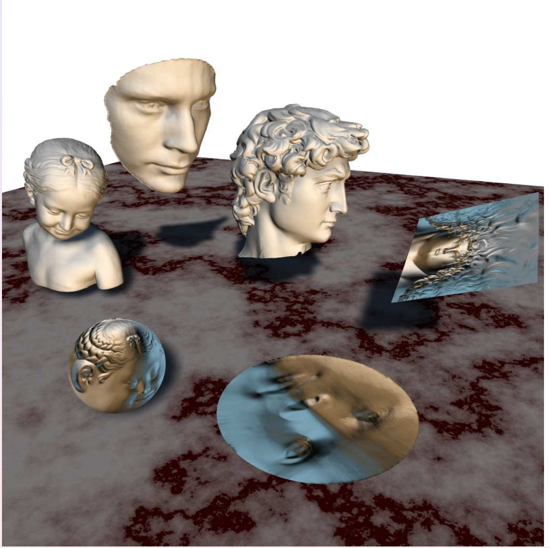
Computer Graphics Application

Graphics

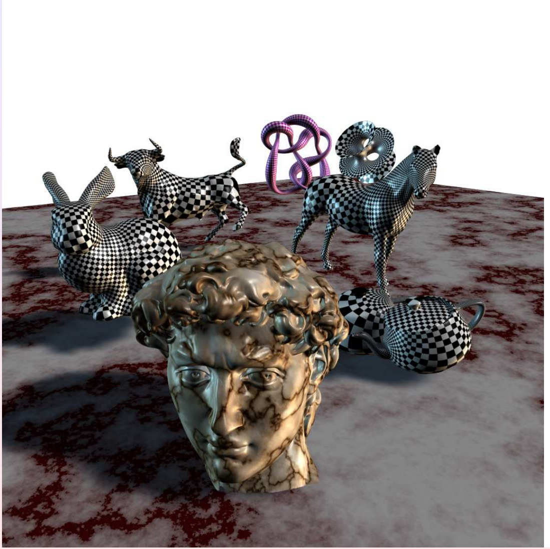
- Surface Parameterization, texture mapping
- Texture synthesis, transfer
- Vector field design
- Shape space and retrieval.

Surface Parameterization

Map the surfaces onto canonical parameter domains

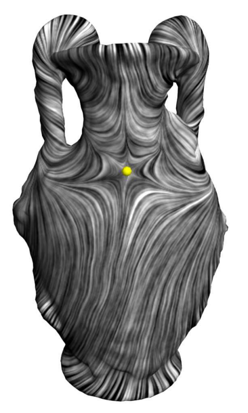


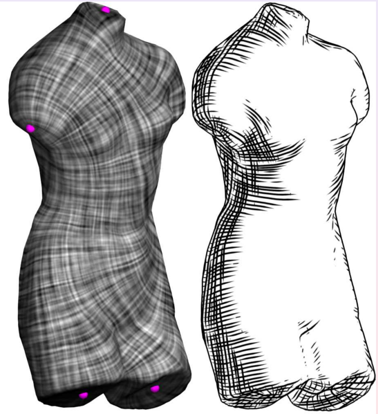
Applied for texture mapping.



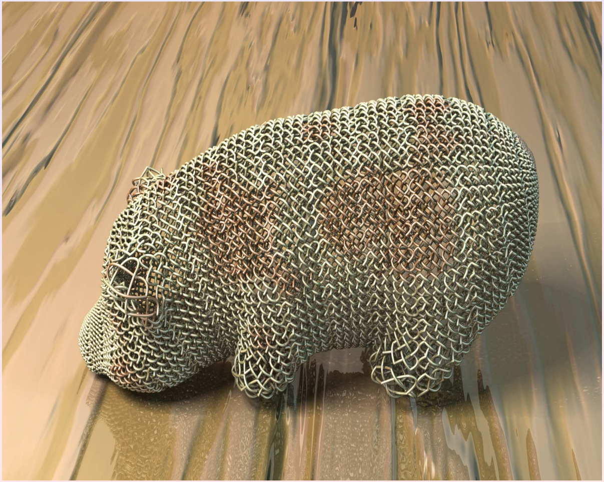
n-Rosy Field Design

Design vector fields on surfaces with prescribed singularity positions and indices.



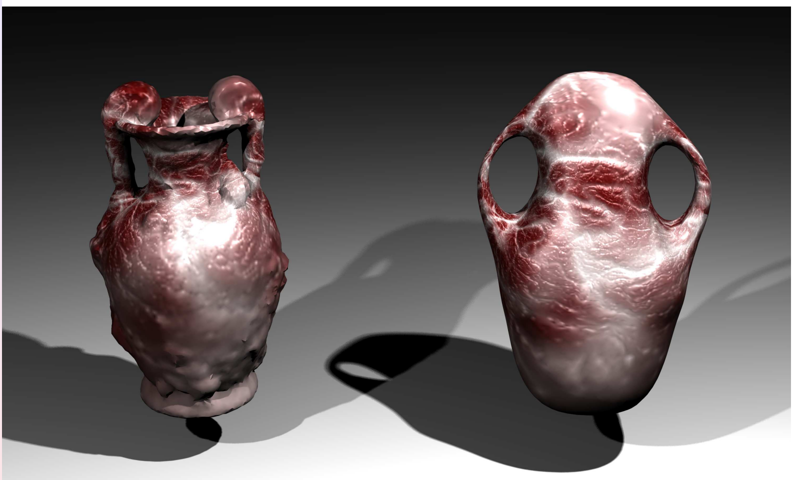


Convert the surface to knot structure using smooth vector fields.



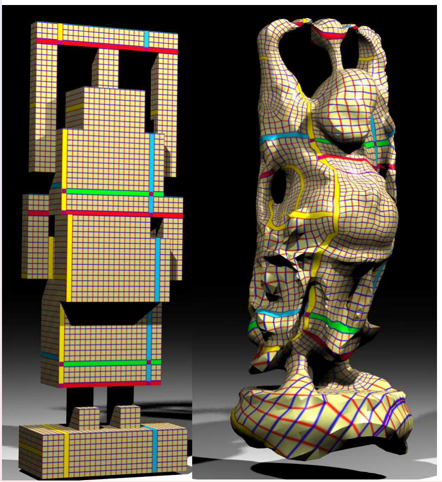
Texture Transfer

Transfer the texture between high genus surfaces.



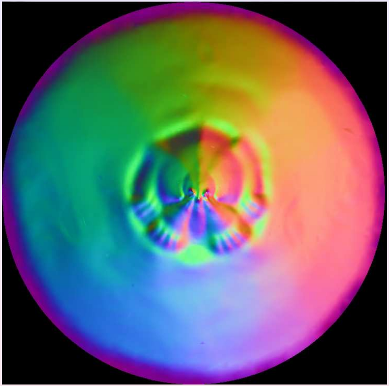
Polycube Map

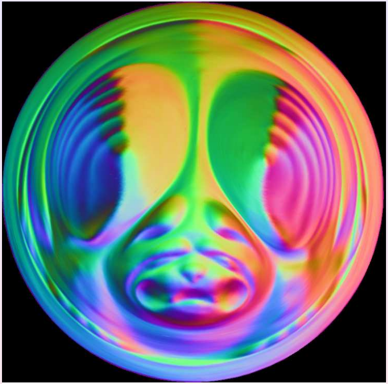
Compute polycube maps for high genus surfaces.



Normal Map

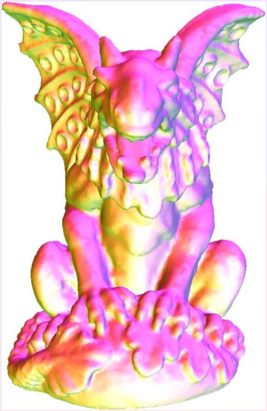
Normal Map

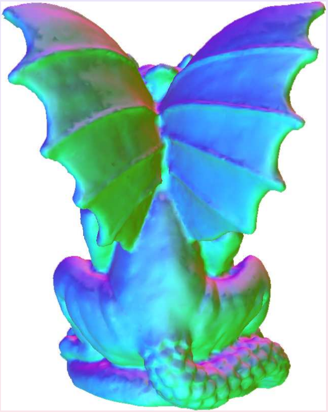




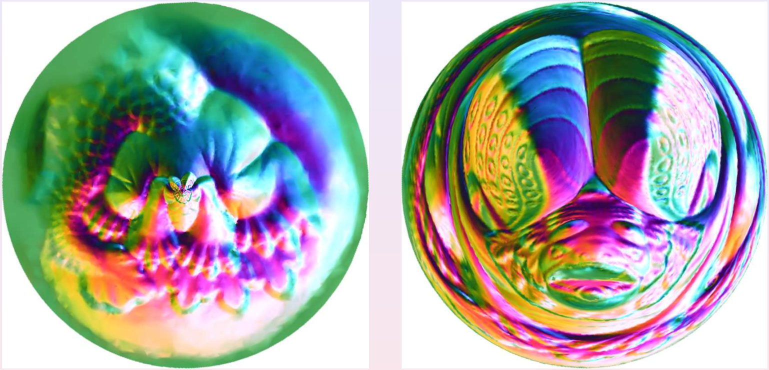
- Quasi-conformal geometry controls angle-distortion;
- Optimal mass transportation map controls area-distortion;

Normal Map





Normal Map



Volumetric Parameterization

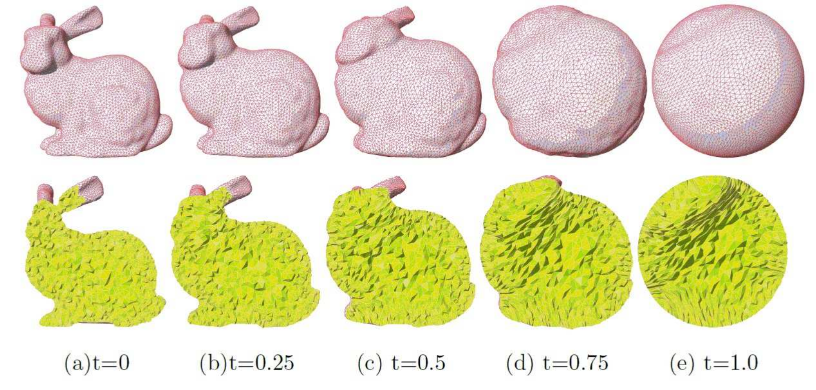


Figure: Volumetric morphing using our method.

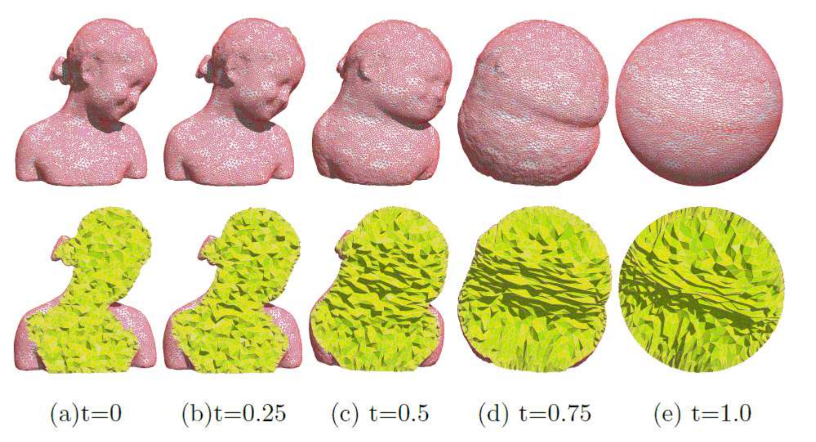


Figure: Volumetric morphing using our method.

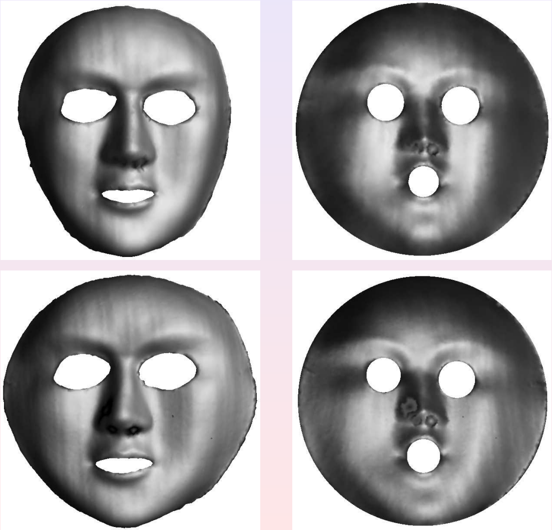
Computer Vision

Computer Vision Application

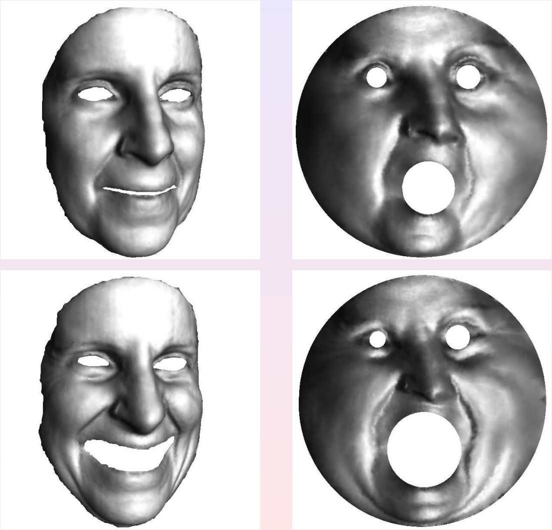
Vision

- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Tracking.

Isometric deformation is conformal. The mask is bent without stretching.

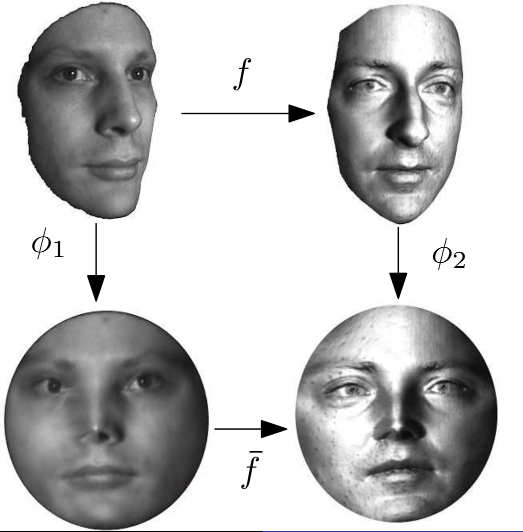


Facial expression change is not-conformal.

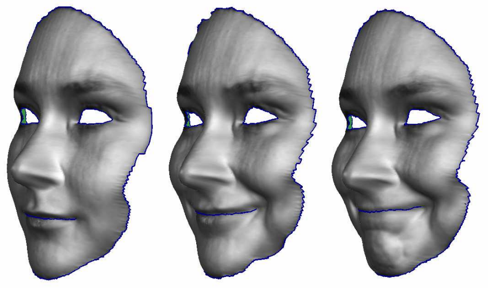


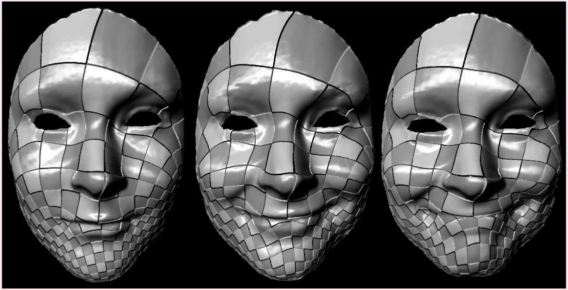
Surface Matching

3D surface matching is converted to image matching by using conformal mappings.

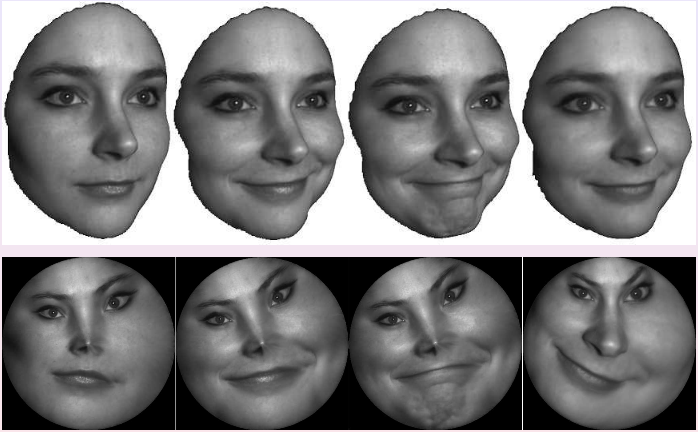


Face Surfaces with Different Expressions are Matched

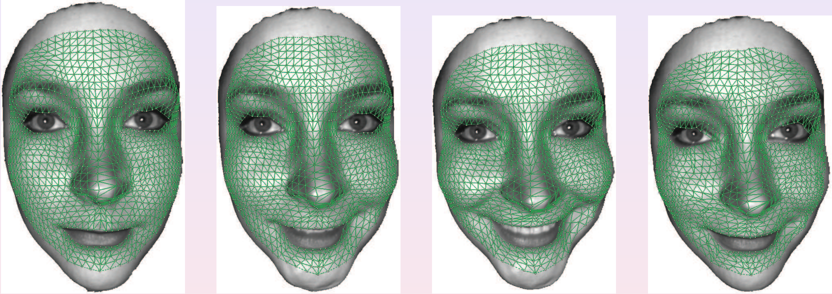




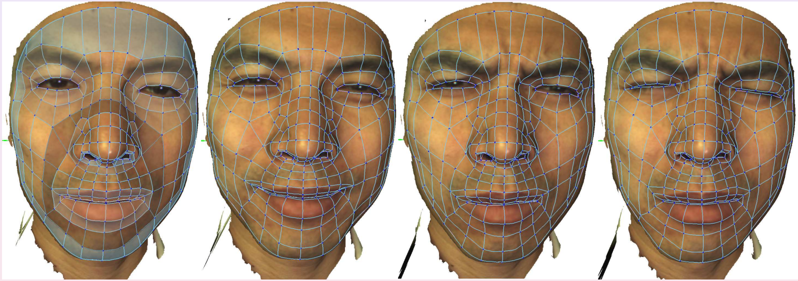
Face Expression Tracking



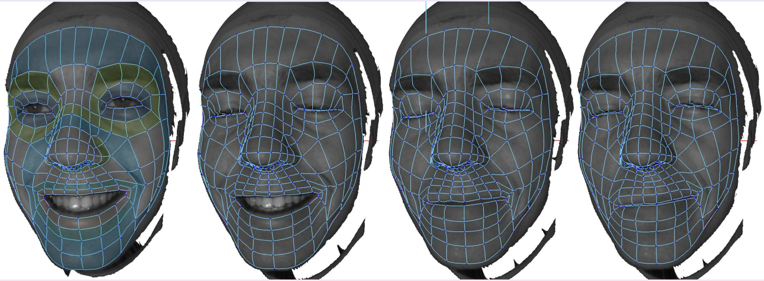
Face Expression Tracking



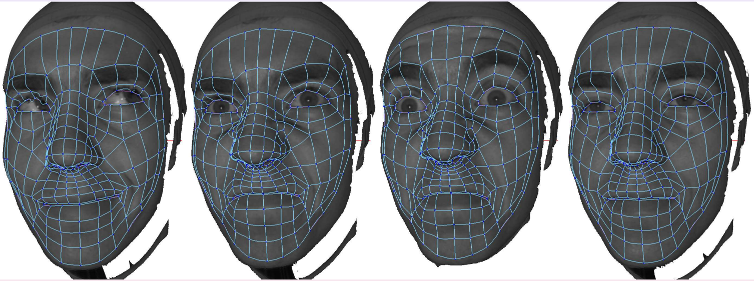
Single Mesh for facial expression transfer.



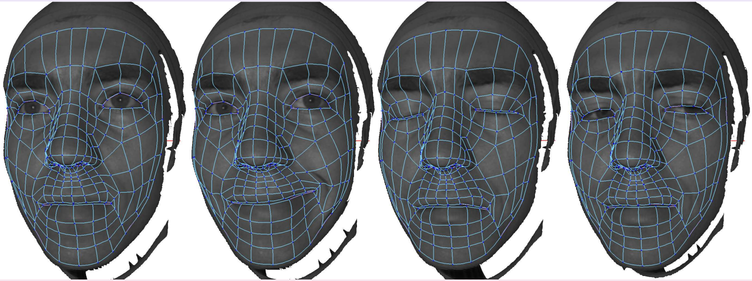
Face Expression Tracking



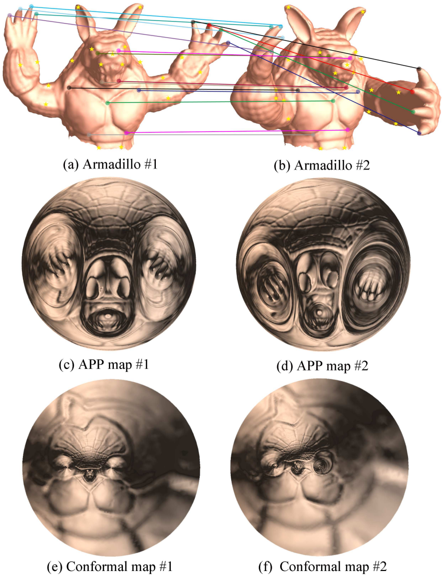
Face Expression Tracking



Face Expression Tracking



Surface Registration



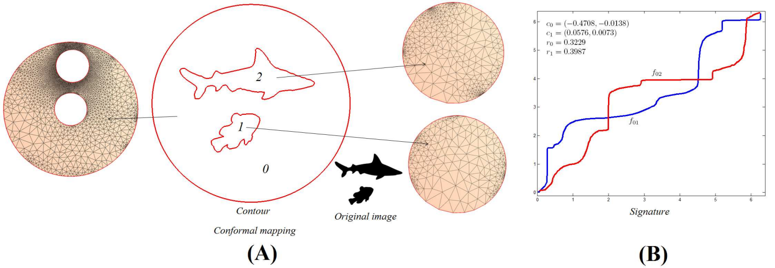
(a) Armadillo #1 (b) Armadillo #2

(c) APP map #1 (d) APP map #2

(e) Conformal map #1 (f) Conformal map #2

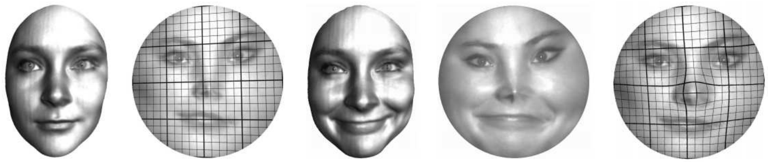
2D Shape Space-Conformal Welding

$$\{2D\ Contours\} \cong \frac{\{Diffeomorphism\ on\ S^1\} \cup \{Conformal\ Module\}}{\{Mobius\ Transformation\}}$$



Wasserstein Distance

Given a metric surface $(S, \mathbf{g})$, a Riemann mapping $\varphi : (S, \mathbf{g}) \to \mathbb{D}^2$, the conformal factor $e^{2\lambda}$ gives a probability measure on the disk. The shape distance is given by the Wasserstein distance.



(a) (b) (c) (d) (e)

Fig. 9: **The computation of Wasserstein distance**

Expression Classification

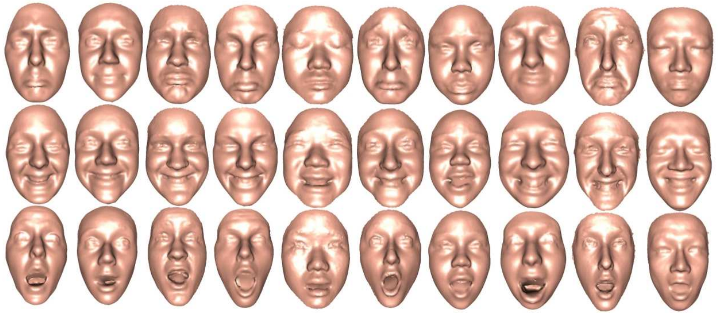
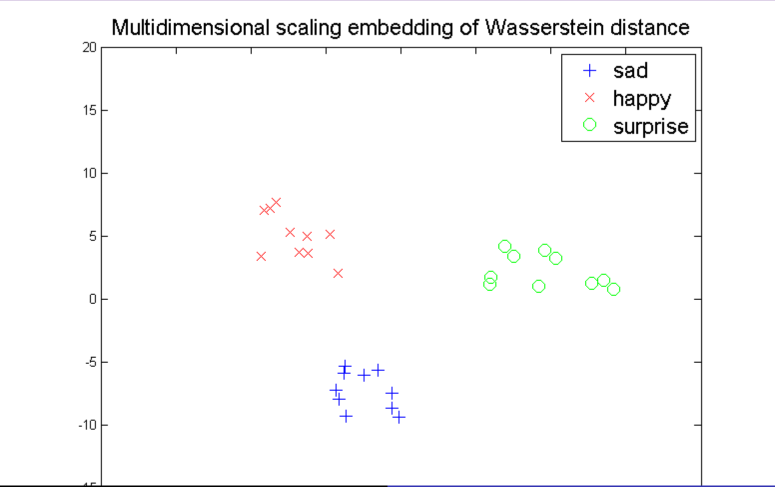


Fig. 10: **Face surfaces for expression clustering. The first row is “sad”, the second row is “happy” and the third row is “surprise”.**

Expression Classification

Compute the Wasserstein distances among all the facial surfaces, isometrically embed on the plane using MDS method, perform clustering.

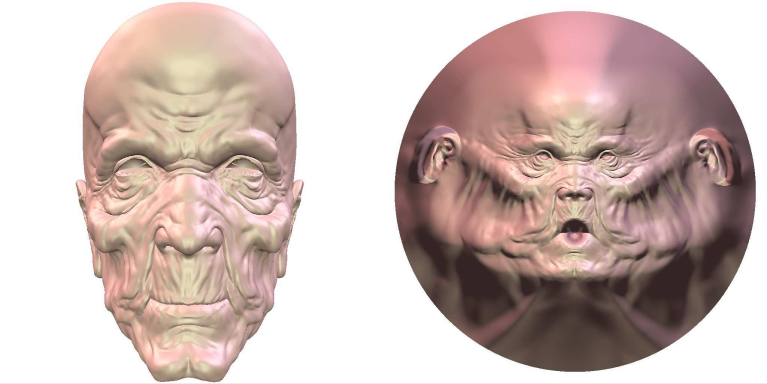


Curvature Sensitive Remeshing

Algorithm Pipeline

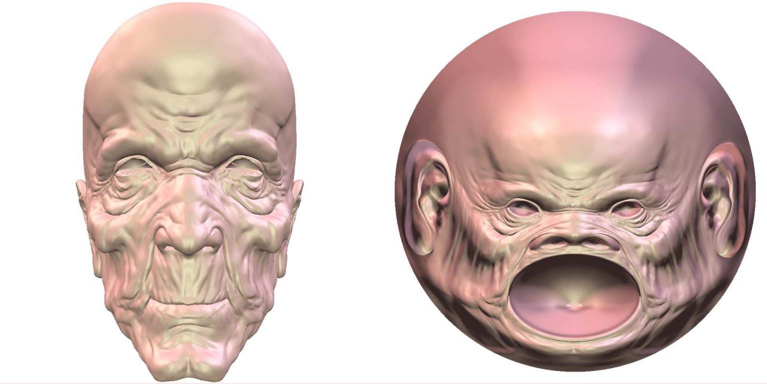
1. Compute the conformal parameterization of the input surface $\varphi : (S, \mathbf{g}) \to (\mathbb{D}, dzd\bar{z})$,
2. Compute the optimal mass transportation map $\psi : (\mathbb{D}, dxdy) \to (\mathbb{D}, \mu)$, where μ is the combination of the surface area element $e^{2\lambda}dxdy$ and the absolute value of the Gaussian curvature measure $|K(x,y)|dxdy$,
3. Uniformly sample on the preimage of the OMT map $\psi^{-1}(\mathbb{D})$,
4. Pull back the samples to the conformal parameter domain $\varphi(S)$, compute the Delaunay triangulation T,
5. Pull back the triangulation T to the original surface S, which induces the remeshing of S.

Curvature Sensitive Parameterization(CSP)



Original face mesh Conformal parameteraiztion

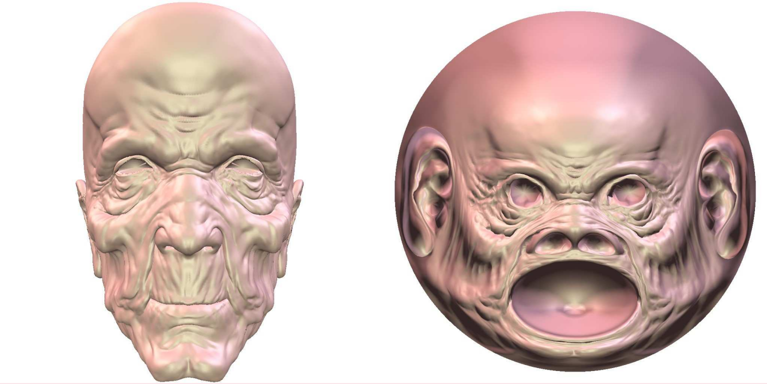
Curvature Sensitive Parameterization(CSP)



Original face mesh

Area preserving parameteraiztion

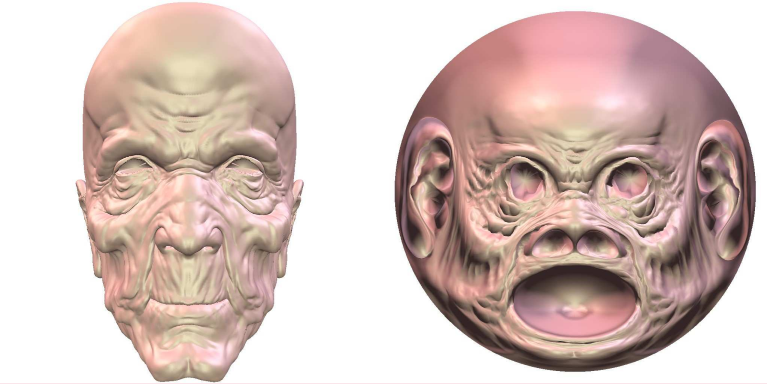
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.1

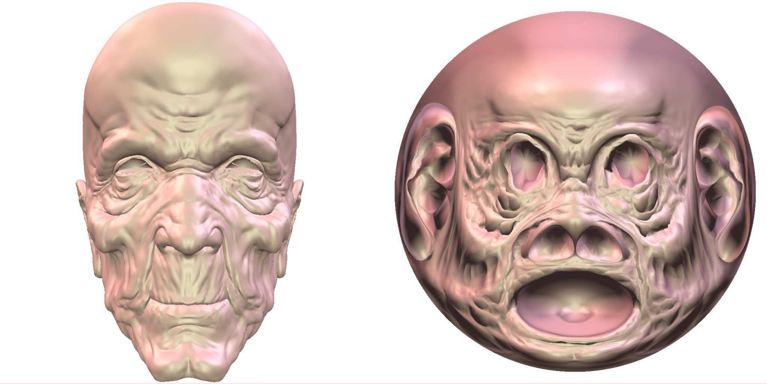
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.2

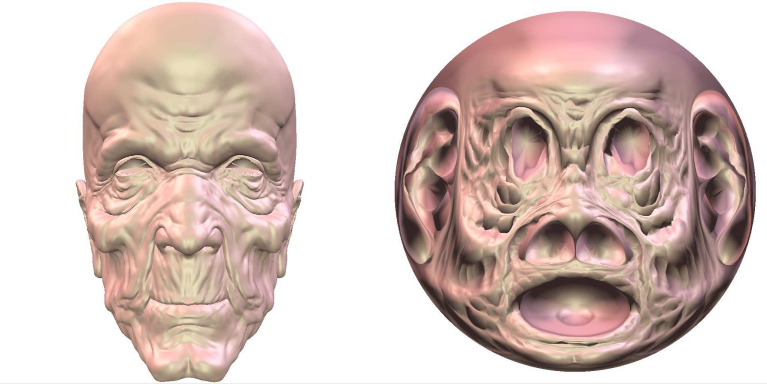
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.4

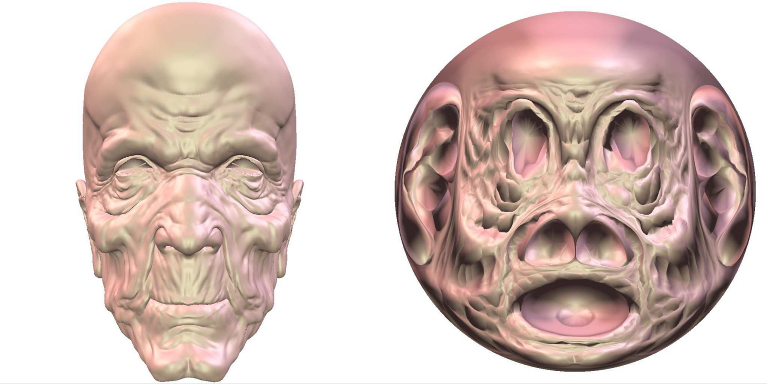
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.8

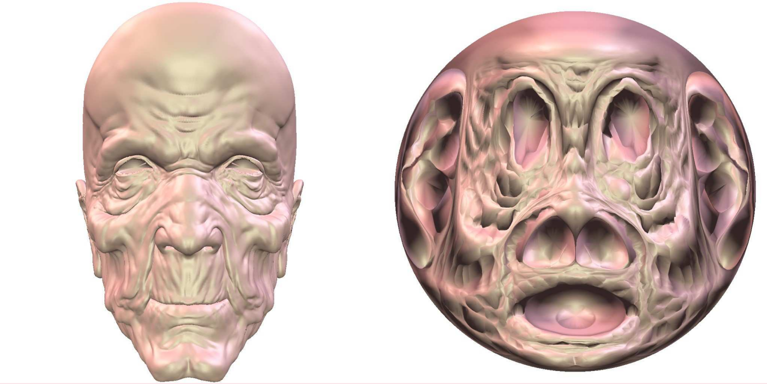
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$1.0

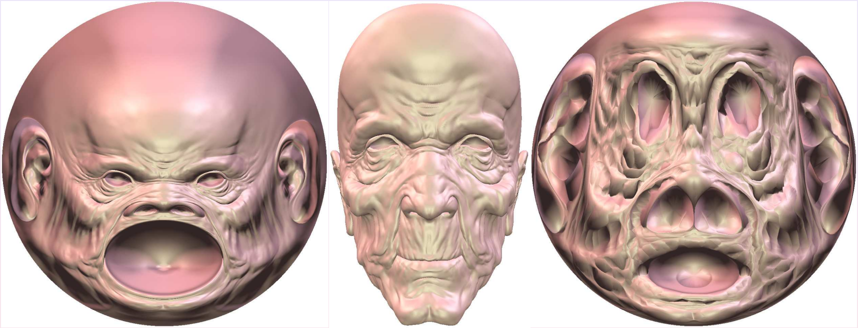
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature $\times$ 2.0

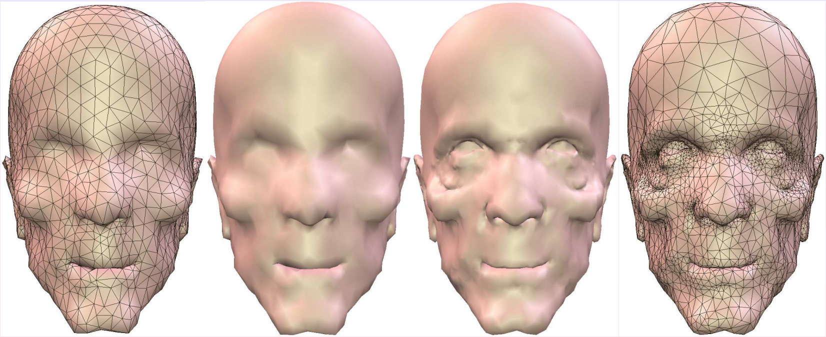
Compare Multi-scale remeshing between APP and CSP



(a) APP (b) original mesh(140K) (c) CSP

Compare Multi-scale remeshing between APP and CSP

Remeshing: 1K vertices



(a) APP wireframe

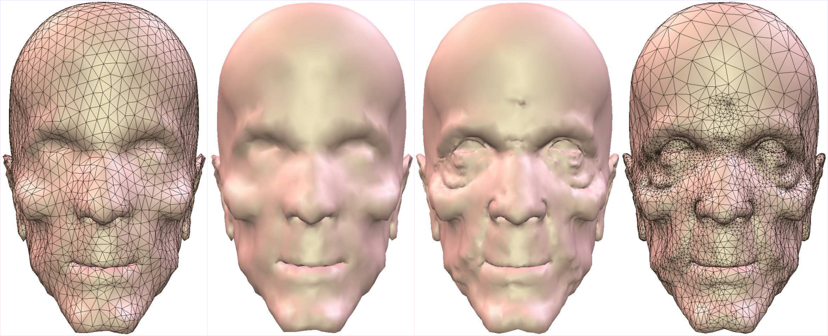
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 2K vertices



(a) APP wireframe

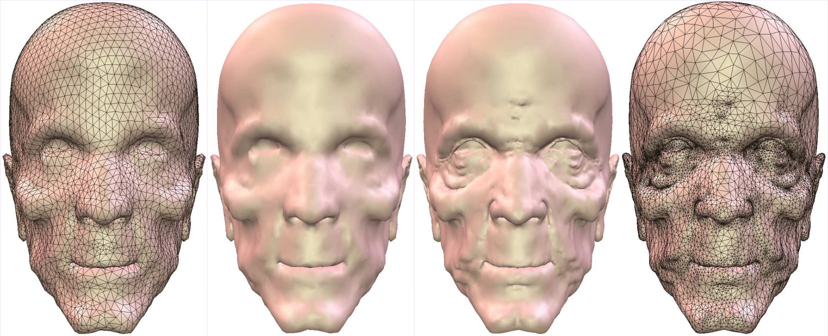
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

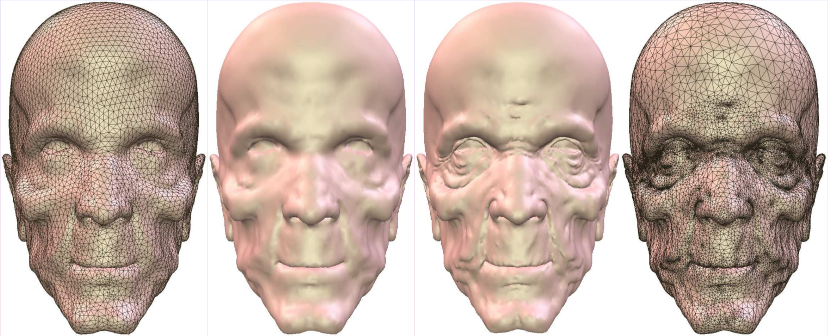
Remeshing: 4K vertices



(a) APP wireframe (b) APP smooth (c) CSP smooth (d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

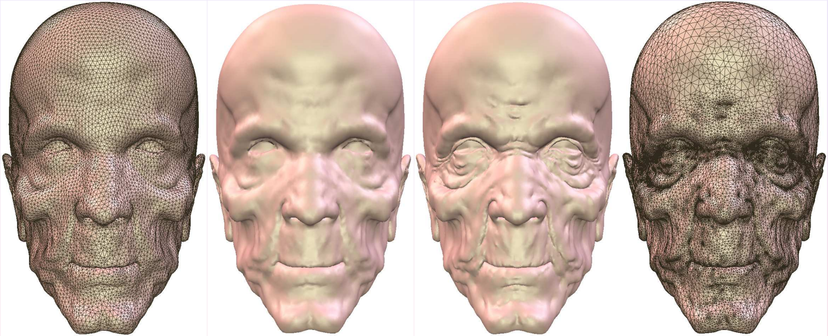
Remeshing: 8K vertices



(a) APP wireframe (b) APP smooth (c) CSP smooth (d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 16K vertices



(a) APP wireframe

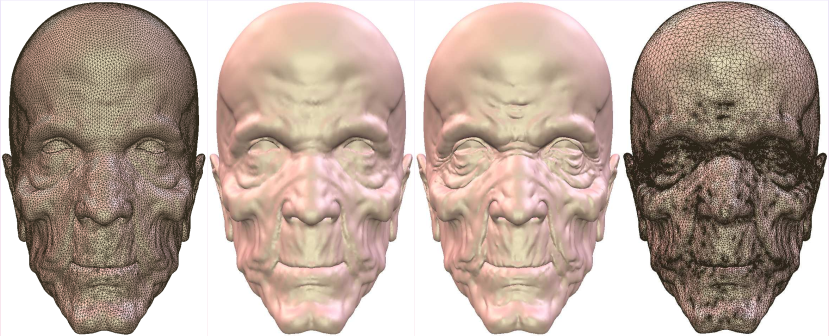
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 32K vertices



(a) APP wireframe

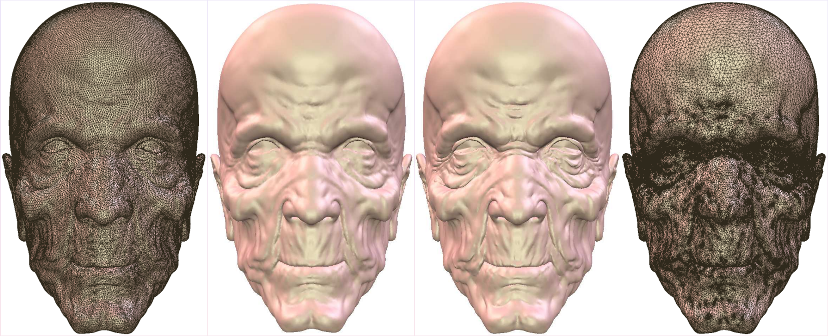
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 64K vertices



(a) APP wireframe

(b) APP smooth

(c) CSP smooth

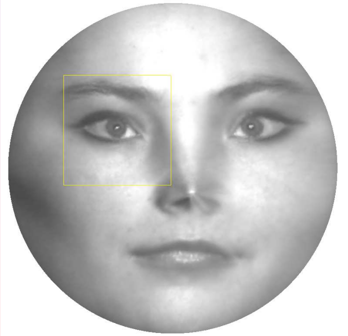
(d) CSP wireframe

Meshing

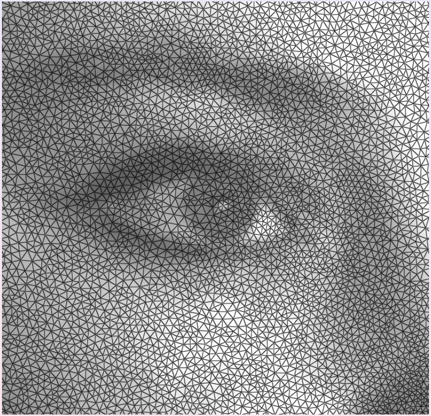
Theorem

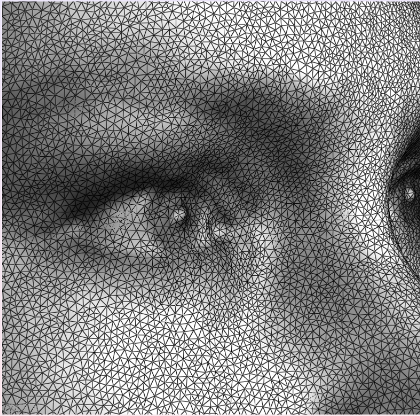
Suppose S is a surface with a Riemannian metric. Then there exist meshing method which ensures the convergence of curvatures.

Key idea: Delaunay triangulations on uniformization domains. Angles are bounded, areas are bounded.



Meshing





Theorem

Let M be a compact Riemannian surface embedded in $\mathbb{E}^3$ with the induced Euclidean metric, T the triangulation generated by Delaunay refinement on conformal uniformization domain, with circumradius bound ε. If B is the relative interior of a union of triangles of T, then

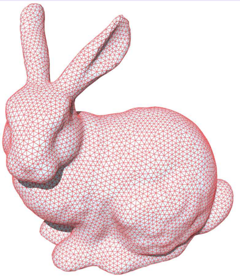
$$\begin{aligned} |\phi_T^G(B) - \phi_M^G(\pi(B))| &\leq K\varepsilon \\ |\phi_T^H(B) - \phi_M^H(\pi(B))| &\leq K\varepsilon \end{aligned}$$

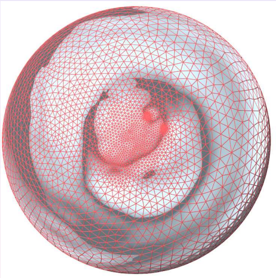
where $\pi : T \to M$ is the closest point projection, ϕ^H, ϕ^G are the mean and Gaussian curvature measures, where

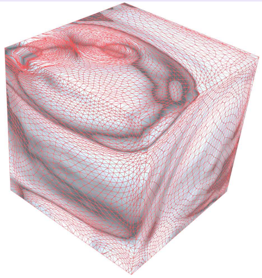
$$K = O(area(B)) + O(length(\partial B)).$$

Hexahedral Meshing

Genus Zero Case

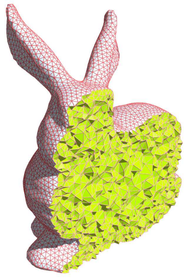




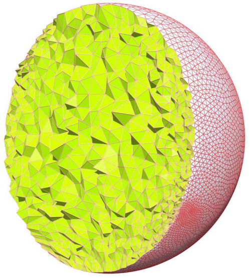


(a) Stanford bunny (b) Spherical mapping (c) Cube mapping

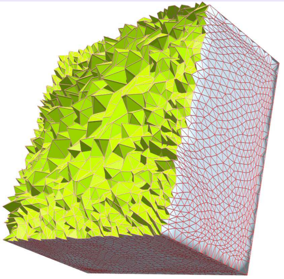
Genus Zero Case



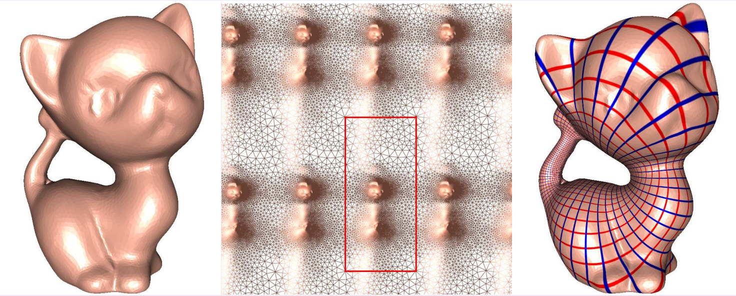
(d) Solid bunny



(e) Solid ball mapping



(f) Solid cube mapping



(a) Kitten surface (b) Flat torus (c) Quad-mesh

Figure: A genus one closed surface can be conformally and periodically mapped onto the plane, each fundamental domain is a parallelogram. The subidvision of the parallelogram induces a quad-mesh of the surface.

Genus One Case

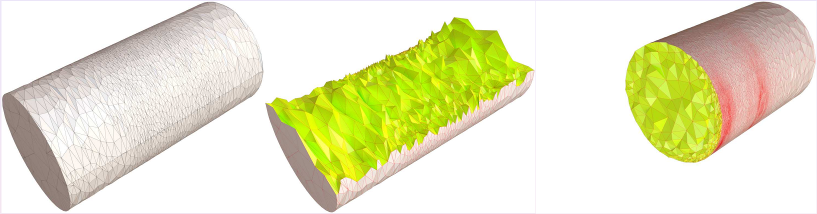


Figure: The interior of the kitten surface is mapped onto a canonical solid cylinder.

Algorithmic Pipeline

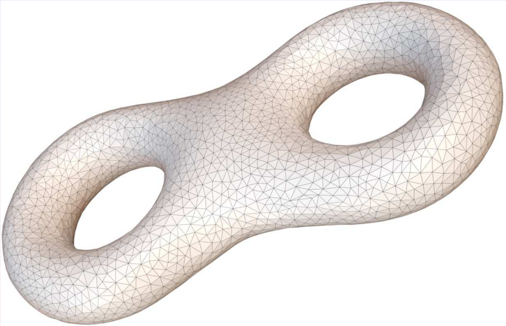


Figure: Input Surface $\partial\Omega$.

Algorithmic Pipeline

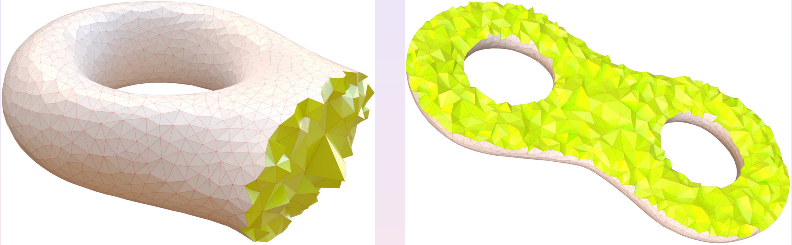


Figure: Tetrahedral meshing Ω.

Algorithmic Pipeline

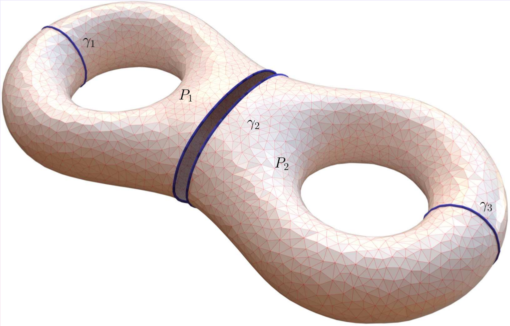


Figure: Admissible curve system $\{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$, pants decomposition $\{P_1, P_2, \cdots, P_{2g-2}\}$.

Algorithmic Pipeline

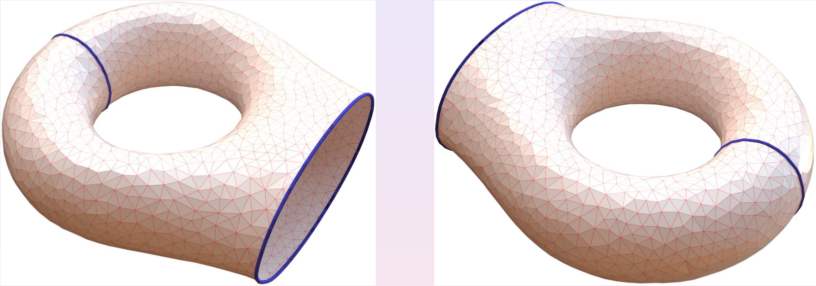


Figure: Pairs of pants $\{P_1, P_2, \cdots, P_{2g-2}\}$.

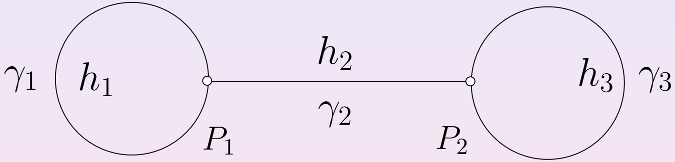


Figure: Pants decomposition graph.

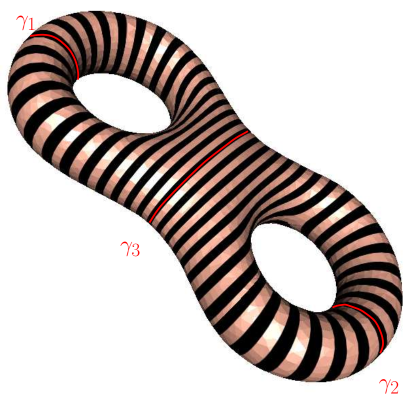


Figure: Foliation, Holomorphic quadratic differential.

Algorithmic Pipeline

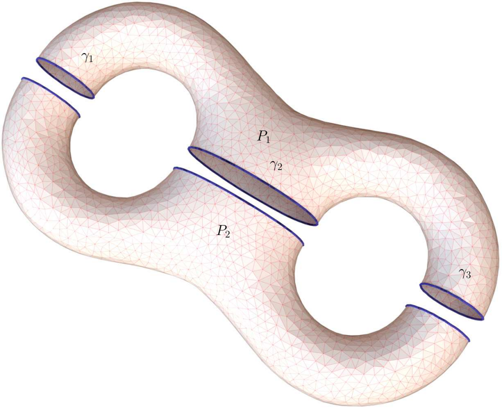


Figure: Admissible curve system, pants decomposition.

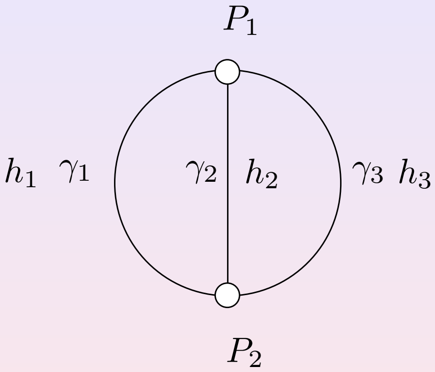


Figure: Pants decomposition graph.

Algorithmic Pipeline

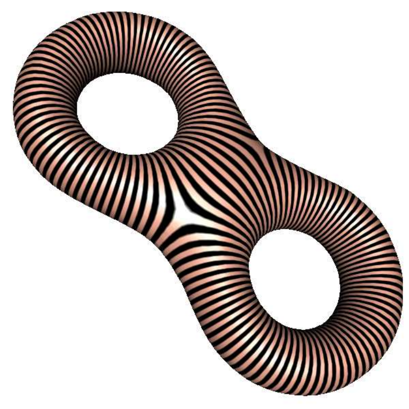


Figure: Foliation, Holomorphic quadratic differential.

Algorithmic Pipeline

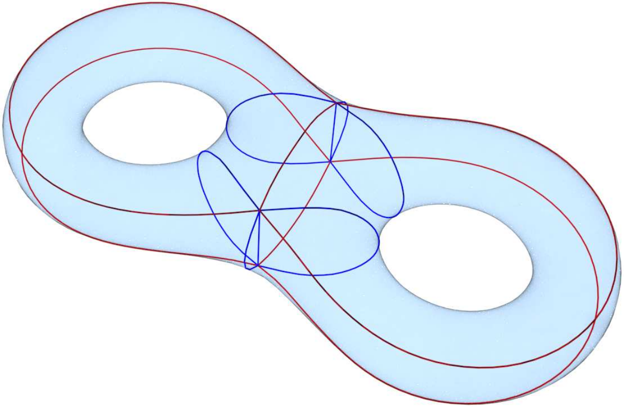


Figure: Critical horizontal trajectories and vertical trajectories.

Algorithmic Pipeline

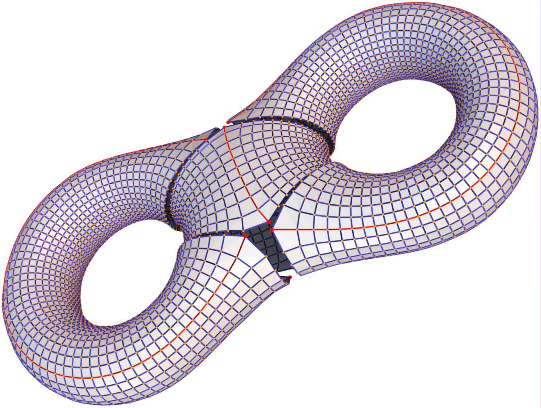


Figure: Quad-Mesh $\mathscr{Q}$ and the cylindrical decomposition.

Algorithmic Pipeline

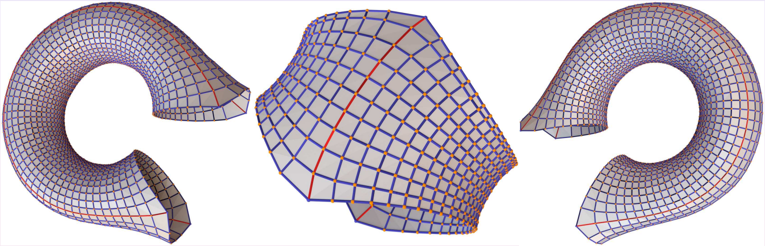


Figure: Cylindrical decomposition.

Algorithmic Pipeline

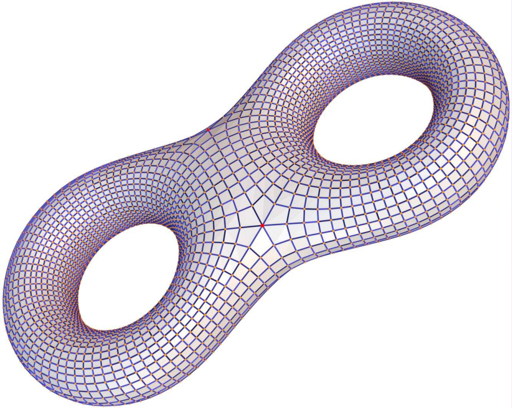
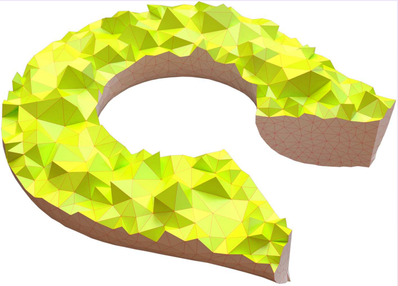


Figure: Colorable quadrilateral mesh, all vertex valences are even.

Algorithmic Pipeline



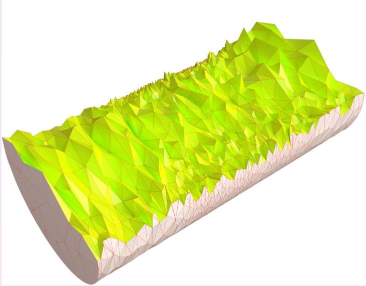


Figure: Left solid cylinder, maps to the canonical solid cylinder.

Algorithmic Pipeline

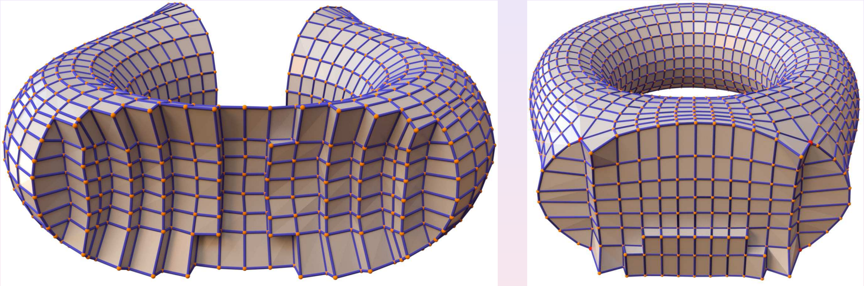


Figure: Hexahedral meshing of solid cylinders.

Algorithmic Pipeline

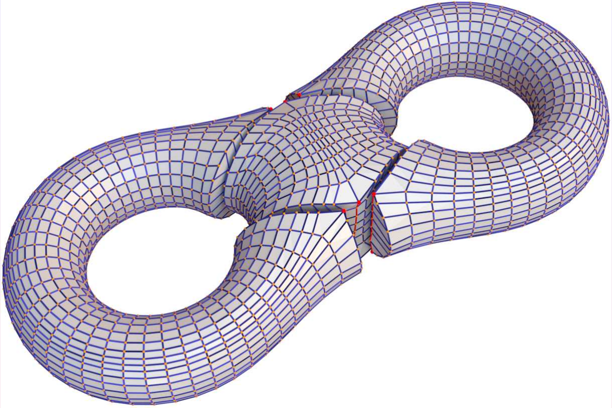


Figure: Hexahedral mesh $\mathscr{H}$ of the interior volume Ω.

Algorithmic Pipeline

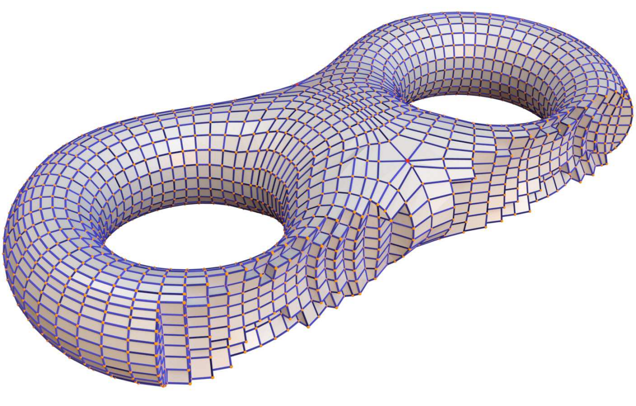


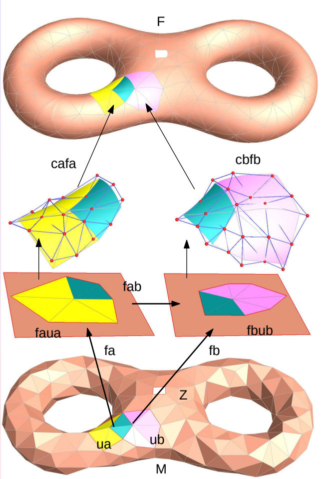
Figure: Hexahedral mesh $\mathscr{H}$ of the interior volume Ω.

Manifold Spline

- Convert scanned polygonal surfaces to smooth spline surfaces.
- Conventional spline scheme is based on affine geometry. This requires us to define affine geometry on arbitrary surfaces.
- This can be achieved by designing a metric, which is flat everywhere except at several singularities (extraordinary points).
- The position and indices of extraordinary points can be fully controlled.

Extraordinary Points

- Fully control the number, the index and the position of extraordinary points.
- For surfaces with boundaries, splines without extraordinary point can be constructed.
- For closed surfaces, splines with only one singularity can be constructed.


F
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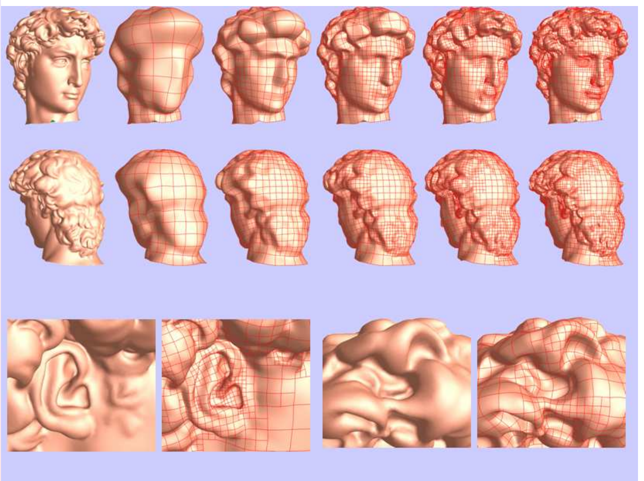
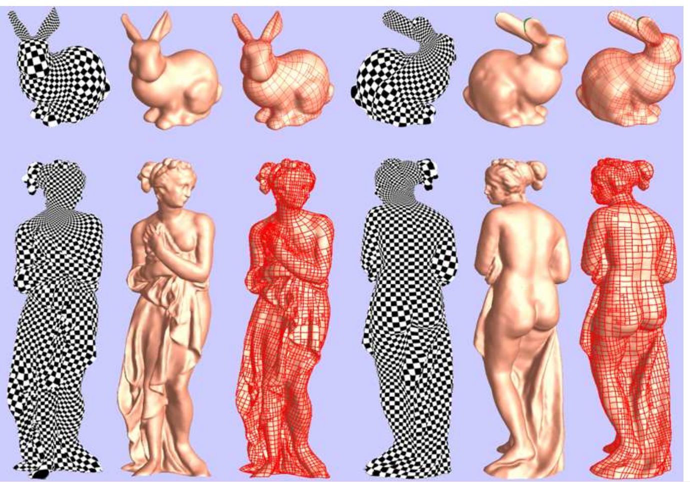
M

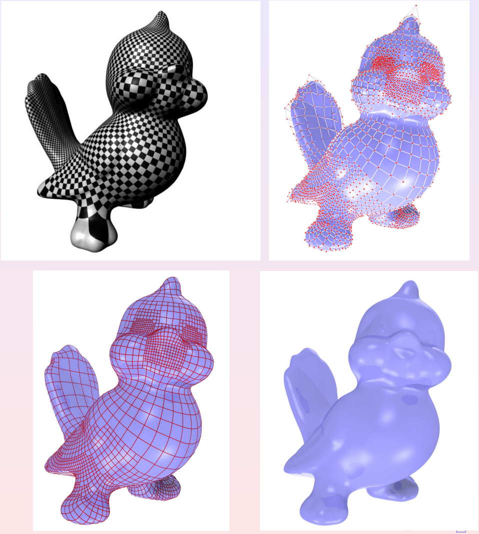
Converting a polygonal mesh to TSplines with multiple resolutions.



Converting scanned data to spline surfaces.

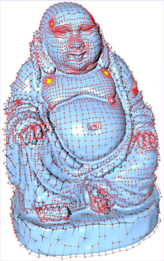


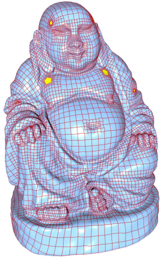
Converting scanned data to spline surfaces, the control points, knot structure are shown.

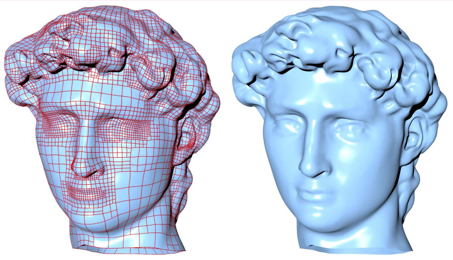


Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.

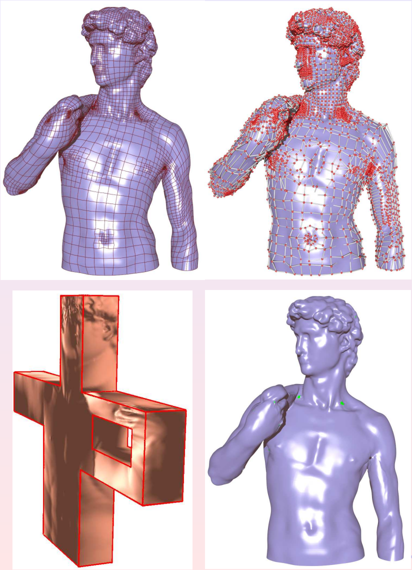




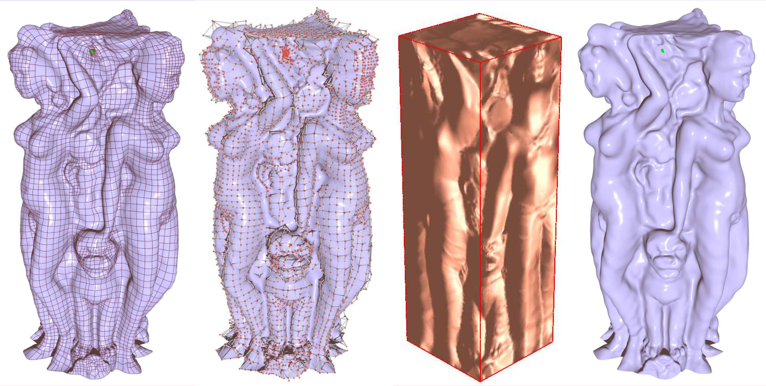


Manifold Spline

Polygonal mesh to spline, control net and the knot structure.

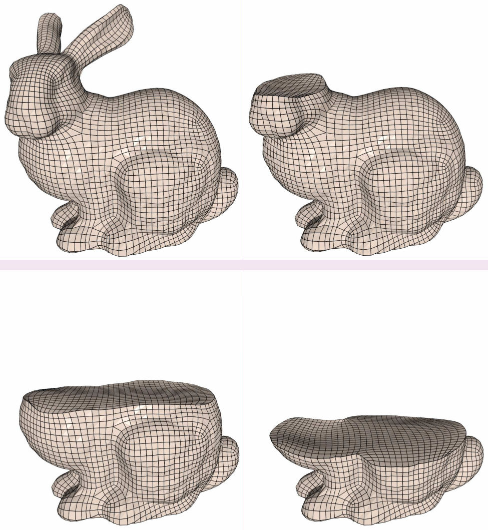


Manifold Spline



Manifold Spline

volumetric spline.



Visualization

(a) Front view (b) Angle-preserving (c) Area-preserving (d) Back view

Angle-perserving parameterization vs. area-preserving parameterization

(a) 2x (b) 3x (c) 4x (d) 6x

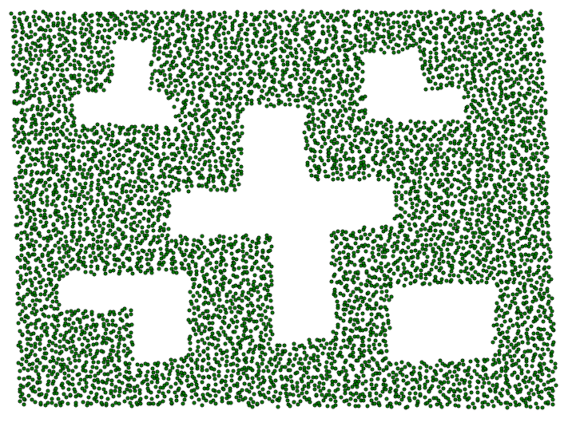
Importance driven parameterization. The Buddha's head region is magnified by different factors

Wireless Sensor Network Application

Wireless Sensor Network

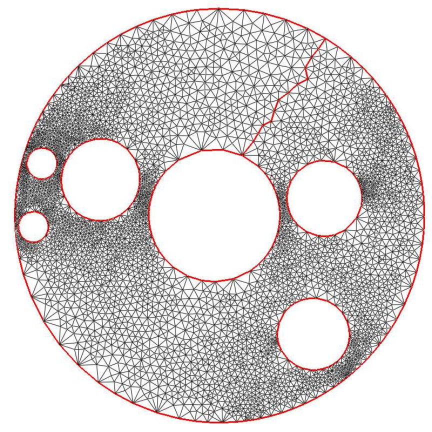
- Detecting global topology.
- Routing protocol.
- Load balancing.
- Isometric embedding.

Given sensors on the ground, because of the concavity of the boundaries, greedy routing doesn't work.

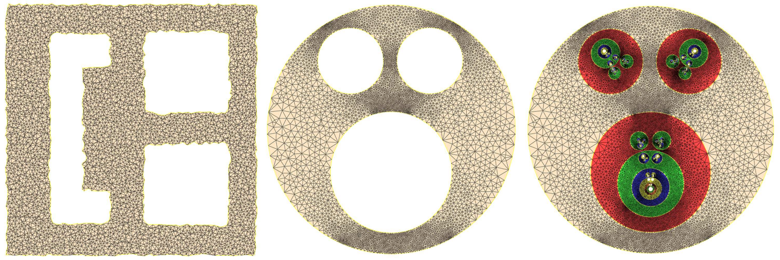


Greedy Routing

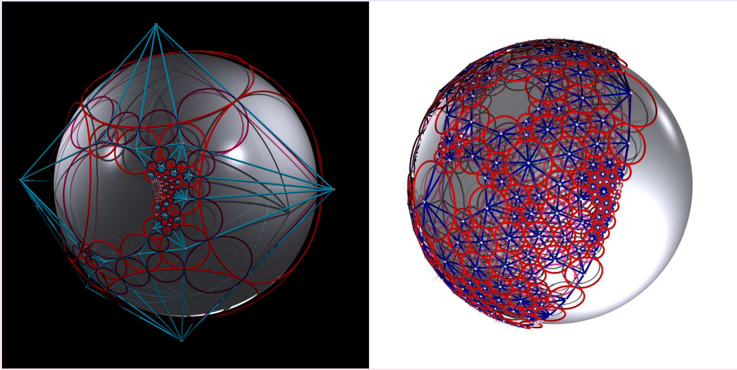
Map the network to a circle domain, all boundaries are circles, greedy routing works.



Schoktty Group - Circular Reflection



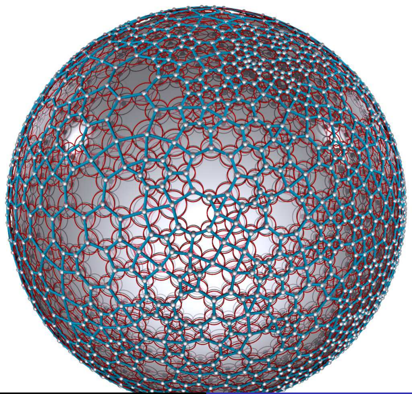
Optimal Planar Graph Embedding.



Graph Embedding

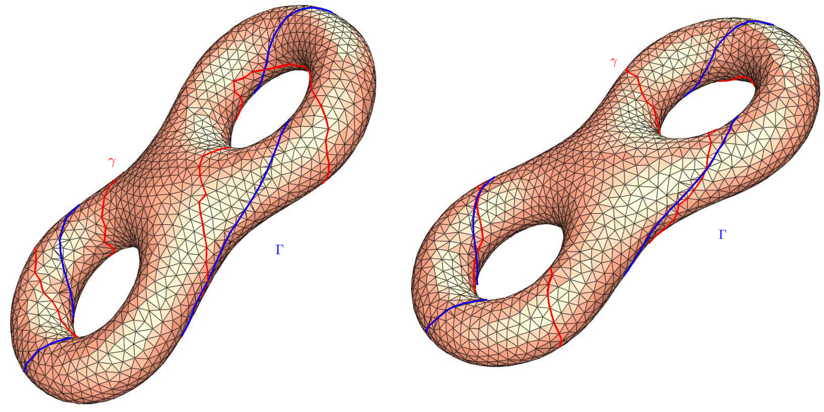
Thurston-Andreev Theorem

A planar graph can be embedded on the unit sphere, such that the face circles are orthogonal to vertex circles; the circles at the vertices of an edge are tangent to each other. Such kind of embedding differ by a Möbius transformation.

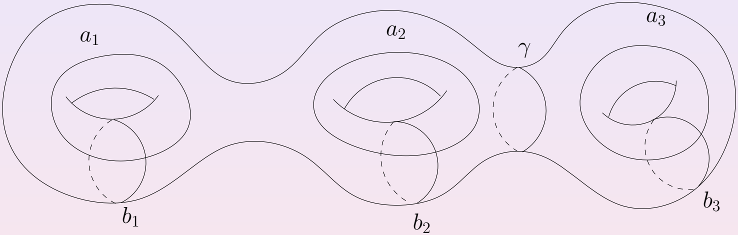


Canonical Homotopy Class Representative

Under hyperbolic metric, each homotopy class has a unique geodesic, which is the representative of the homotopy class.

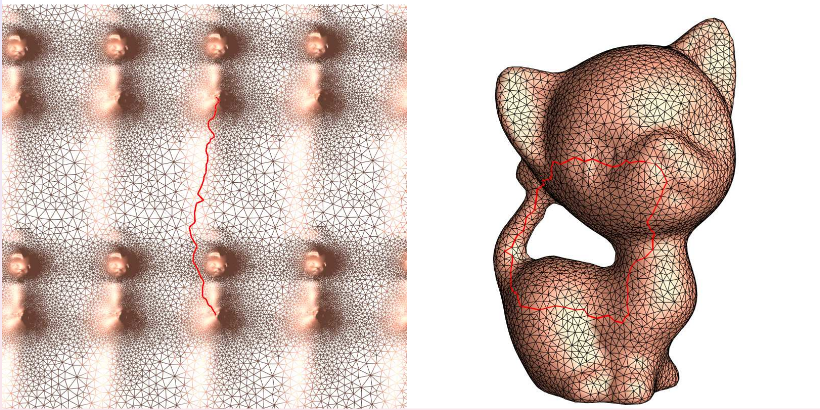


Shortest word Problem (NP Hard):

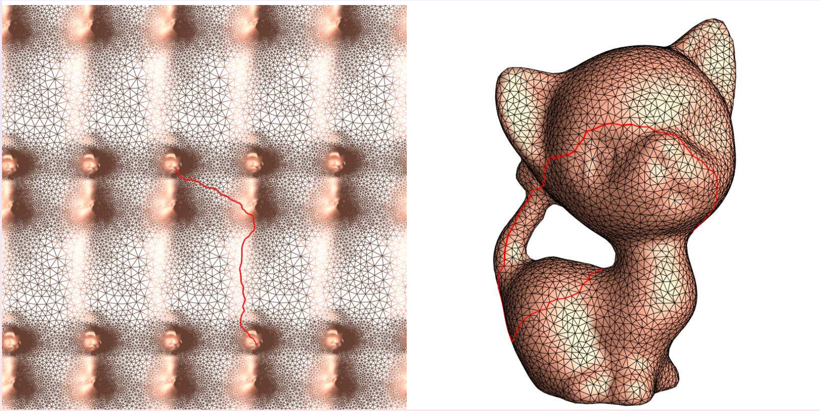


$$\gamma = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = (a_3 b_3 a_3^{-1} b_3^{-1})^{-1}$$

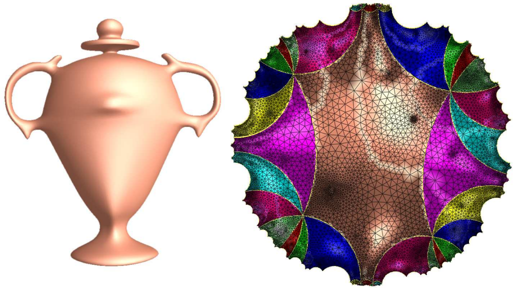
Loop Lifting

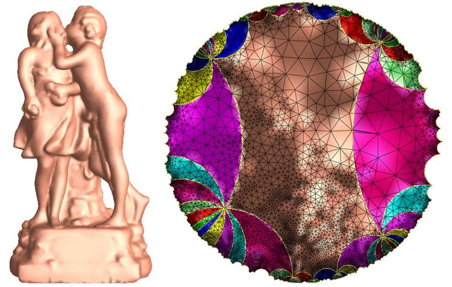


Loop Lifting

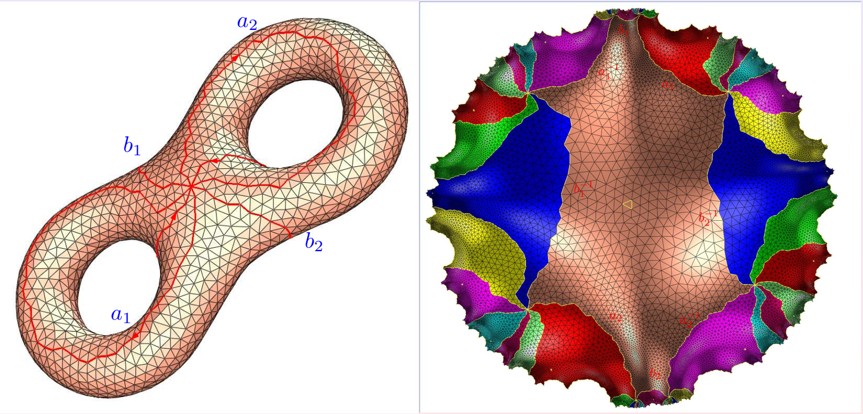


Hyperbolic Ricci Flow



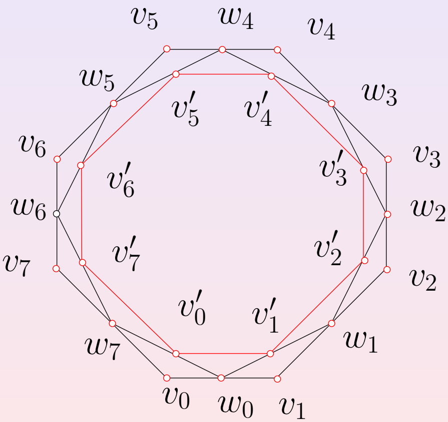


Lifting a loop from base surface to the universal covering space.



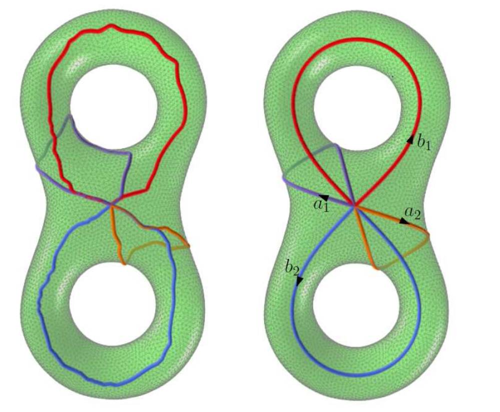
Birkoff Curve Shorting

Birkoff curve shortening deforms a loop to a geodesic.



Birkoff Curve Shorting

Birkoff curve shortening deforms a loop to a geodesic.



Computing Shortest Word

1. Compute the uniformization metric using Ricci flow.
2. Compute the geodesic loop by Birkoff curve shortening.
3. Lift the geodesic loop to the universal covering space.
4. Trace the lifted loop to compute the word.

Medical Imaging Application

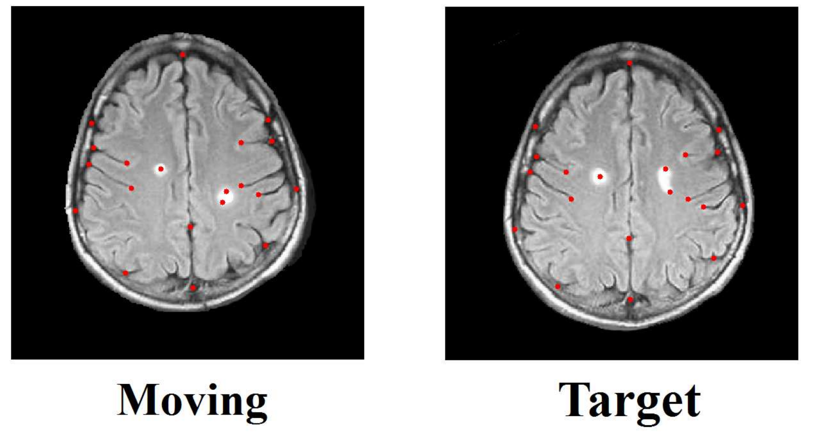
Medical Imaging

Quantitatively measure and analyze the surface shapes, to detect potential abnormality and illness.

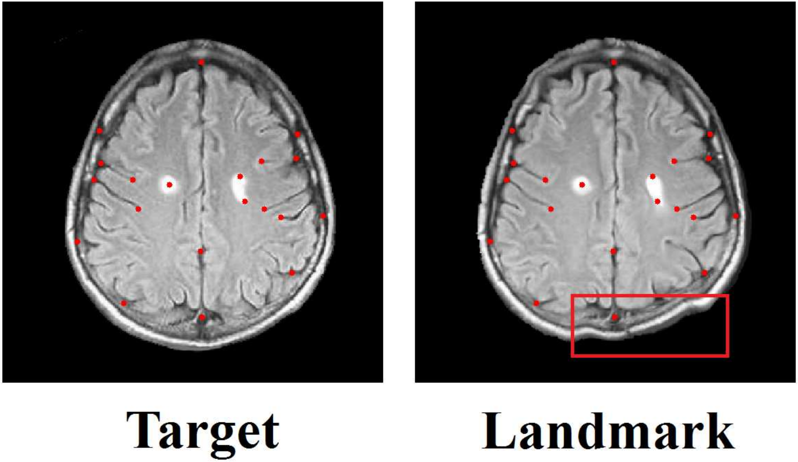
- Shape reconstruction from medical images.
- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Shape retrieval.

Image registration : Brain MRI

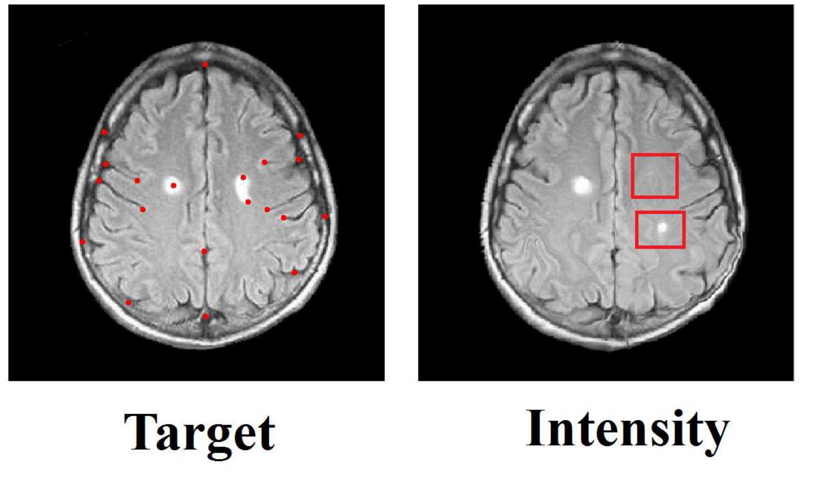
Registration problem :



Landmark only :



Intensity only :



Landmark + intensity only :

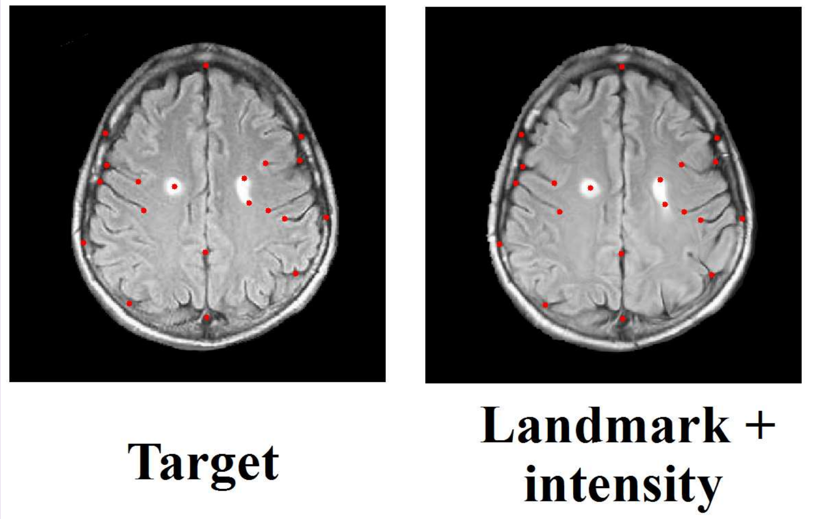
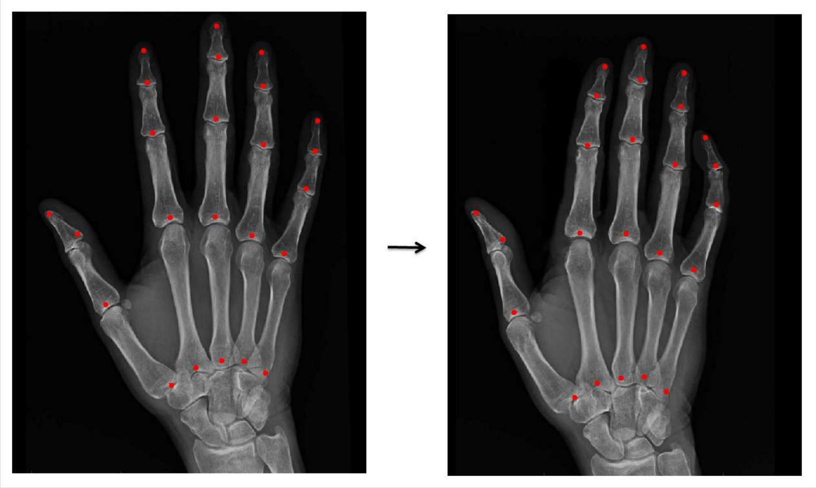
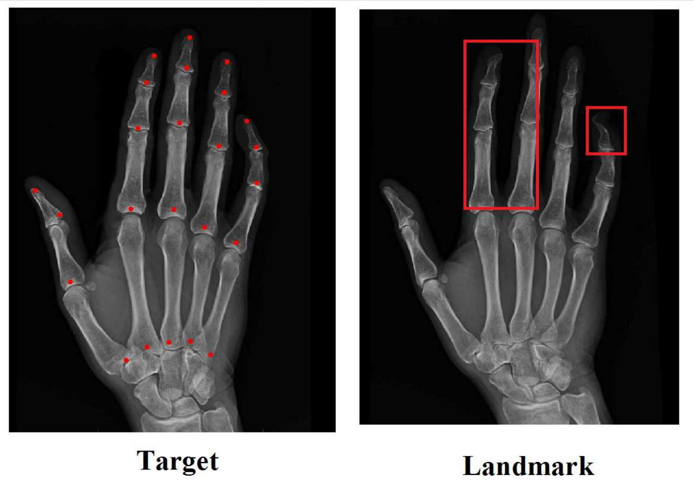


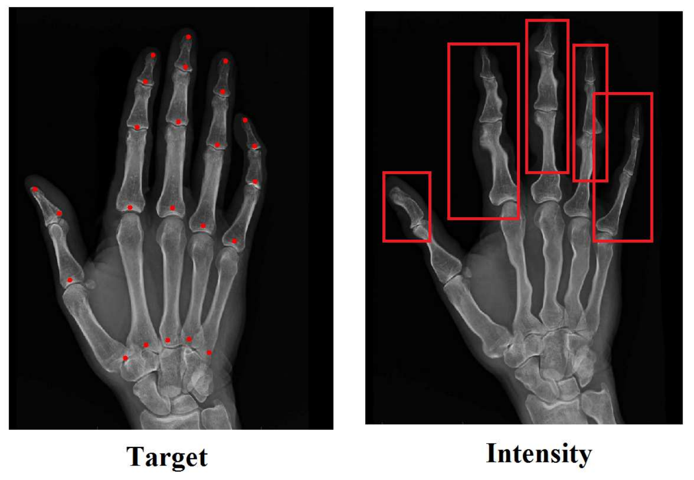
Image registration : X-ray bone



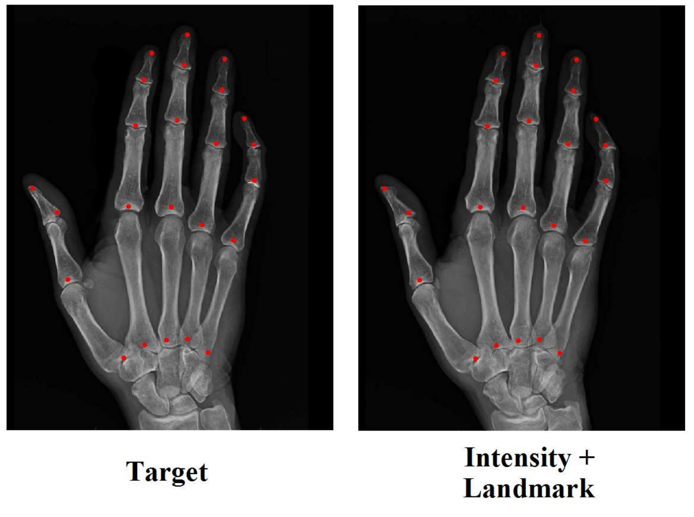
Landmark only :



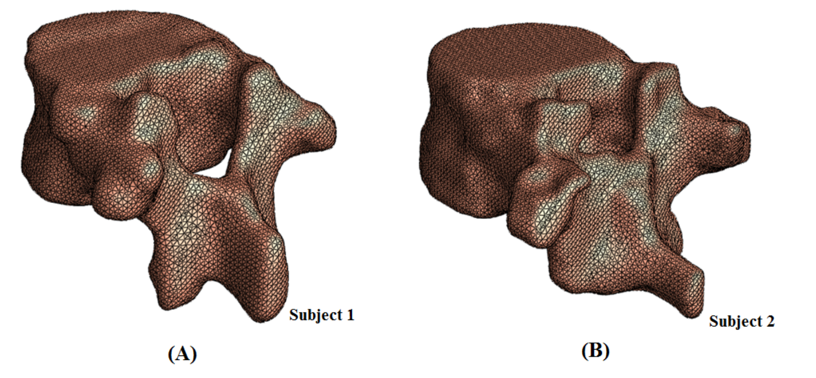
Intensity only :



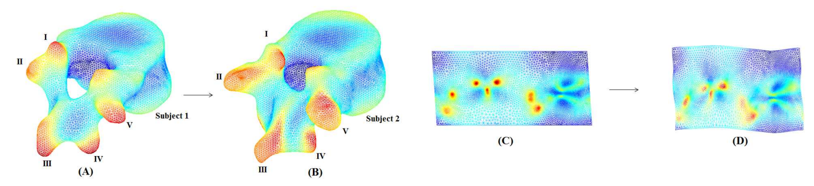
Landmark + Intensity :

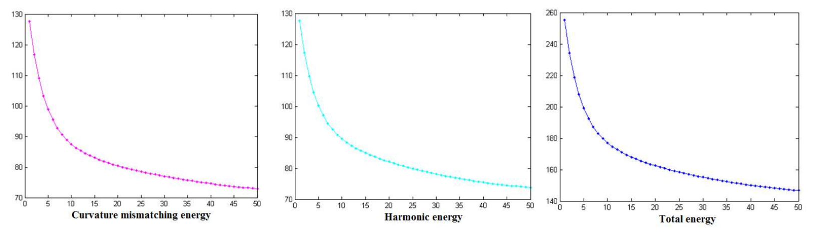


Medical registration : Vertebral bone

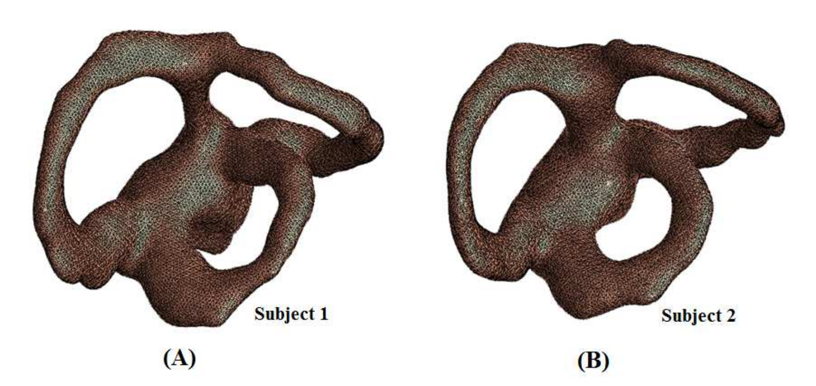


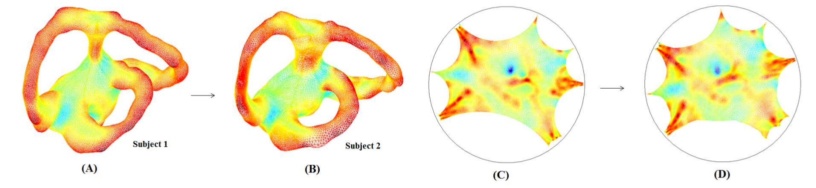
Registration result:

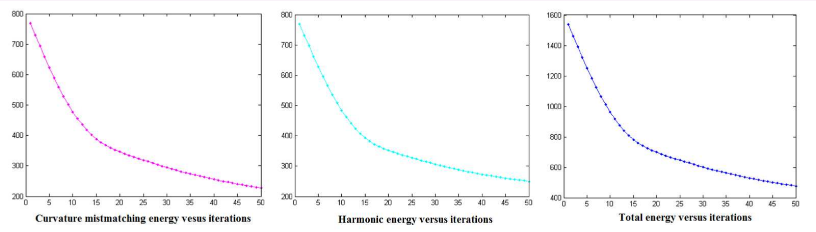




Medical registration : Vestibular system

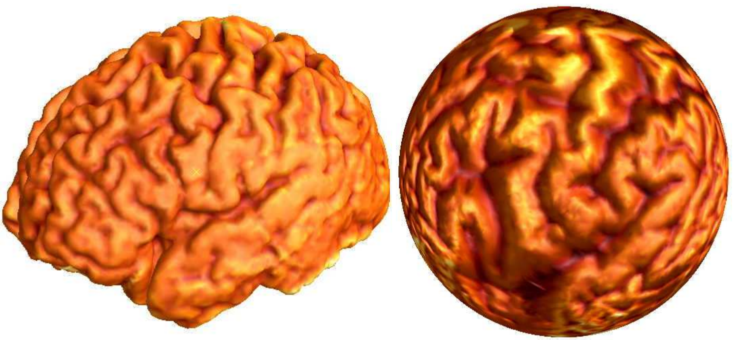



Subject 1
Subject 2
(A)
(B)
(C)
(D)


Curvature mistmatching energy vesus iterations
Harmonic energy versus iterations
Total energy versus iterations

Brain Cortex Surface

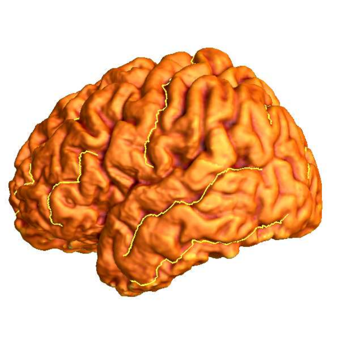
Conformal Brain Mapping for registration, matching, comparison.

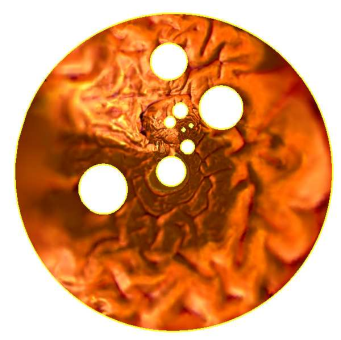


Conformal Brain Mapping

Using conformal module to analyze shape abnormalities.

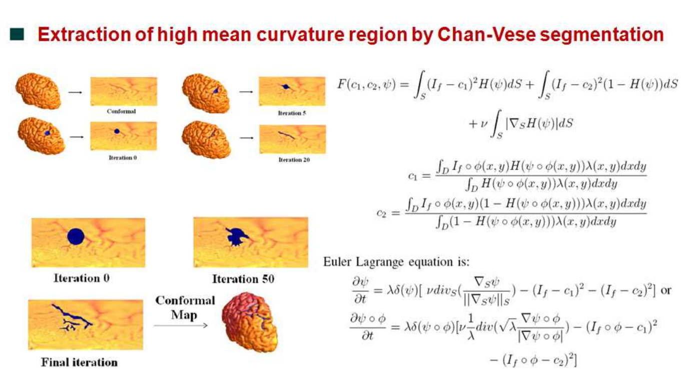
Brain Cortex Surface





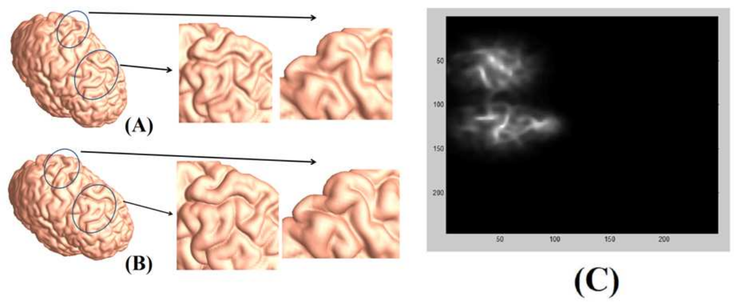
Automatic sulcal landmark Tracking

- With the conformal structure, PDE on Riemann surfaces can be easily solved.
- Chan-Vese segmentation model is generalized to Riemann surfaces to detect sulcal landmarks on the cortical surfaces automatically



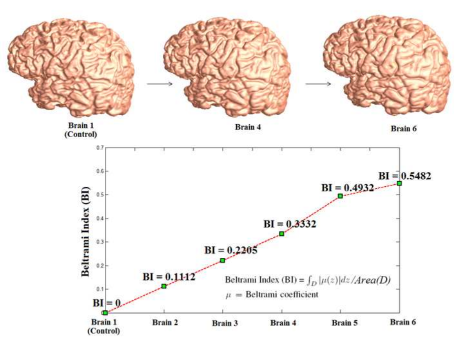
Abnormality detection on brain surfaces

The Beltrami coefficient of the deformation map detects the abnormal deformation on the brain.

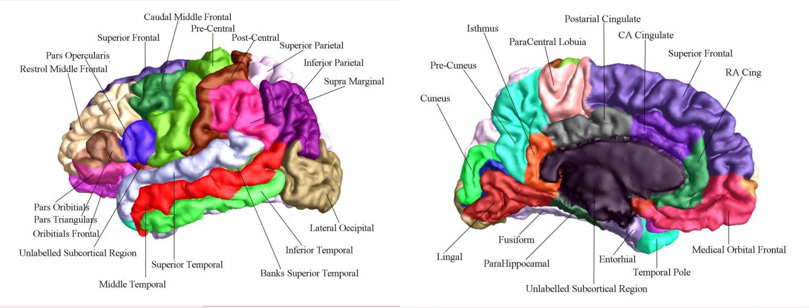


Abnormality detection on brain surfaces

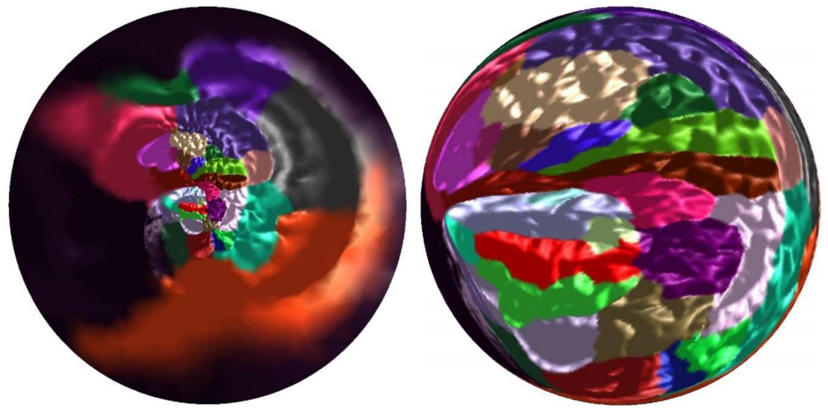
The brain is undergoing gyri thickening (commonly observed in Williams Syndrome) The Beltrami index can effectively measure the gyrification pattern of the brain surface for disease analysis.



Alzheimer Study

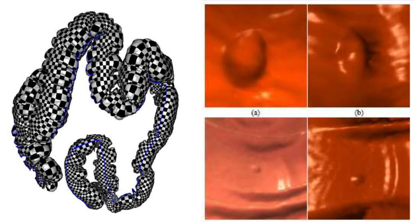


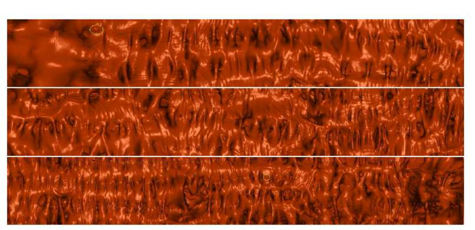
Alzheimer Study



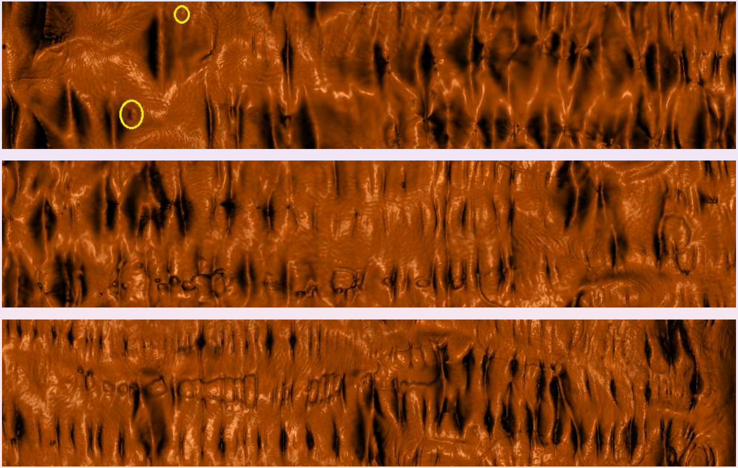
Virtual Colonoscopy

Colon cancer is the 4th killer for American males. Virtual colonosocpy aims at finding polyps, the precursor of cancers. Conformal flattening will unfold the whole surface.





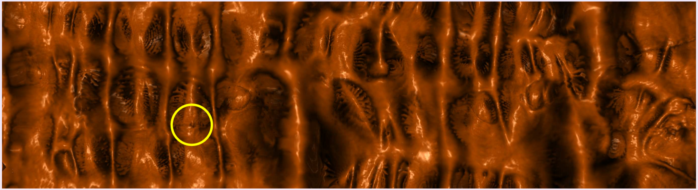
Colon Flattening

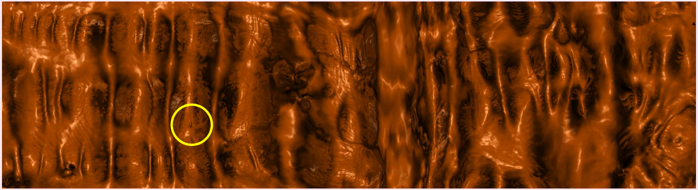


Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.

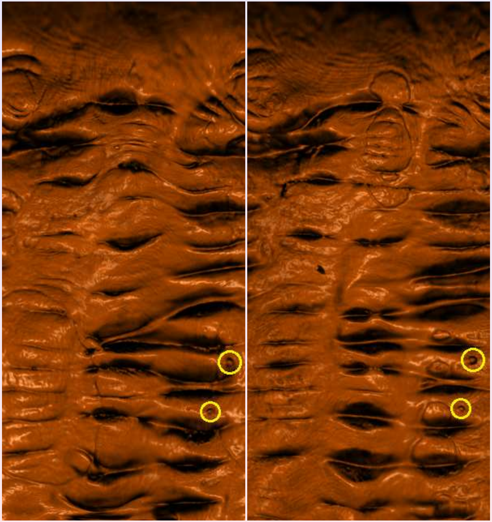
Virtual Colonoscopy

Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.



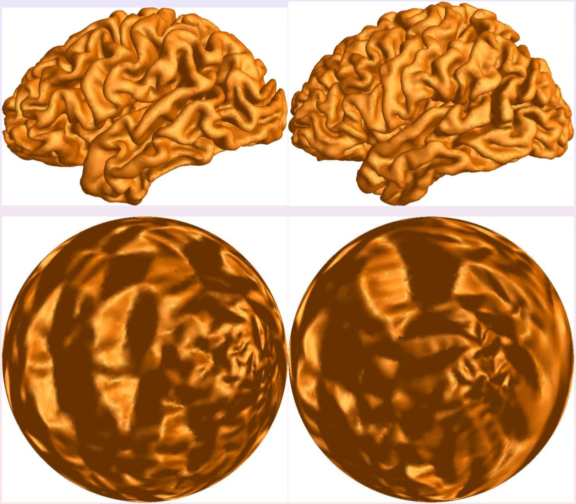


Colon Registration

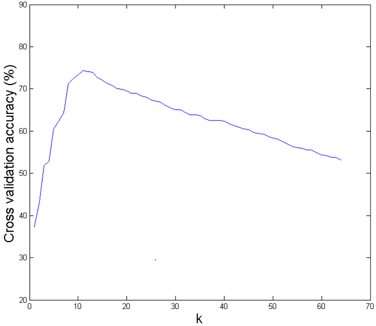


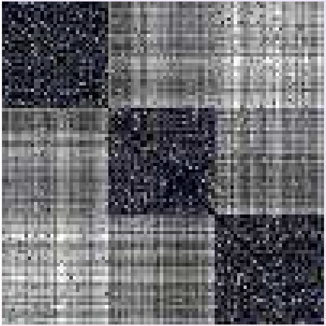
Brain Morphometry

IQ from shape



IQ from shape





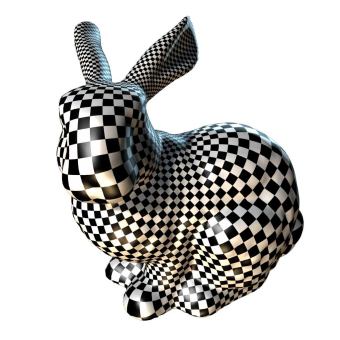
3D Fabrication

3D Fabrication

Figure: Fabrication by plain weaving.

Thanks

For more information, please email to gu@cs.stonybrook.edu.



Thank you!