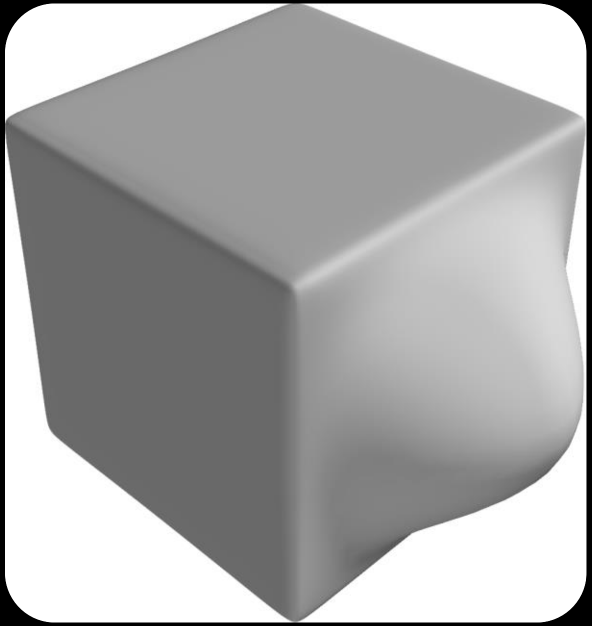
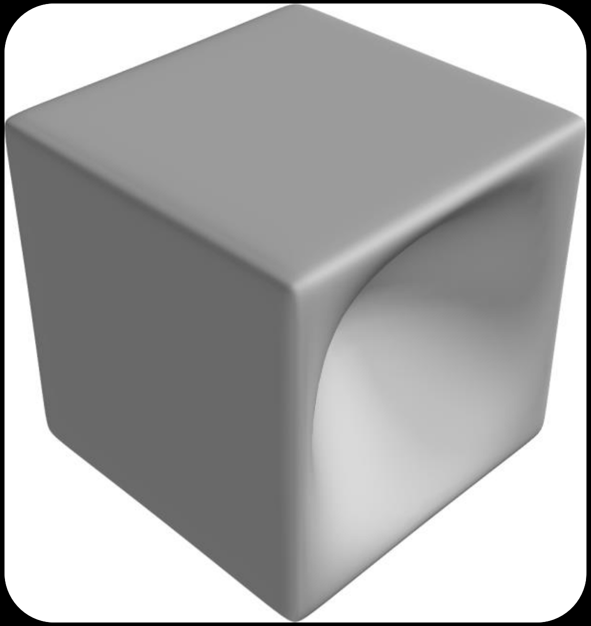


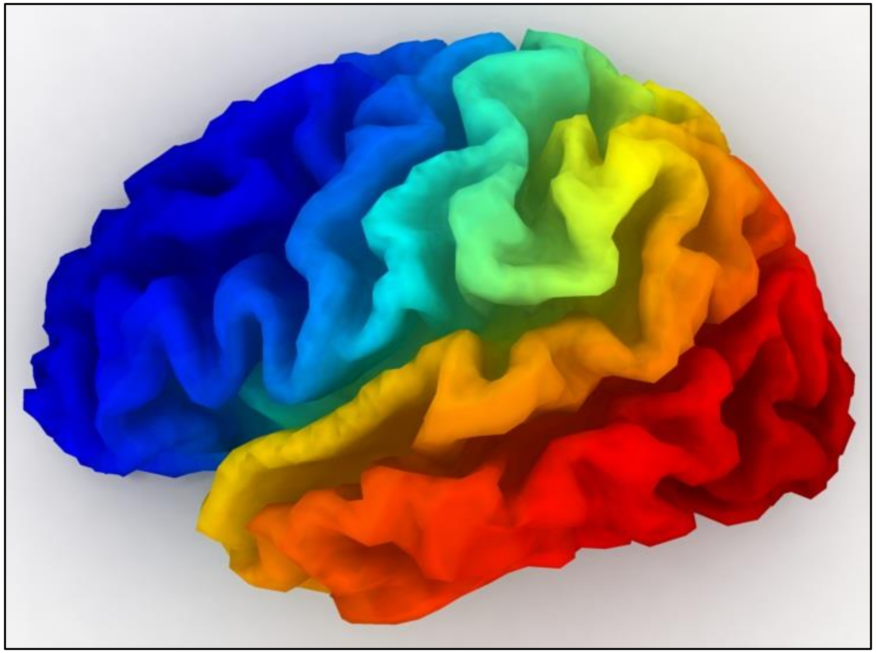


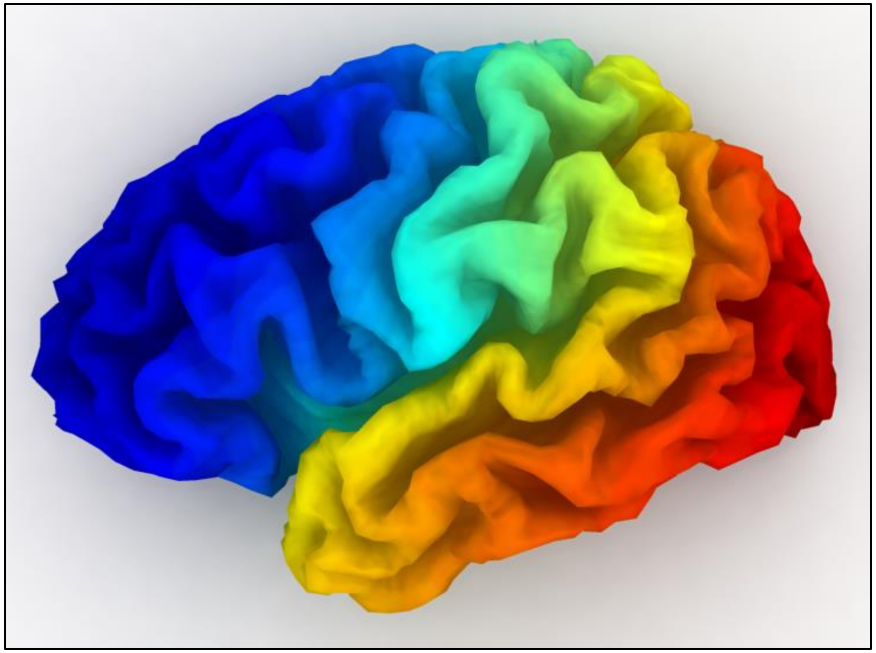
Volumetric Challenges in Shape Analysis

Justin Solomon

MIT EECS

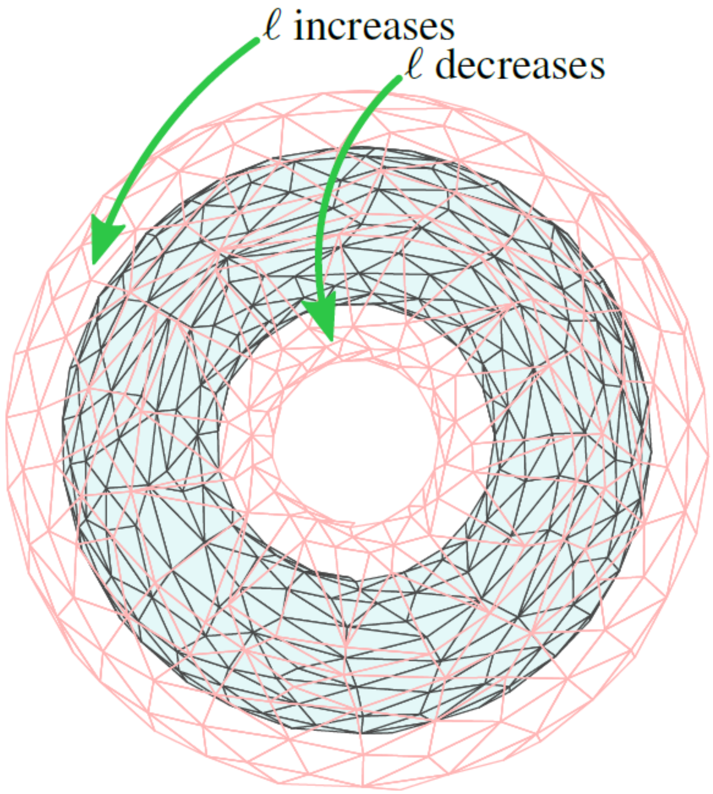
Shape Analysis: Typical Tasks

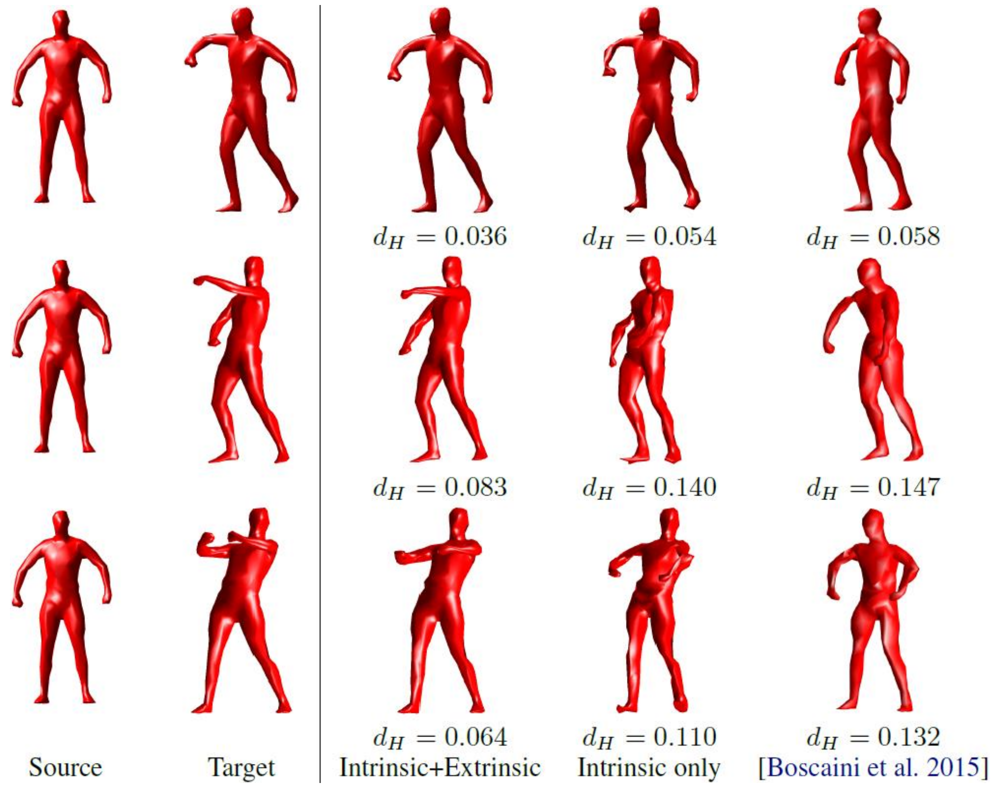




Which points on one object correspond to points on another?

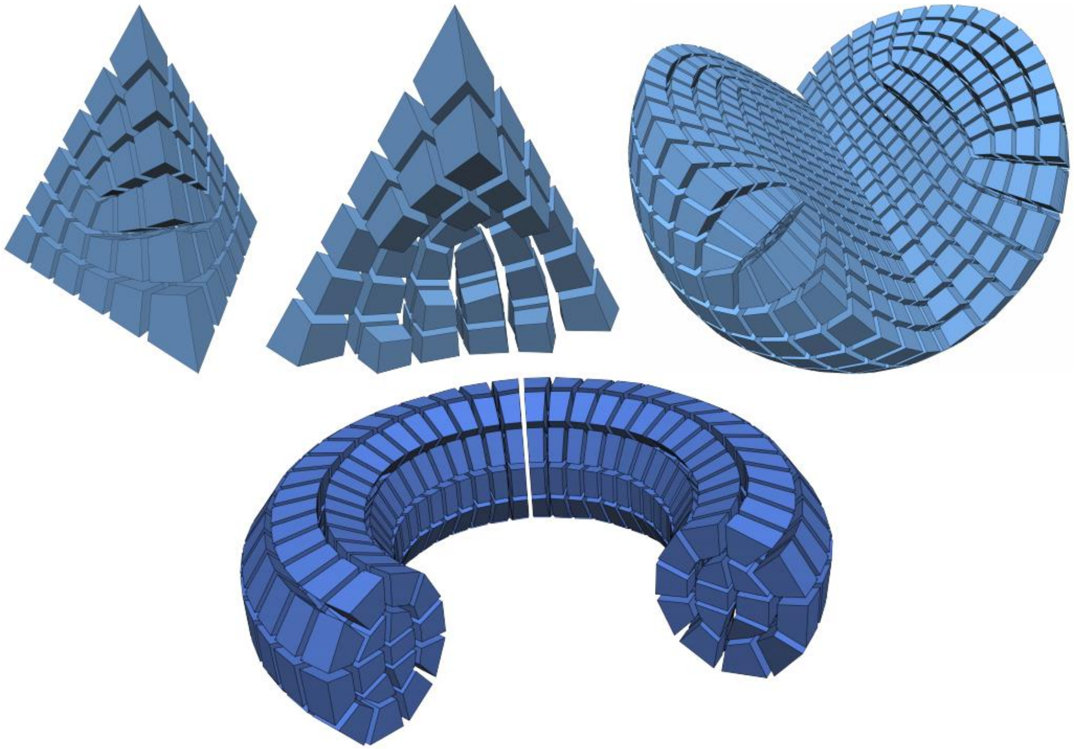
Shape Analysis: Typical Tasks





What distinguishes shapes from one another?

Shape Analysis: Typical Tasks



How can we tile a shape with simpler elements?

Desiderata

- **Efficient**
  Surfaces have many vertices and triangles
- **Discriminative**
  Must be able to distinguish between shapes
- **Multiscale**
  Resilient to noise, small changes
- **Well-justified**
  Connection to differential geometry

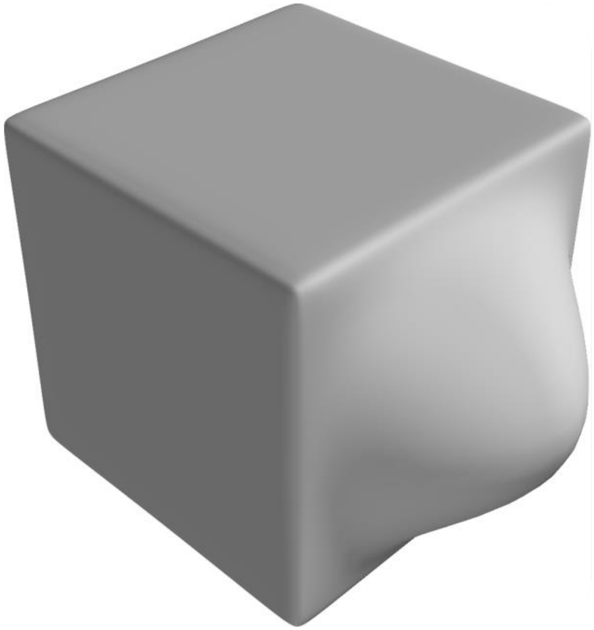
Today's Challenge

Figure 1: Deformations of a glove (left) and a solid hand (right) are an illustration of the difference between boundary and volume isometries.

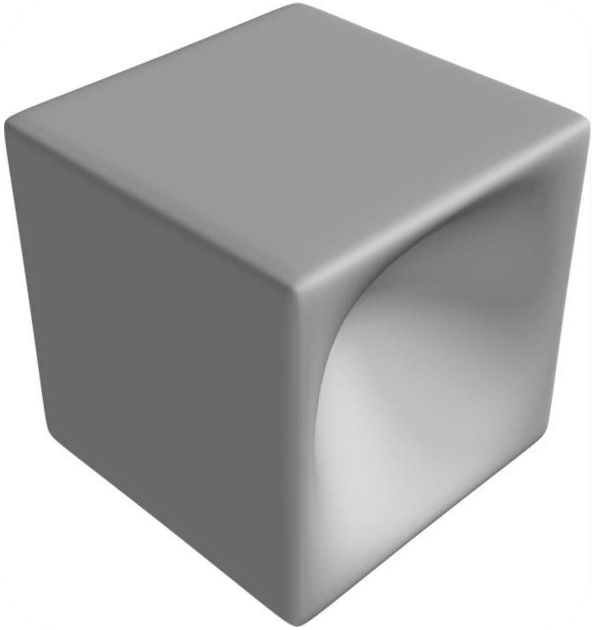
Image from: Raviv et al. "Volumetric Heat Kernel Signatures." 3DOR 2010.

Not the same.

What’s Different?

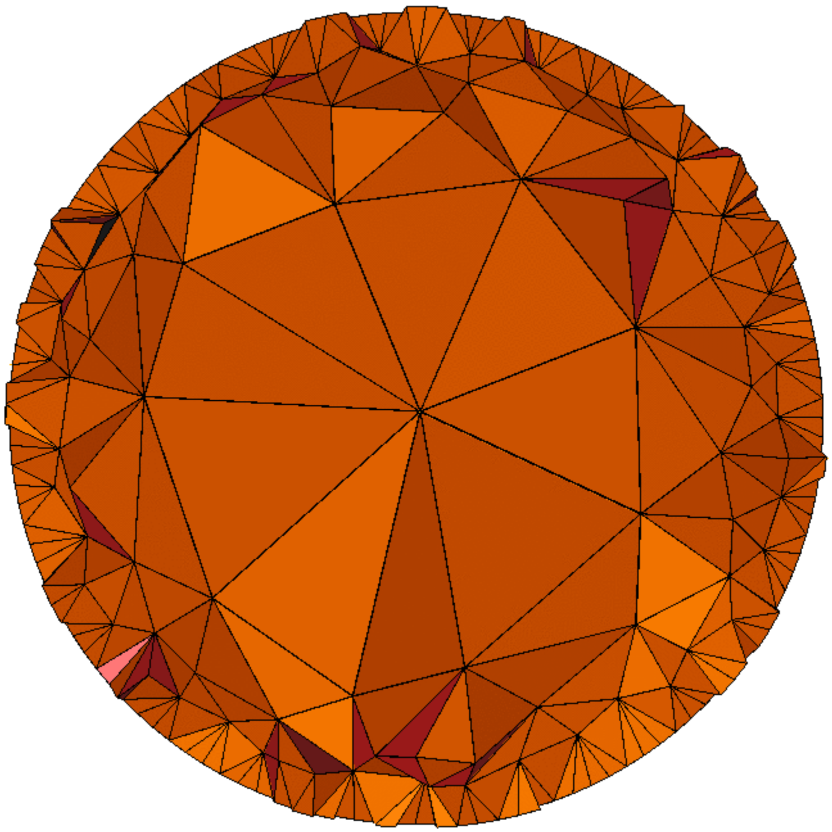


$\ncong$

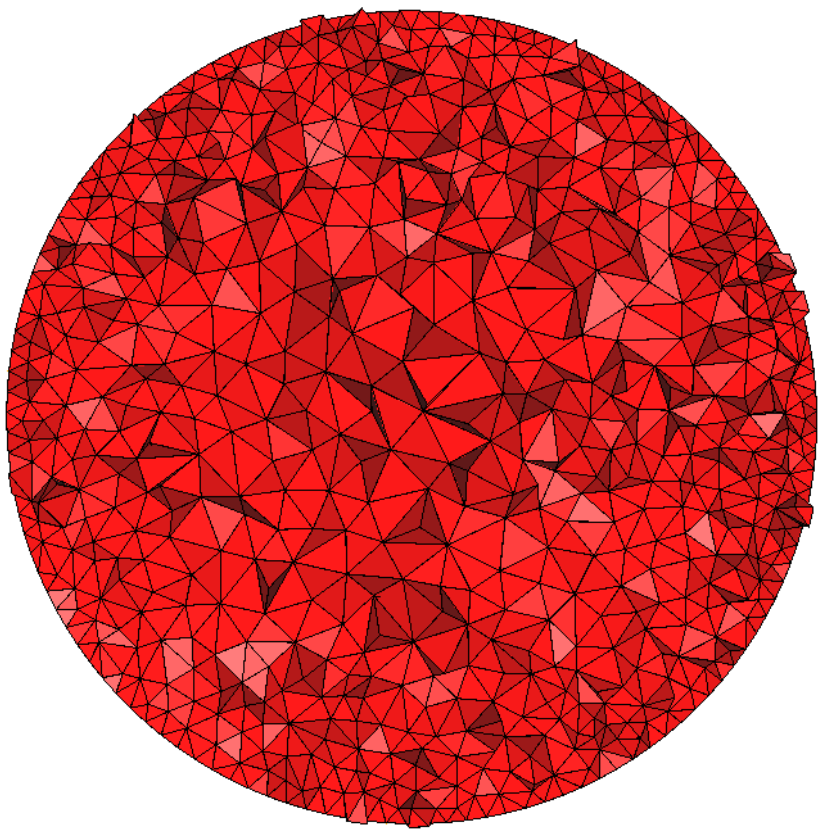


Intrinsic structure is incomplete

What's Different?

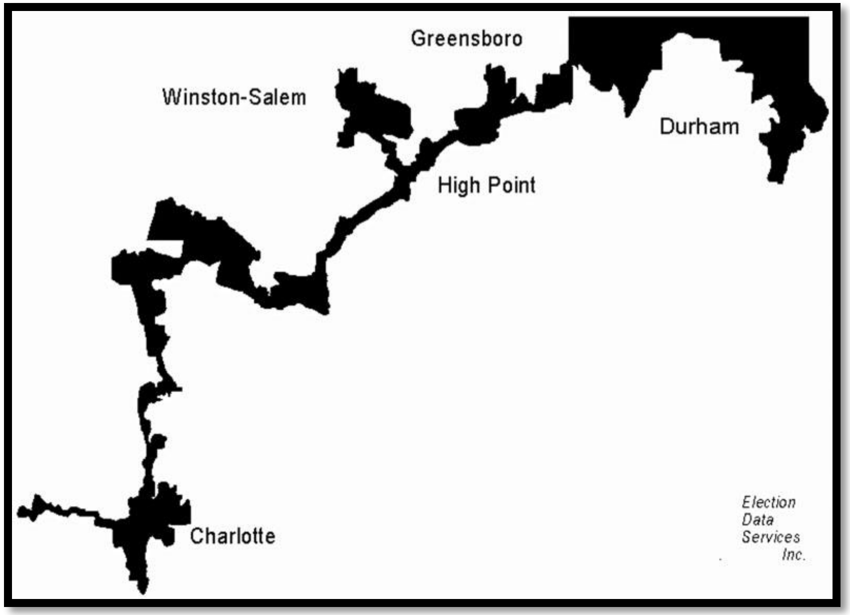


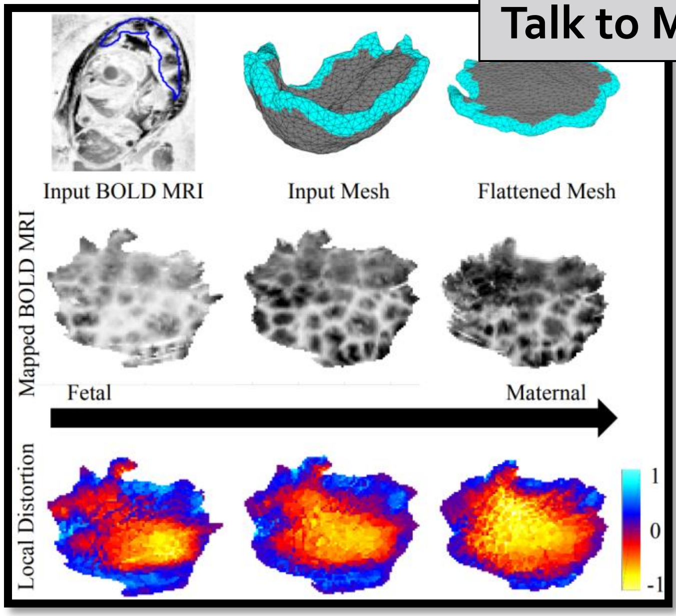
$\cong$



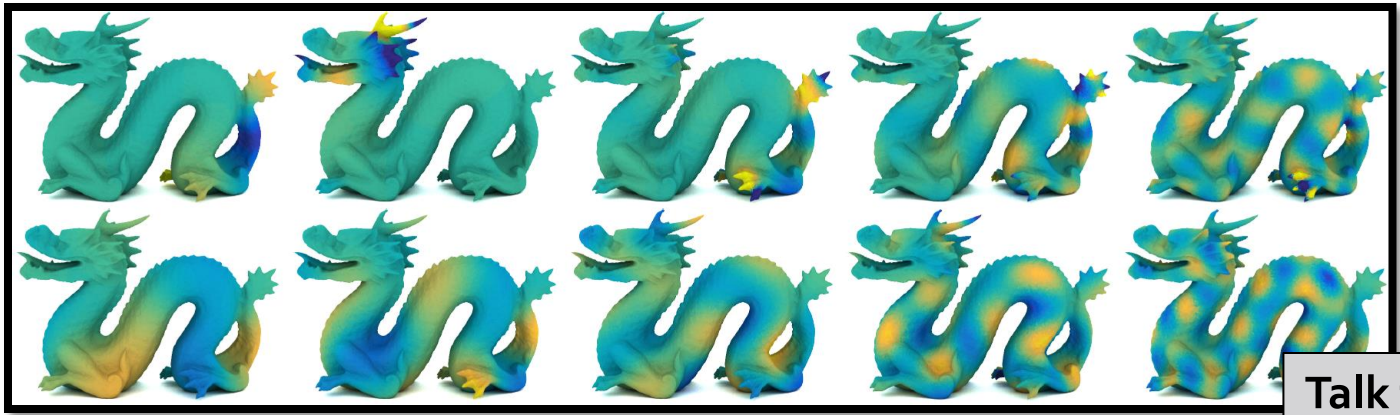
Interesting geometry still outside

Many Applications





Talk to Maz!

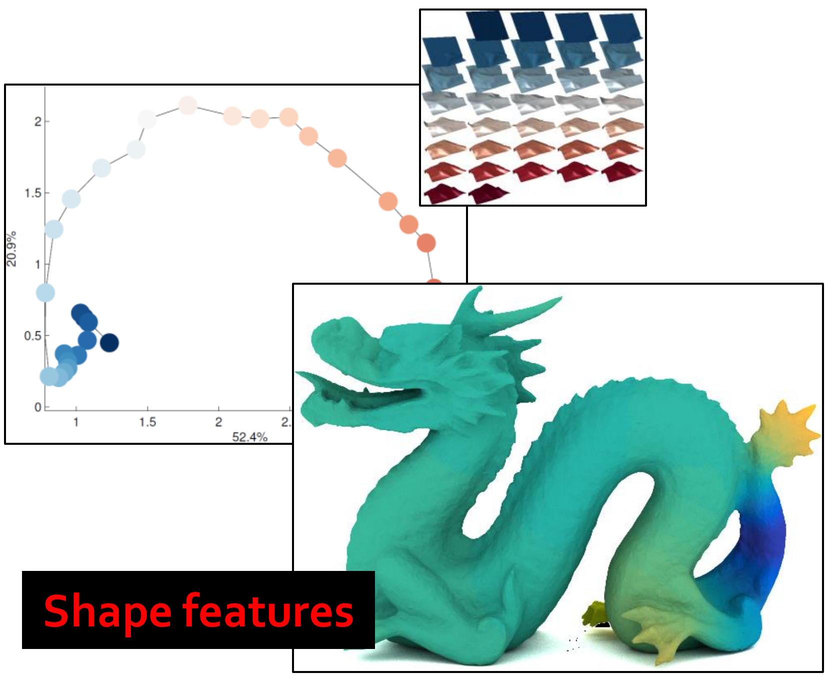


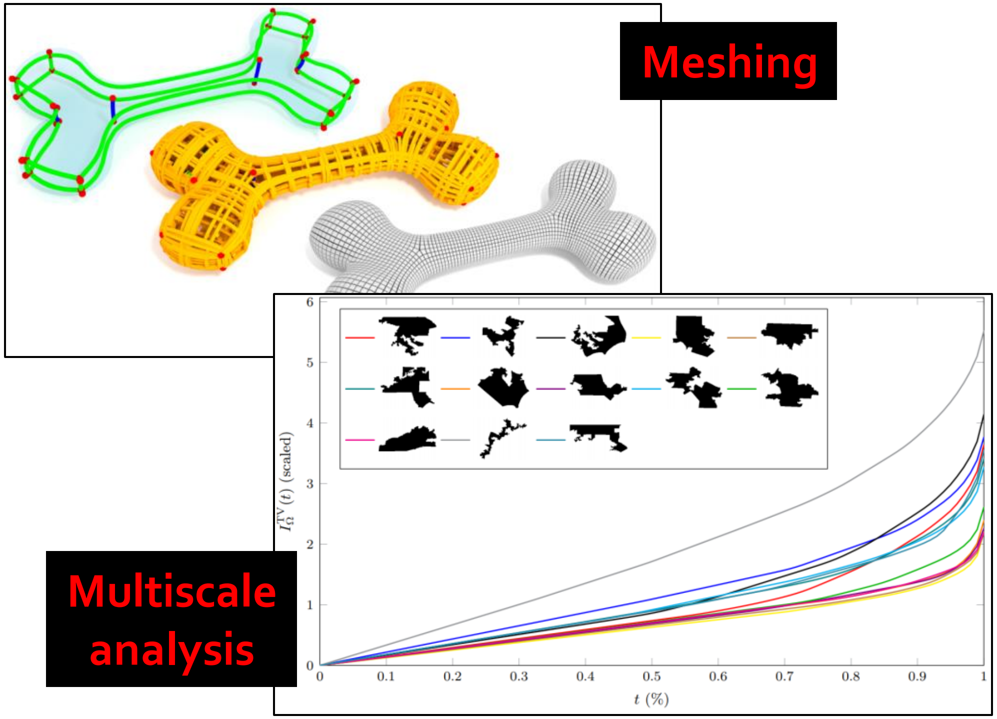
Talk to Yu!

Plan for Today

Geometry processing algorithms

tuned for volumes.

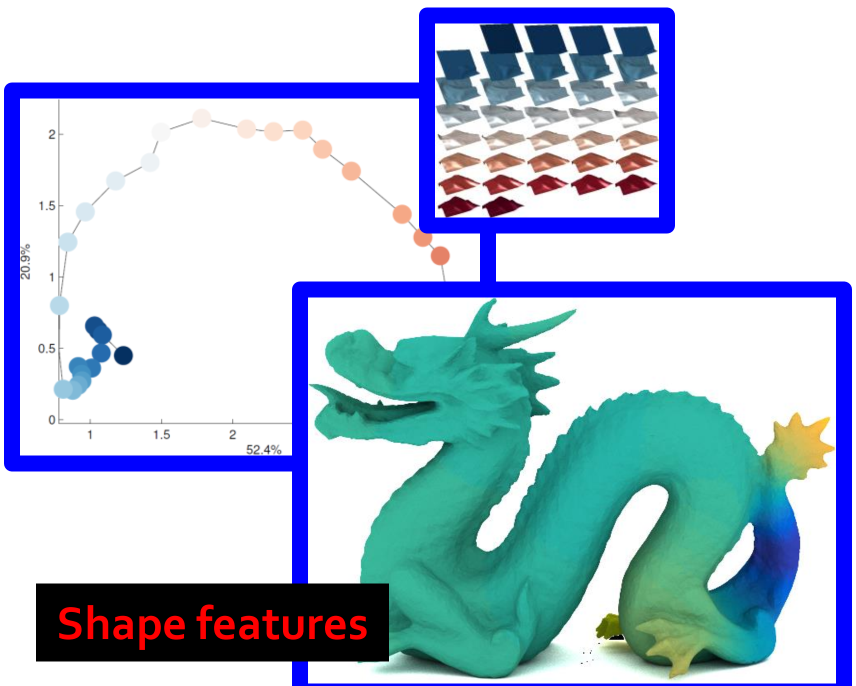




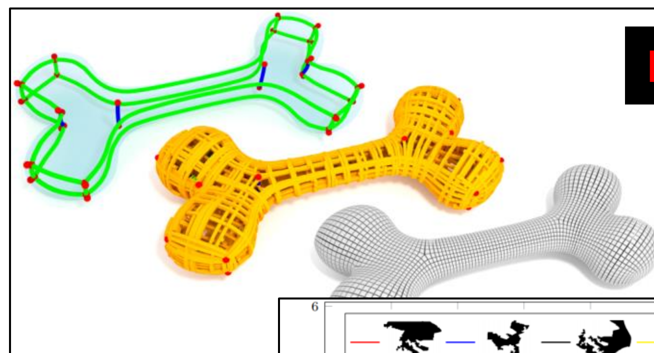
Plan for Today

Geometry processing algorithms

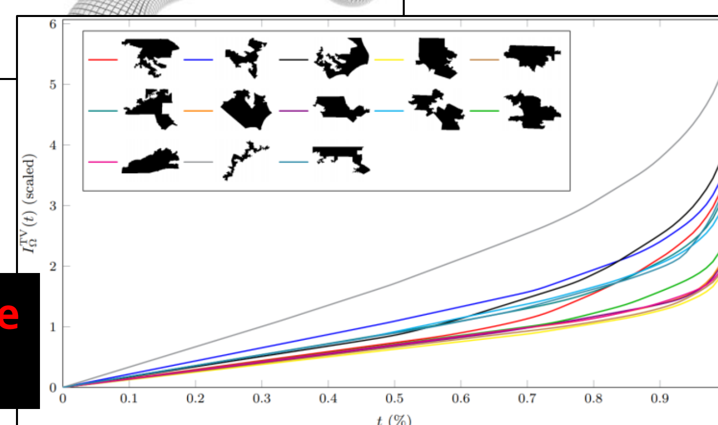
tuned for volumes.



Shape features



Meshing



Multiscale analysis

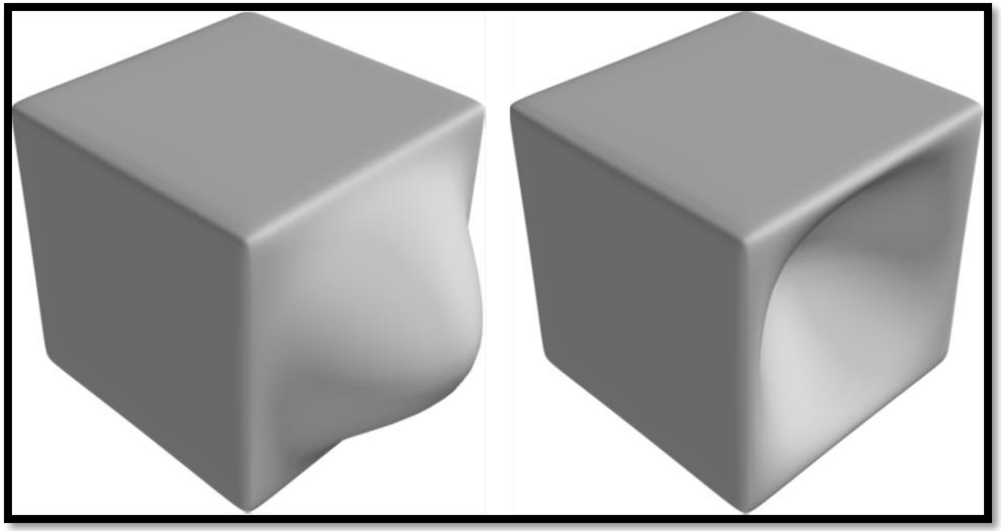
Starting Point: Spectral Geometry

http://pngimg.com/upload/hammer_PNG3886.png

You can learn a lot about a shape by hitting it (lightly) with a hammer!

$$\Delta f = \lambda f$$

Reminder

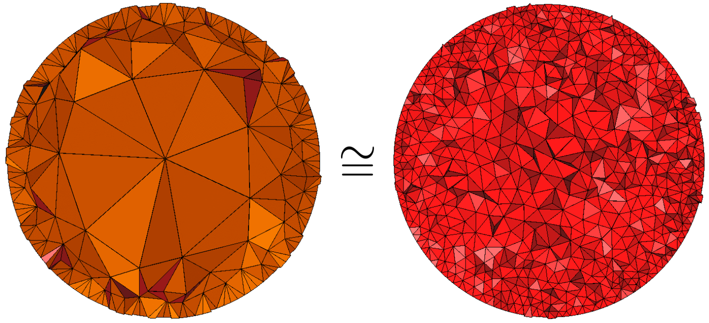


But calculations on a volume are expensive!

Not the same.

Alternative Proposal

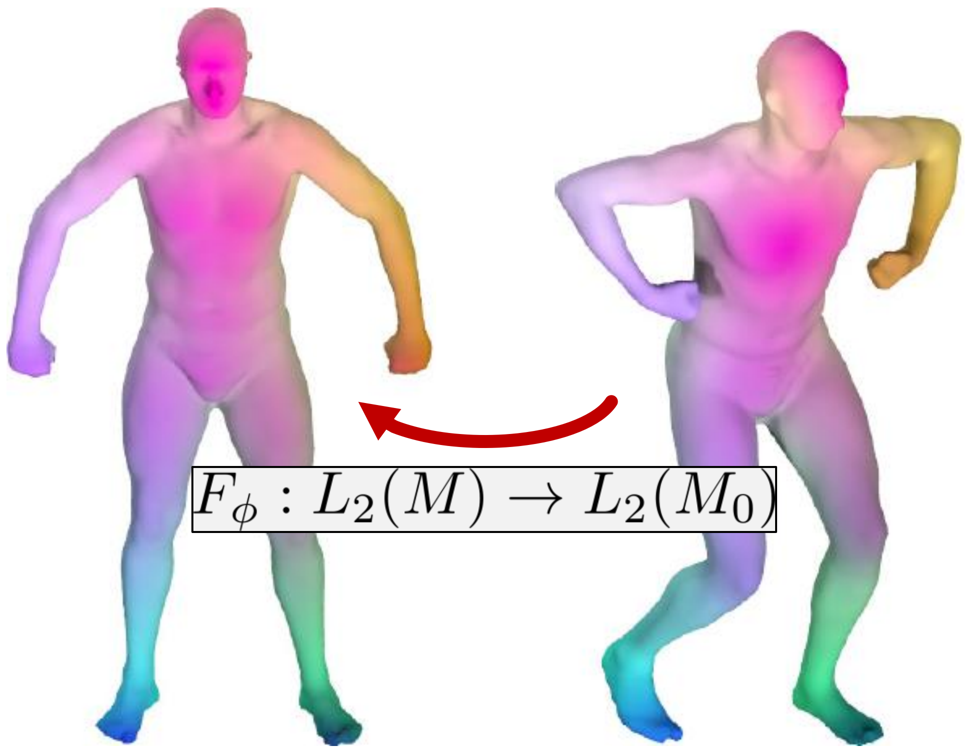
- **Advantages of spectral geometry**
  Multiscale, linear algebra, PDE interpretation
- **Complete characterization of shape**
  Fully encodes geometry, no judgment call about what's relevant for a computational problem
- **Computed from boundary**
  For efficiency *and* consistency



Possible context:

Shape Differences

[Rustamov et al. 2013]



$$\langle f, g \rangle_F^M := \langle F_\phi[f], F_\phi[g] \rangle^{M_0}$$

Trick:

Compare surfaces by comparing inner product matrices.

$$\langle f, g \rangle_A := \int_M f(x) g(x) \, dA$$

$$\langle f, g \rangle_C := \int_M [\nabla f(x) \cdot \nabla g(x)] \, dA$$

Functional map *pulls back* products

Continuous Question

[Rustamov et al. 2013]

Given

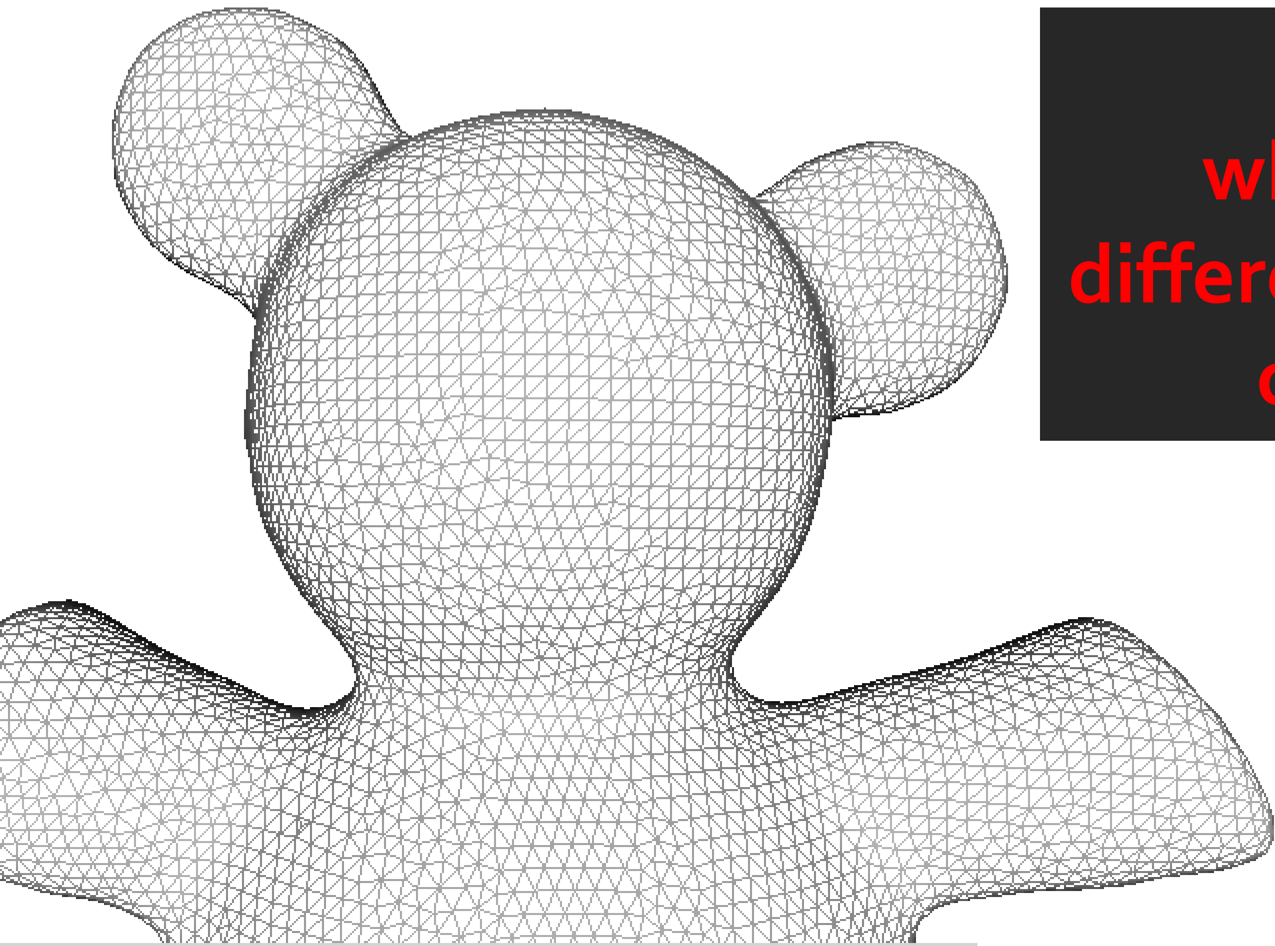
area-based and conformal inner product matrices,

can you compute

lengths and angles?

Discrete Question

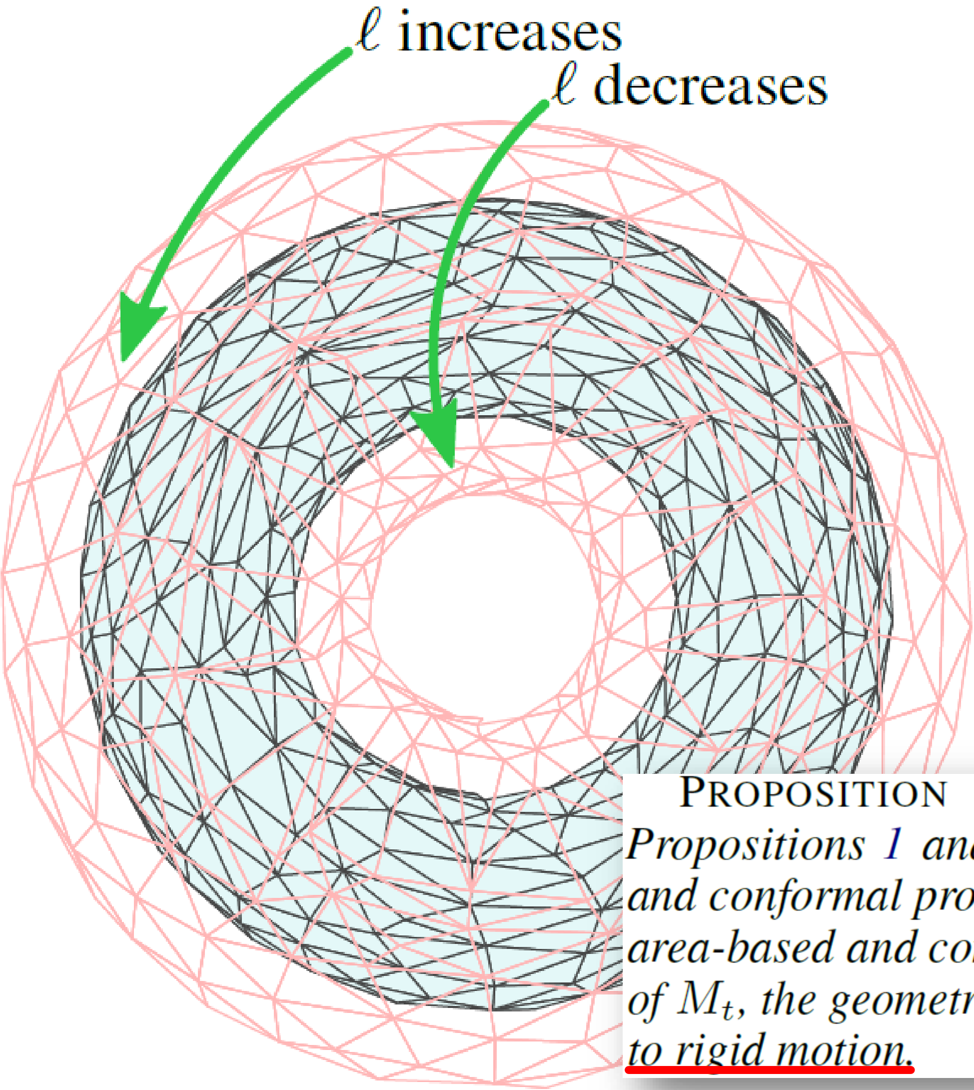
Precisely
what do shape differences determine on meshes?



Edge lengths.

"Functional Characterization of Intrinsic and Extrinsic Geometry." Corman et al. TOG 2017.

Extension to Extrinsic Shape

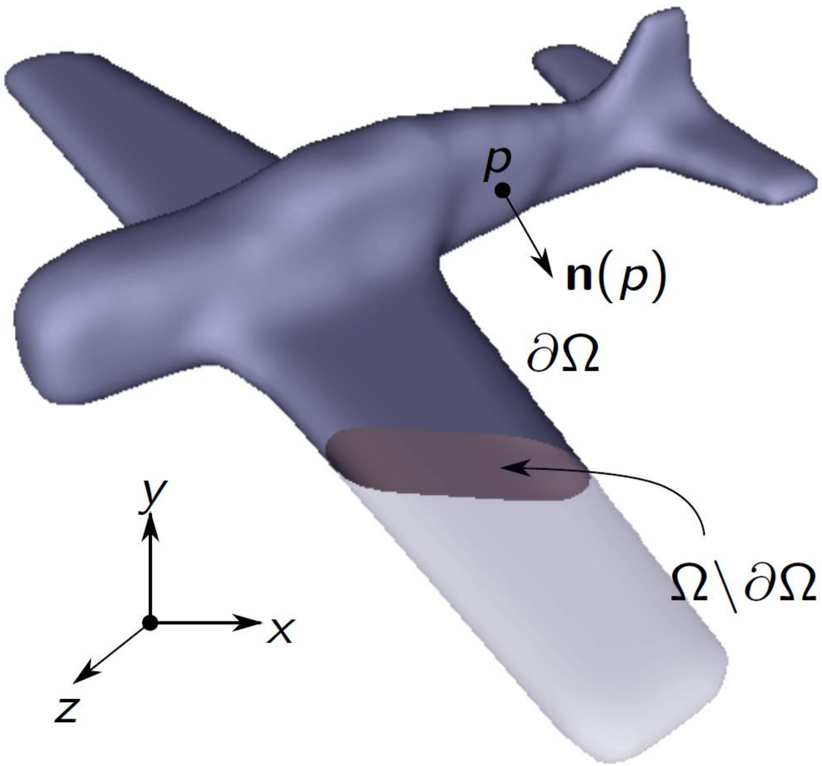


Throw in the offset surface.

Encodes mean curvature!

PROPOSITION 4. *Suppose a mesh M satisfies the criteria in Propositions 1 and 2. Given the topology of M, the area-based and conformal product matrices $A(\mu)$ and $C(\nu;\mu)$ of M, and the area-based and conformal product matrices $A_t(\mu_t)$ and $C(\nu_t;\mu_t)$ of M_t, the geometry of M can (almost always) be reconstructed up to rigid motion.*

Boundary Representations?



Surface eigenfunctions:

$$\begin{aligned}\min_f \ & \int_{\partial\Omega} \|\nabla f\|_2^2 \, dA \\ \text{s.t.} \ & \int_{\partial\Omega} |f|^2 \, dA = 1\end{aligned}$$

- Isometry invariant
- Easy to compute

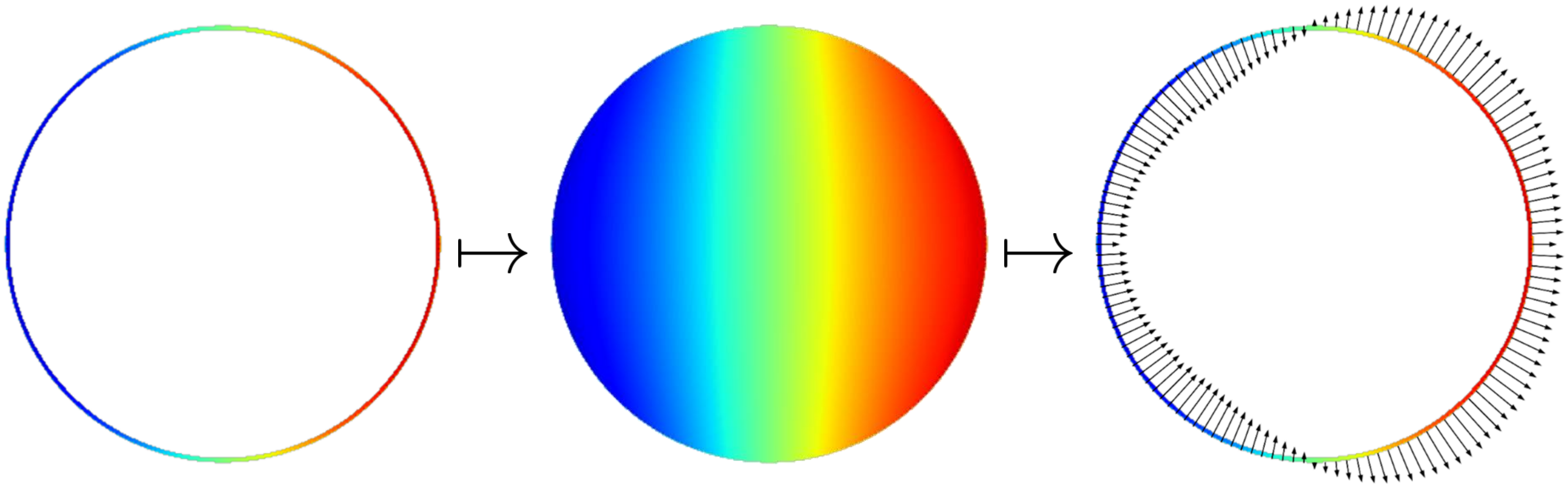
Volume eigenfunctions:

$$\begin{aligned}\min_f \ & \int_{\Omega} \|\nabla f\|_2^2 \, dV \\ \text{s.t.} \ & \int_{\Omega} |f|^2 \, dV = 1\end{aligned}$$

- Volume dependent
- Requires tet mesh

Wang, Ben-Chen, Polterovich, and Solomon. "Steklov Spectral Geometry for Extrinsic Shape Analysis." ACM Transactions on Graphics (TOG), 2019.

Dirichlet-to-Neumann



$$f|_{\partial\Omega} \mapsto \Delta f \equiv 0 \mapsto \nabla f \cdot \hat{n}$$

Surface data to surface data

Theoretical Results

$$e^{-t\mathcal{S}}(x,x) = \sum_{i=0}^{\infty} e^{-t\lambda_i}\psi_i(x)^2 \sim \sum_{k=0}^{\infty} a_k(x)t^{k-2} + \sum_{\ell=1}^{\infty} b_\ell(x)t^\ell \log t$$

$$\begin{aligned} a_0(x) &\equiv \frac{1}{2\pi} \\ a_1(x) &= \frac{H(x)}{4\pi} \\ a_2(x) &= \frac{1}{16\pi}\left(H(x)^2 + \frac{K(x)}{3}\right) \end{aligned}$$

Polterovich & Sher: "Heat invariants of the Steklov problem."
***J. Geometric Analysis* 25.2 (2015): 924-950.**

(Slightly) New Result

PROPOSITION 3.1. *Suppose $\Omega_1, \Omega_2 \subseteq \mathbb{R}^3$ are compact domains with C^∞ boundaries Γ_1, Γ_2 and Dirichlet-to-Neumann operators $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Let $\alpha : \Omega_1 \to \Omega_2$ be a bijection which is C^∞ up to the boundary, and let $\tilde{\alpha} : \Gamma_1 \to \Gamma_2$ be the induced mapping between the boundaries. Suppose that the operators S_1 and S_2 coincide up to composition with $\tilde{\alpha}$, i.e. $S_2 f = \tilde{\alpha}_* S_1 \tilde{\alpha}^* f$, for any $f \in C^\infty(\Gamma_2)$, where $\tilde{\alpha}^* f = f \circ \tilde{\alpha}$, $\tilde{\alpha}_* g = g \circ \tilde{\alpha}^{-1}$ denote the pull-backs by $\tilde{\alpha}$ and $\tilde{\alpha}^{-1}$, respectively. Then α is a rigid motion.*

Proof suggested by Mikhail Karpukhin

Computation

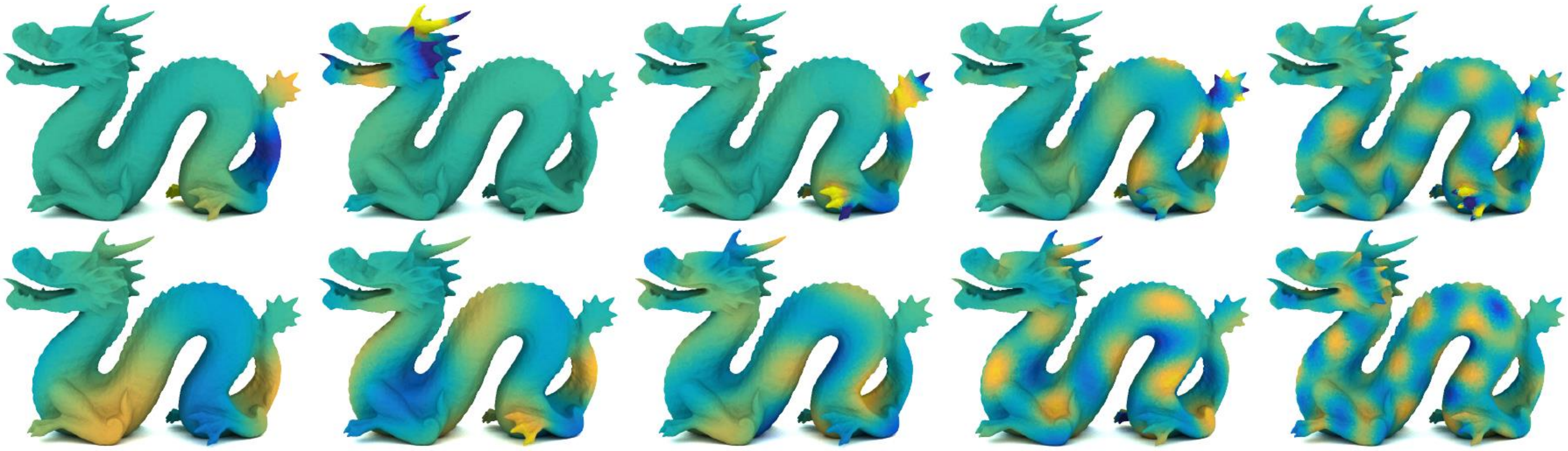
$$u(x) = \int_{\partial\Omega} \left[G(y-x)\frac{\partial u(y)}{\partial \hat{n}} - u(y)\frac{\partial G(y-x)}{\partial \hat{n}} \right] dy \;\forall x \in \operatorname{int}\Omega$$

(see paper for details)

Boundary element method (BEM)

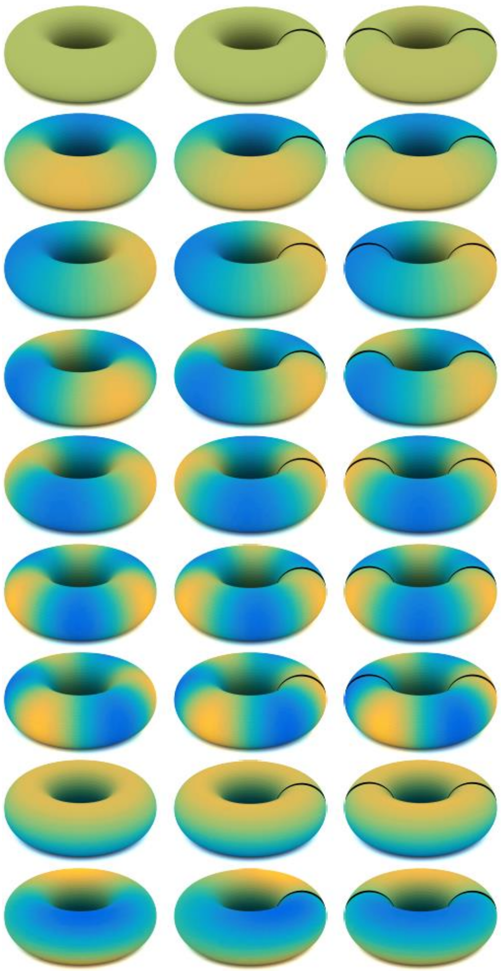
Comparison: Eigenfunctions

Steklov

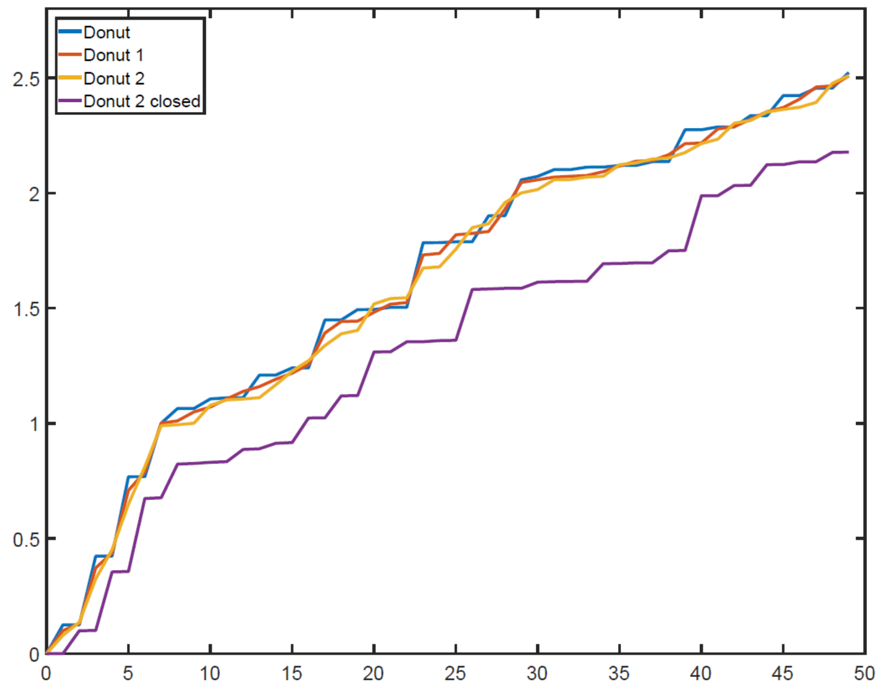


Laplacian

Stability to Cuts



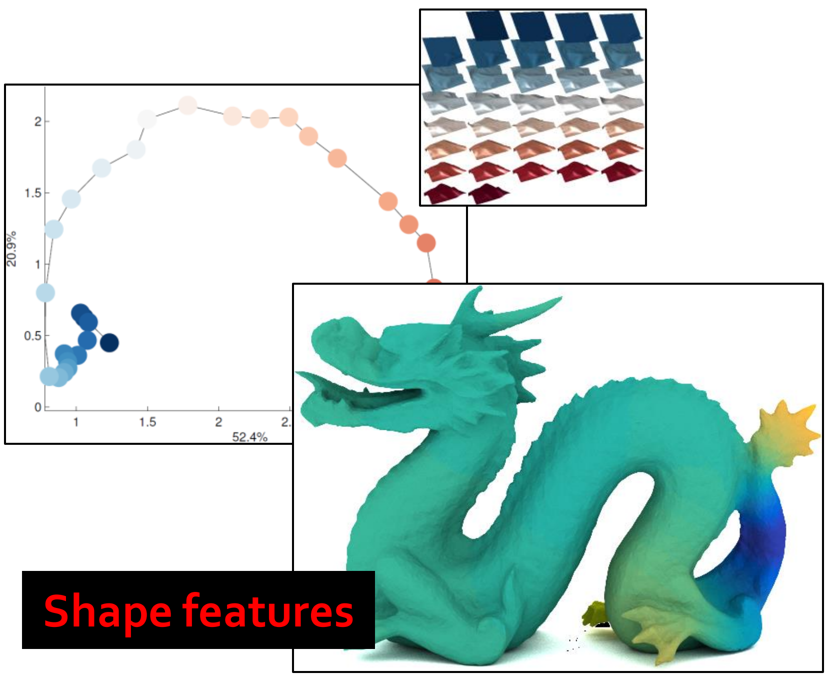
Donut Donut 1 Donut 2



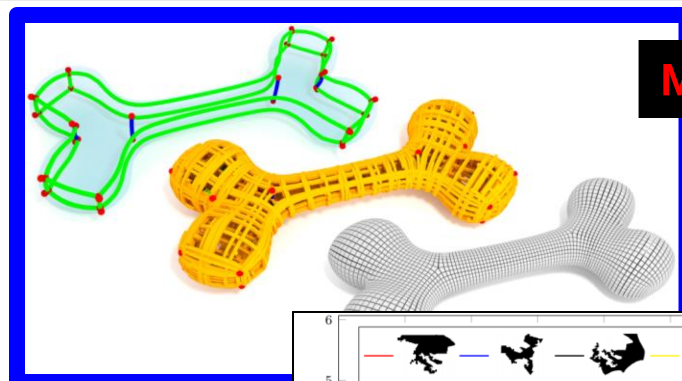
Plan for Today

Geometry processing algorithms

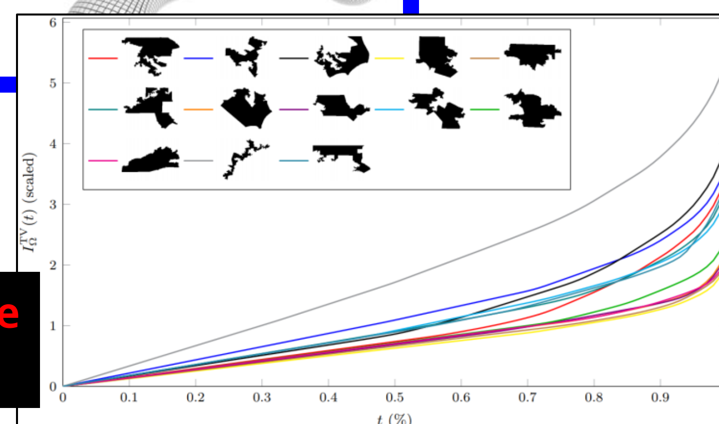
tuned for volumes.



Shape features

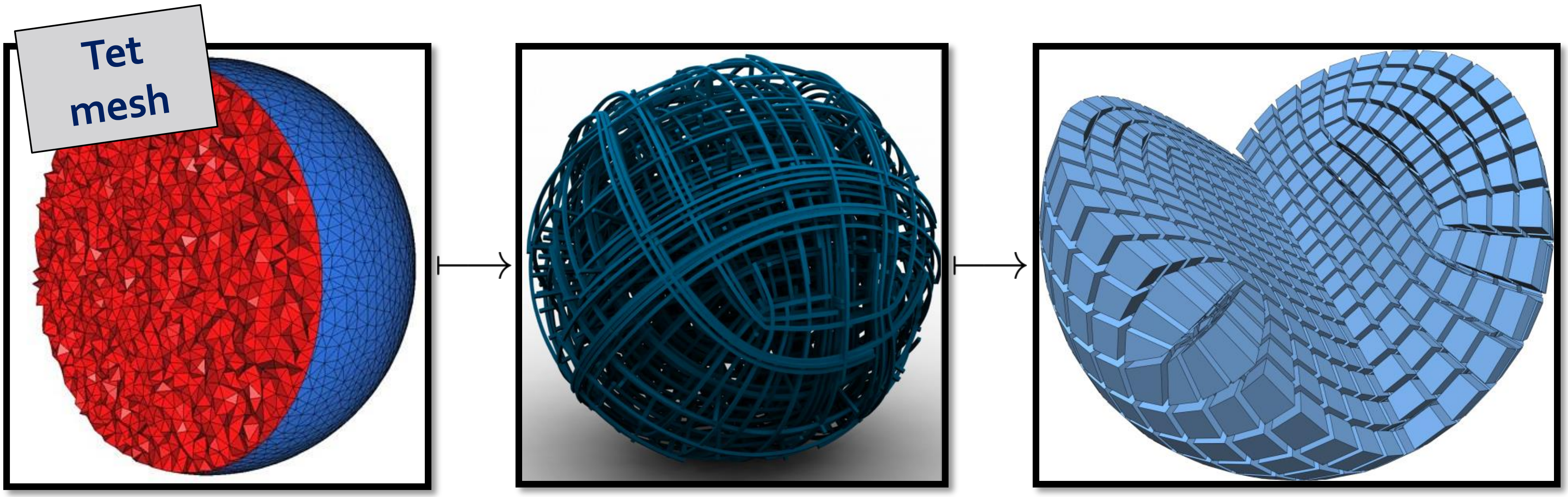


Meshing



Multiscale analysis

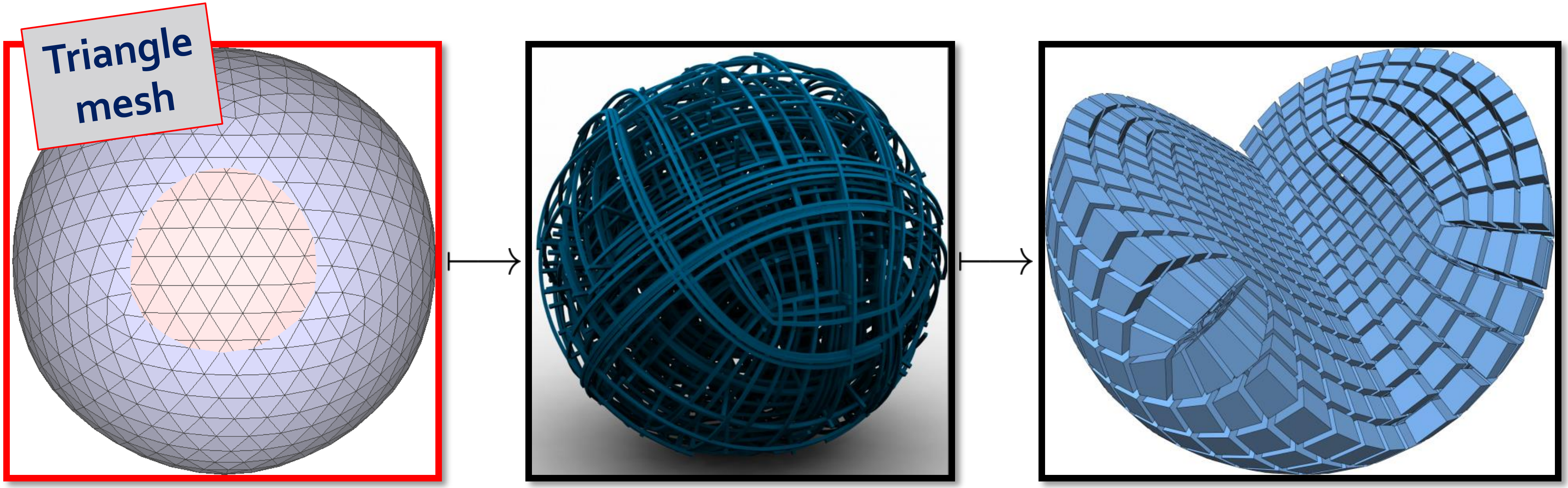
Common Pipeline



Sphere tet mesh from http://doc.cgal.org/latest/Mesh_3/index.html

Frame per element on a tet mesh

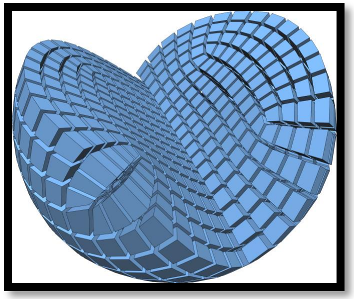
Idea

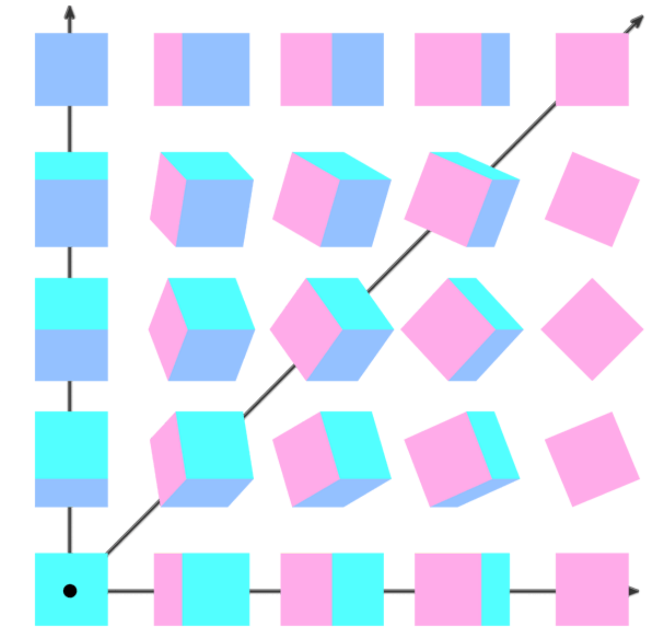


Solomon, Vaxman, and Bommes. "Boundary Element Octahedral Fields in Volumes."
ACM Transactions on Graphics (TOG) 36.3, 2017

Work from boundary representation

Octahedral Field



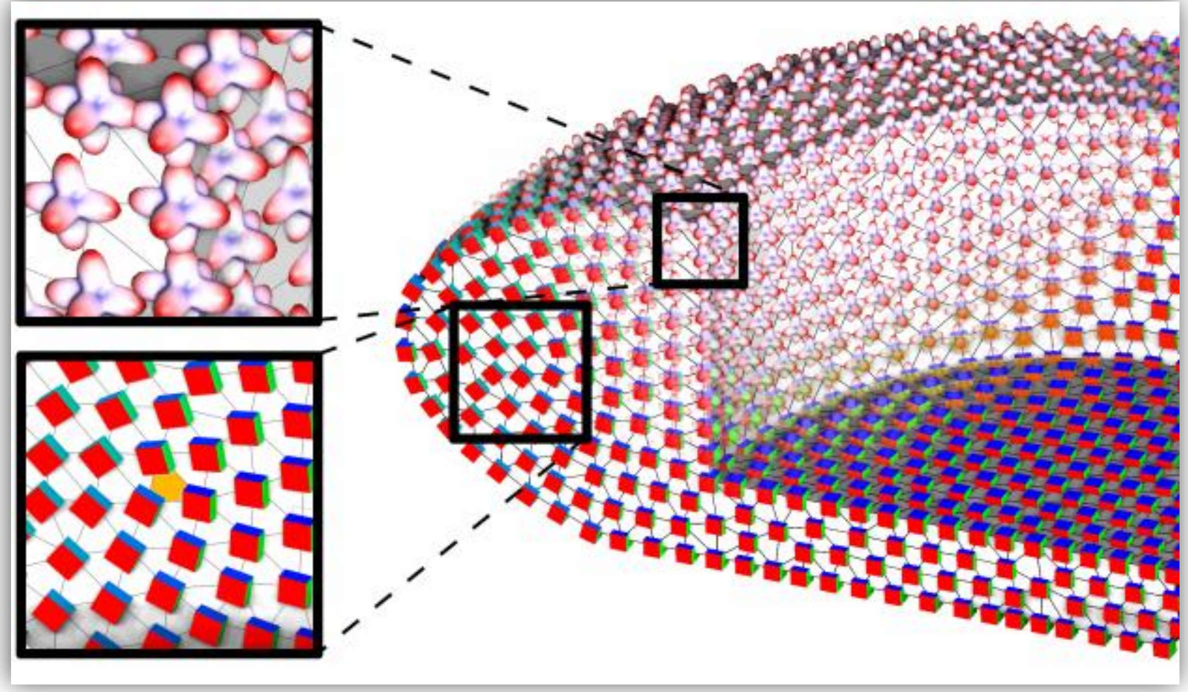


{Rotations of a cube}

https://design.tutsplus.com/

Used to guide meshing

Field-Based Meshing

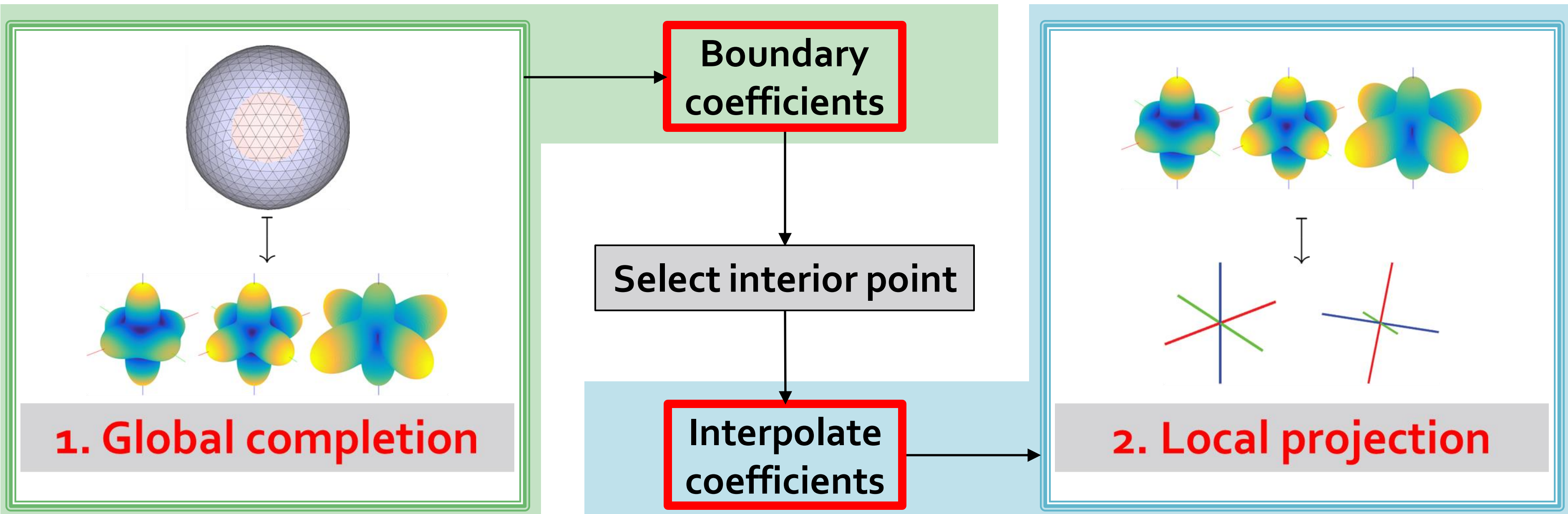


Nine spherical harmonic coefficients **per point**

Original idea in [Huang et al. 2011]
Visualization from [Ray, Sokolov, and Lévy 2016]

$$f(x, y, z) = x^4 + y^4 + z^4$$

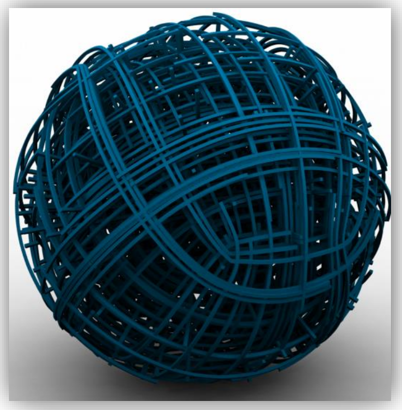
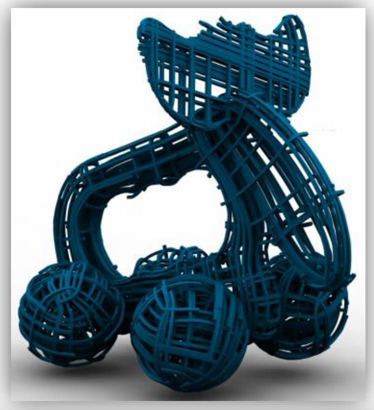
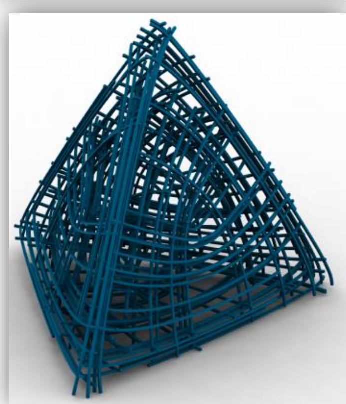
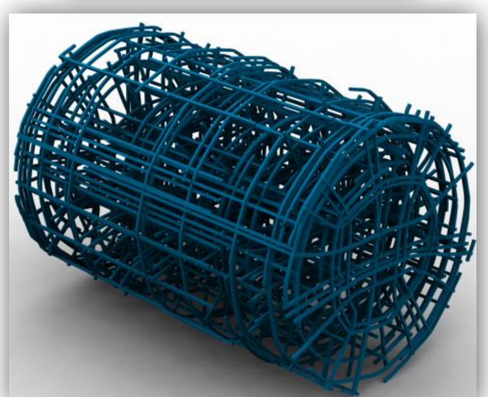
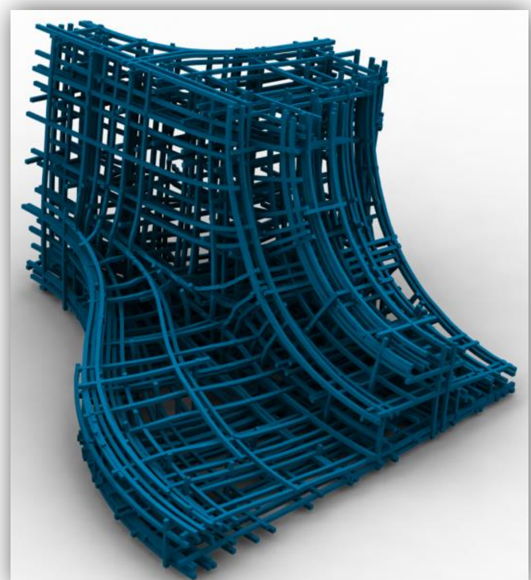
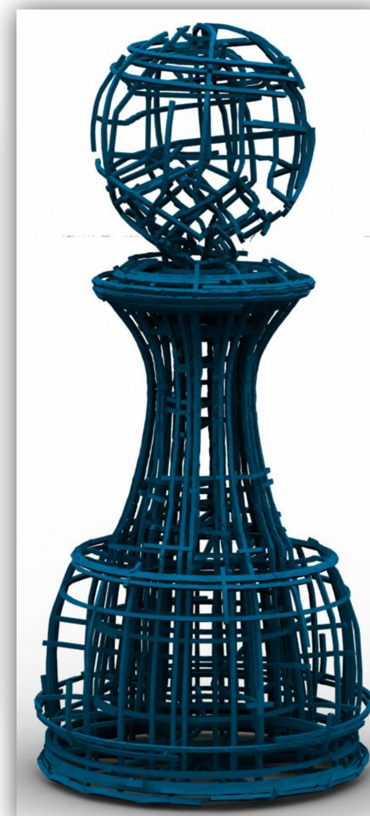
BEM Approach



Once per boundary surface
Uses Dirichlet-to-Neumann!

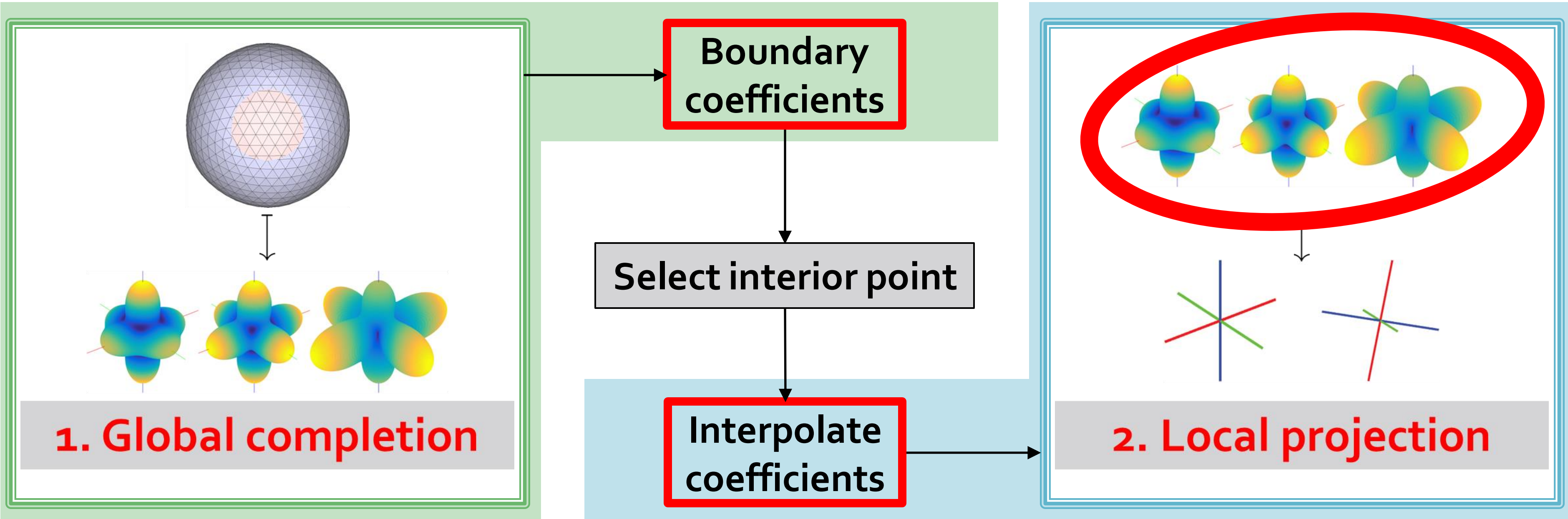
Once per sample point

Example Frame Fields

Non-Tight Relaxation

Not rotations of $x^4 + y^4 + z^4$



Once per boundary surface
Uses Dirichlet-to-Neumann!

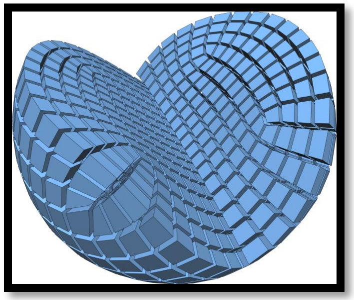
Once per sample point

Issue

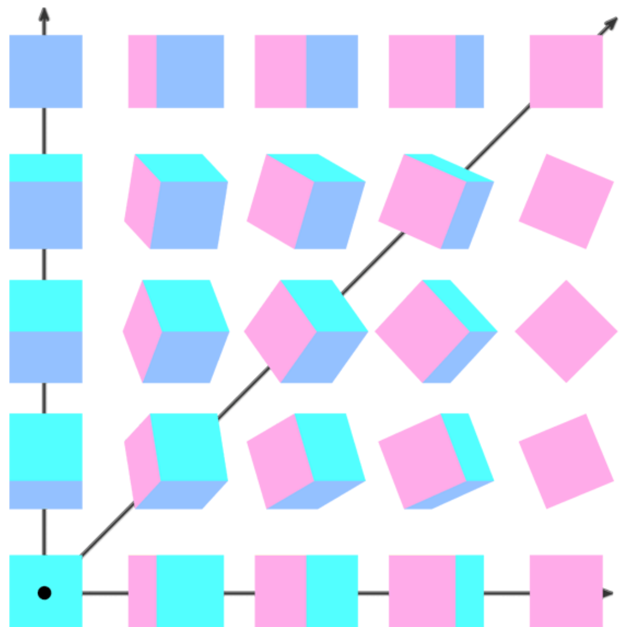
$$f(x, y, z) = x^4 + y^4 + z^4$$

$$\{\text{rotations of } f(x, y, z)\} \not\cong \{\text{degree-4 polynomials}\}$$

Backtracking



https://design.tutsplus.com/



$$\{\text{Rotations of a cube}\} \cong \mathrm{SO}(3)/O$$

Rotations

Octahedral group

Algebraic variety!

"Algebraic Representations for Volumetric Frame Fields." Palmer, Bommes, & Solomon.

Octahedral variety

Representation Theory Perspective

Space of rotations

Wigner d-matrices

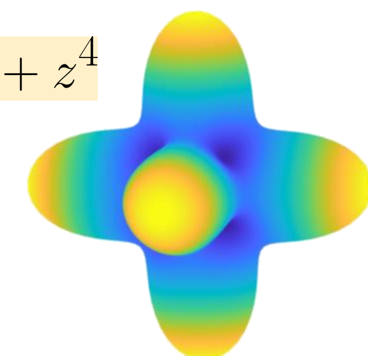
$$\begin{tikzcd} SO(3) \arrow[r, "\rho"] \arrow[d, two heads] & GL(9) \arrow[d, two heads, "\cdot q_0"] \\ \mathcal{F} \arrow[r, dashed, "\phi"] & F \end{tikzcd}$$

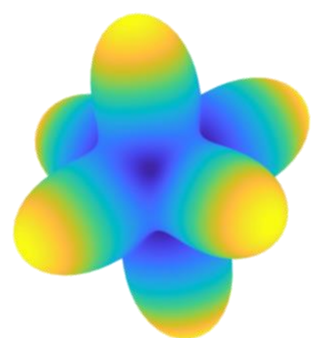
Octahedral variety

Orbit of *f*

Isometry (up to scale)

$f(x, y, z) = x^4 + y^4 + z^4$

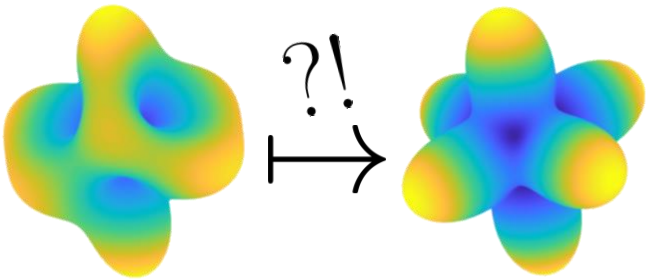




Roughly: Coefficients of $f(R^\top \mathbf{x})$

Two Optimization Algorithms

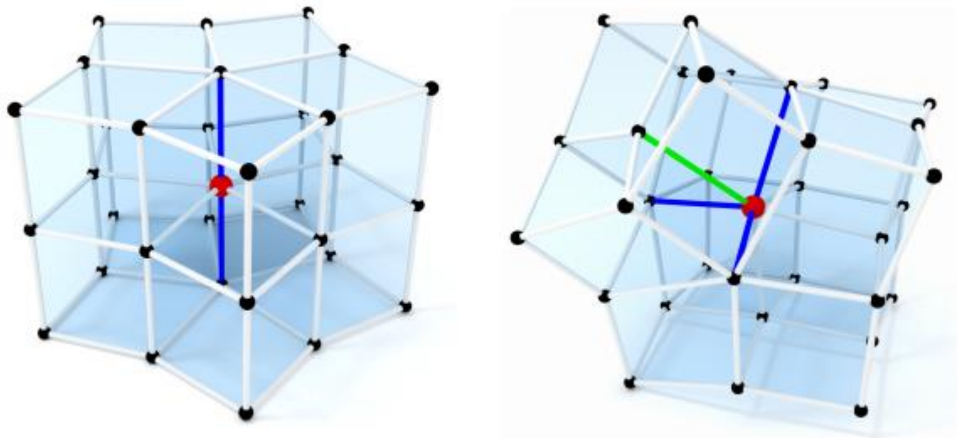
- **MBO**
  - Diffuse-and-project
  - SDP relaxation of projection operator
    - Open problem: Exact recovery?
- **Riemannian trust region (RTR)**
  - Gradient descent along constraint manifold
  - Closed-form exponential map



But:
Both require a tet mesh

Extension: Odeco Frames

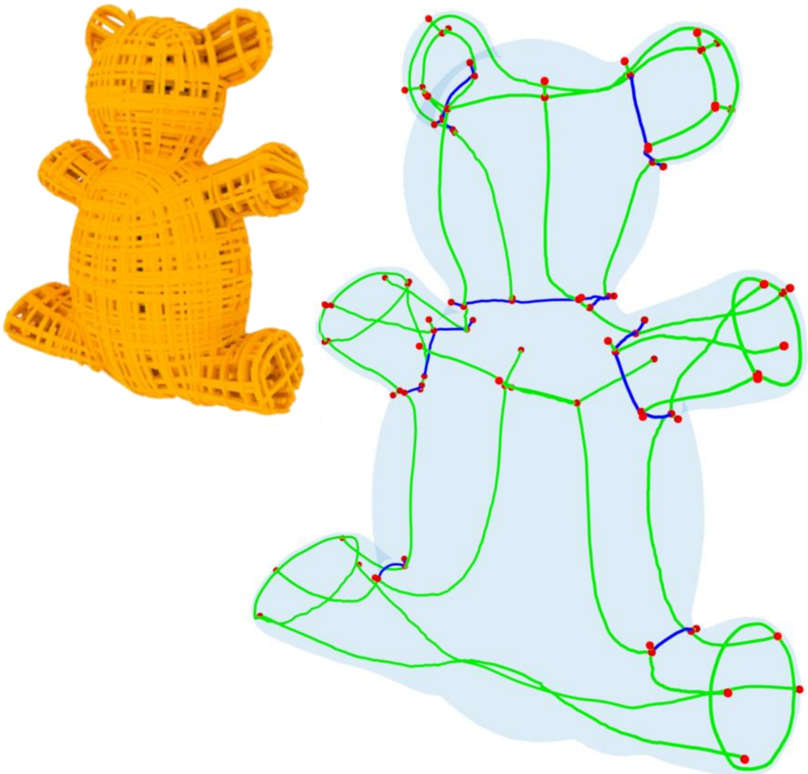
$$\sum_i \boxed{\lambda_i}(v_i^\top x)^d$$



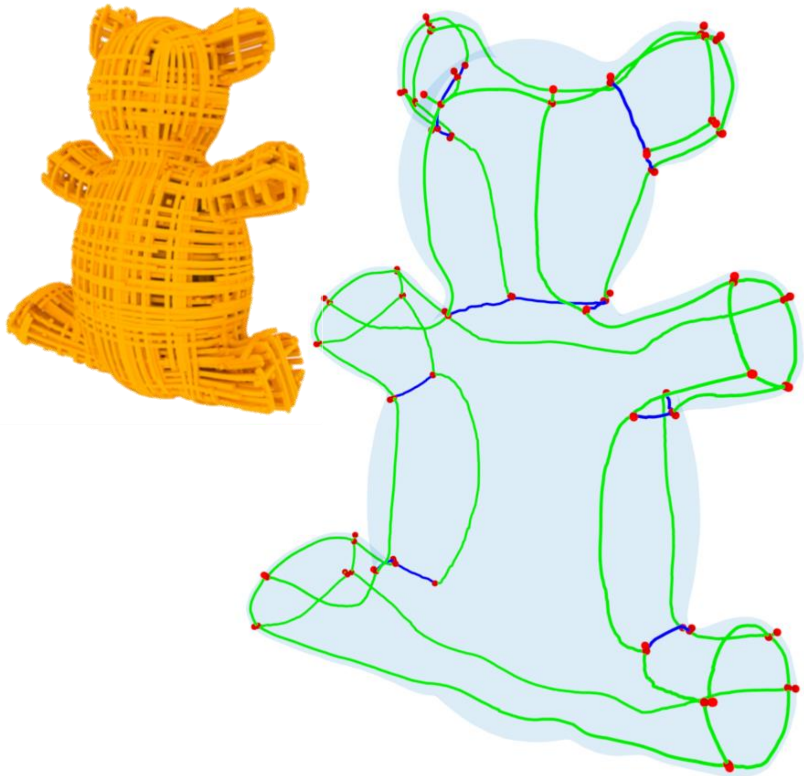
(0,5,2) (1, 3, 3)

Orthogonally-decomposable tensors

Improved Practical Result

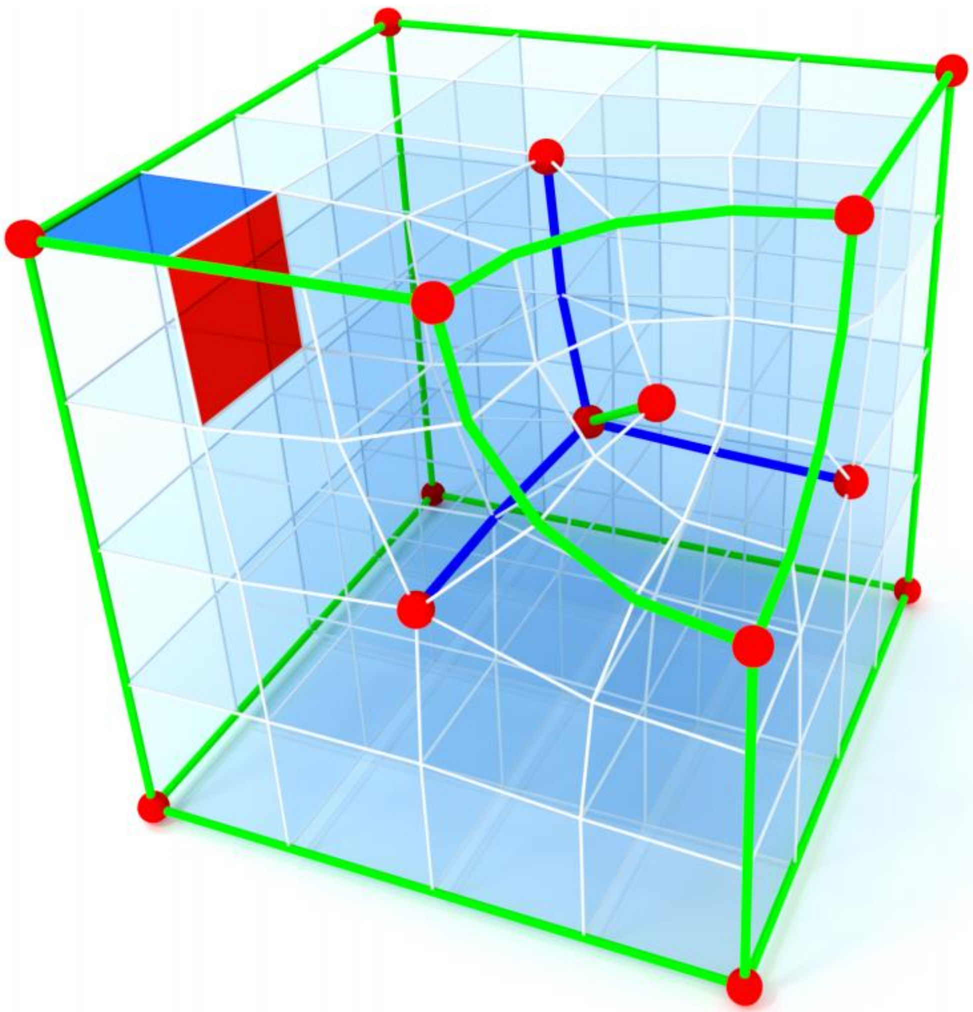


[Ray et al. 2016]



MBO+RTR

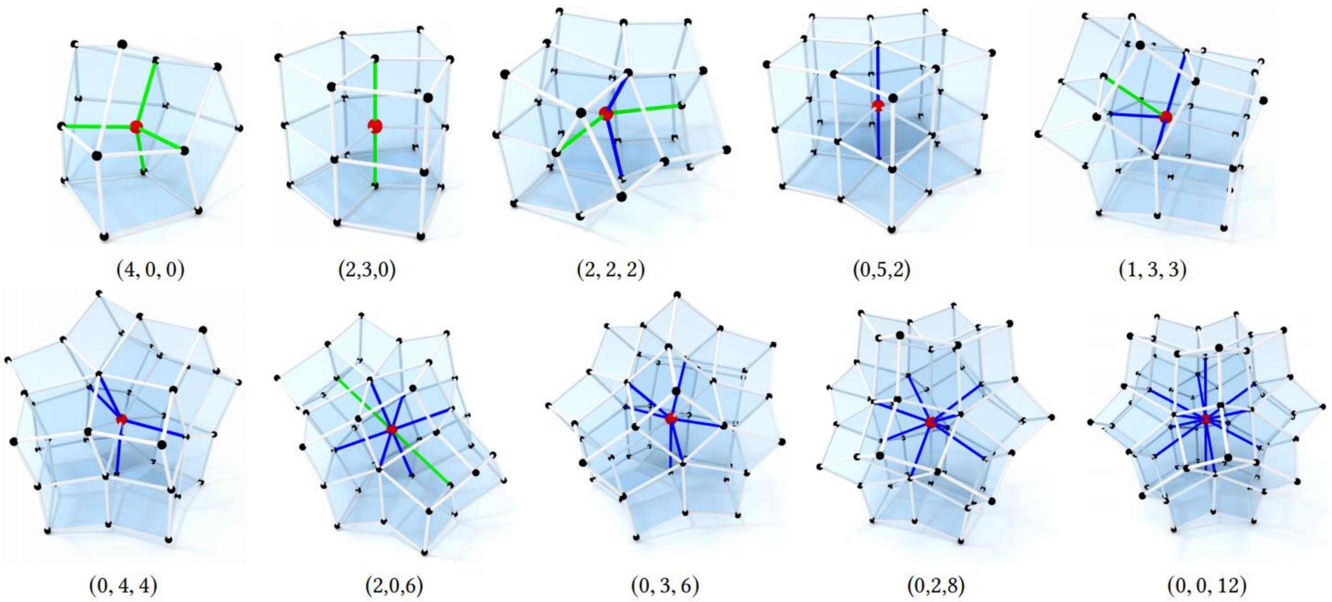
From Local to Global

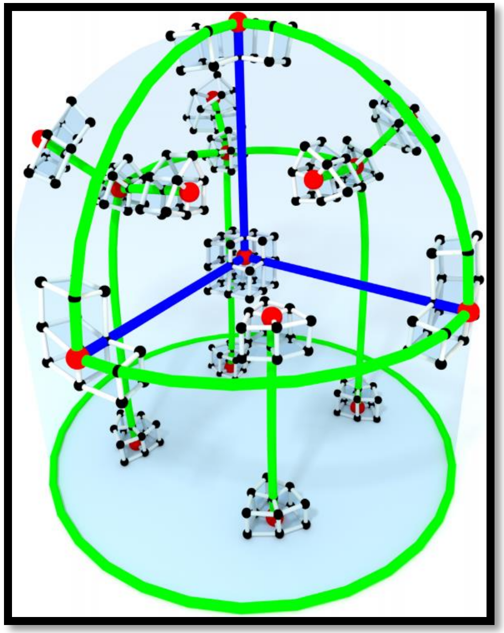


What singular structures are possible?

What is the relationship between meshes and fields?

Complete Set: Bounded Degree



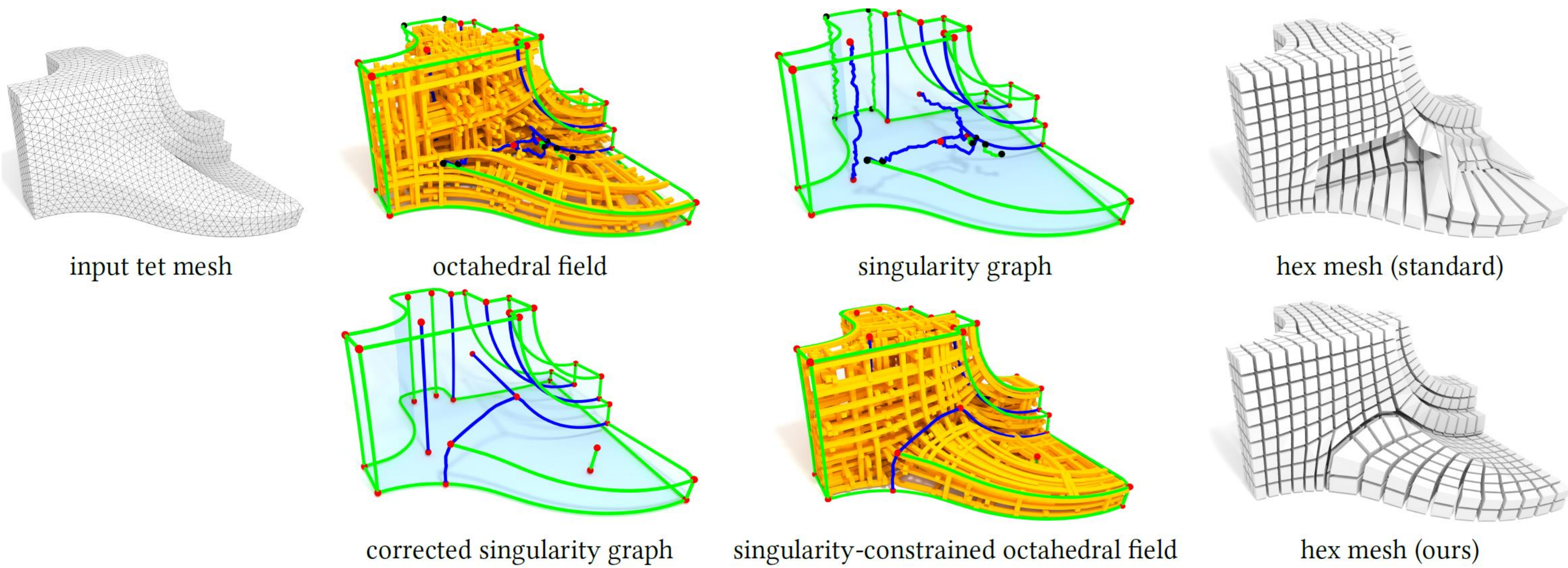


Complete local theory; global necessary condition; repair algorithm

Liu, Zhang, Chien, Solomon, and Bommes.
"Singularity-Constrained Octahedral Fields for Hexahedral Meshing." SIGGRAPH 2018.

Realize singular graph as a mesh?

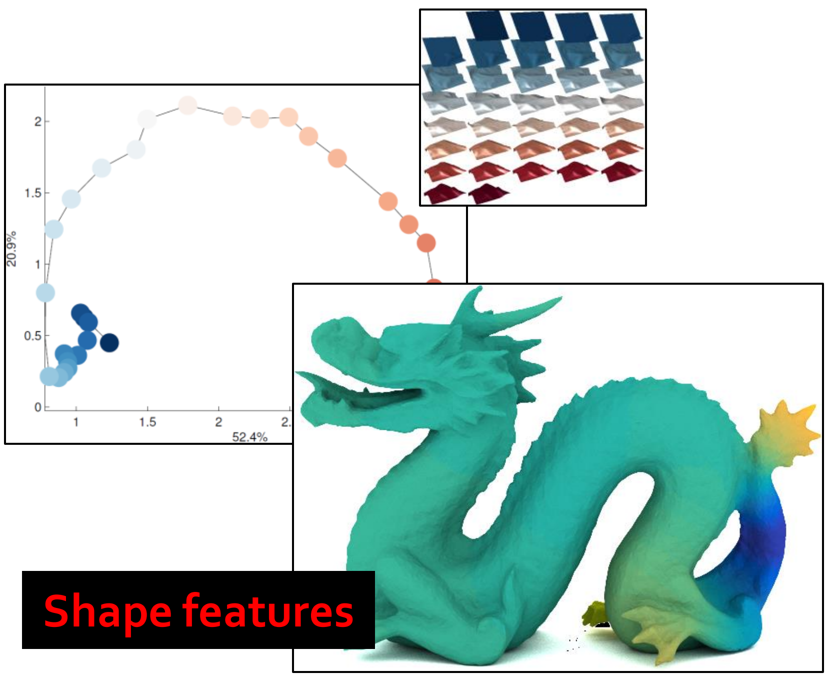
New Pipeline



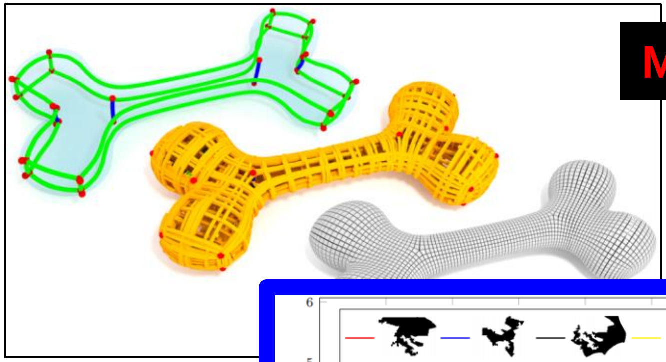
Plan for Today

Geometry processing algorithms

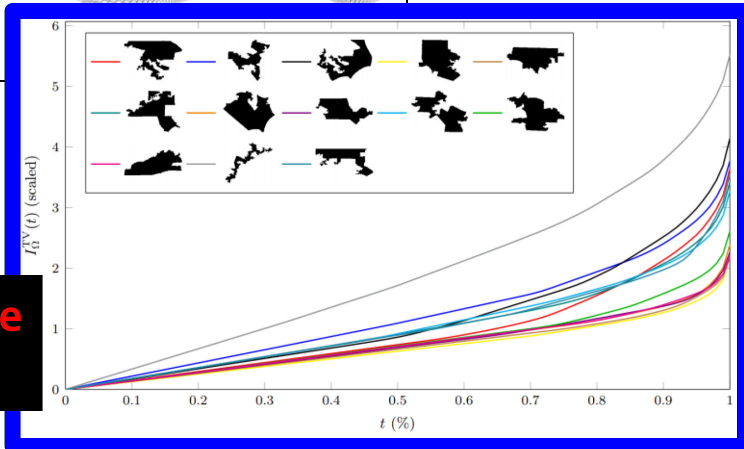
tuned for volumes.



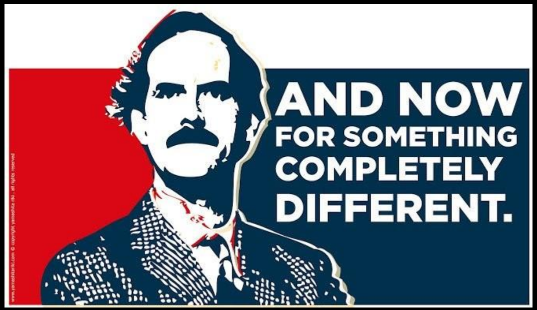
Shape features

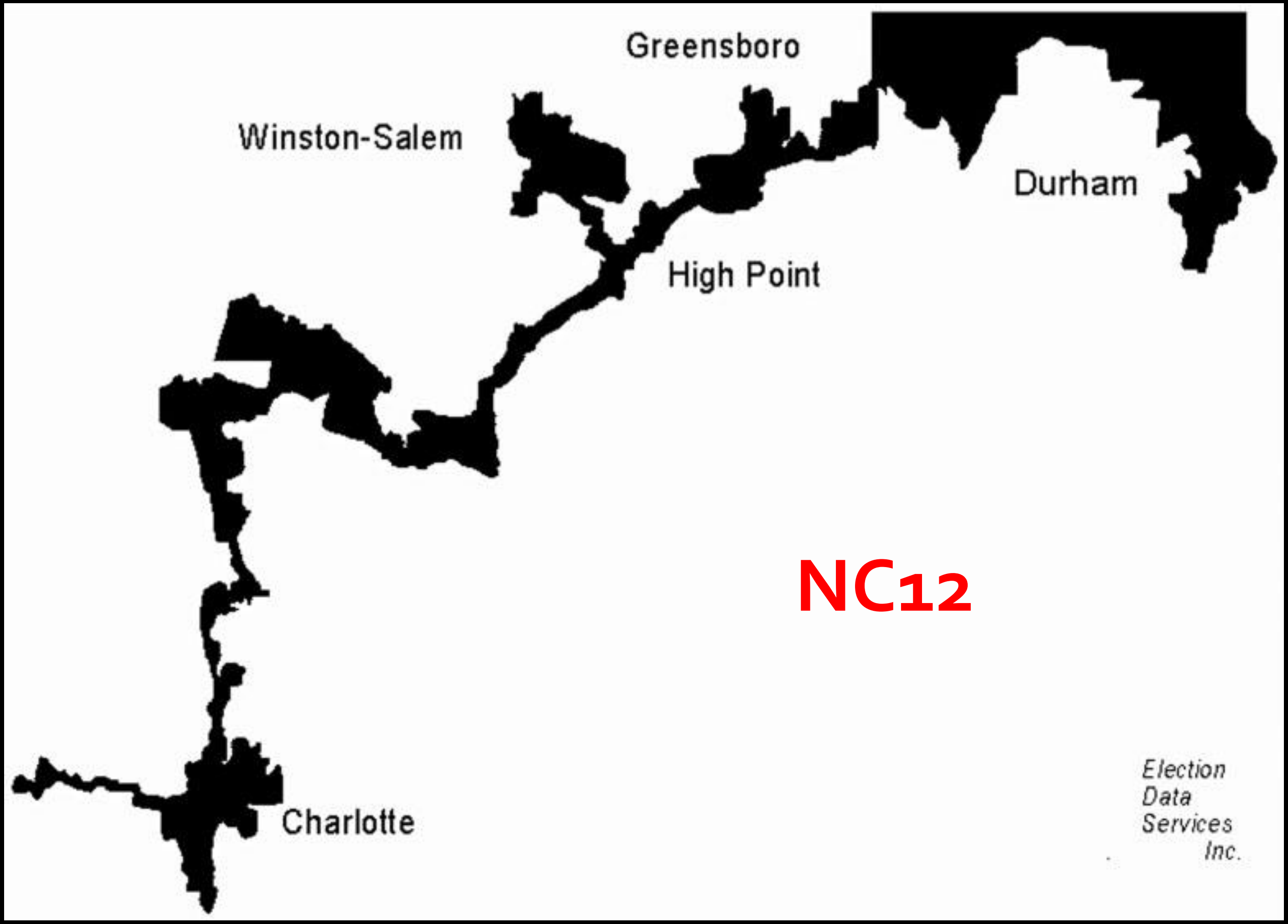


Meshing

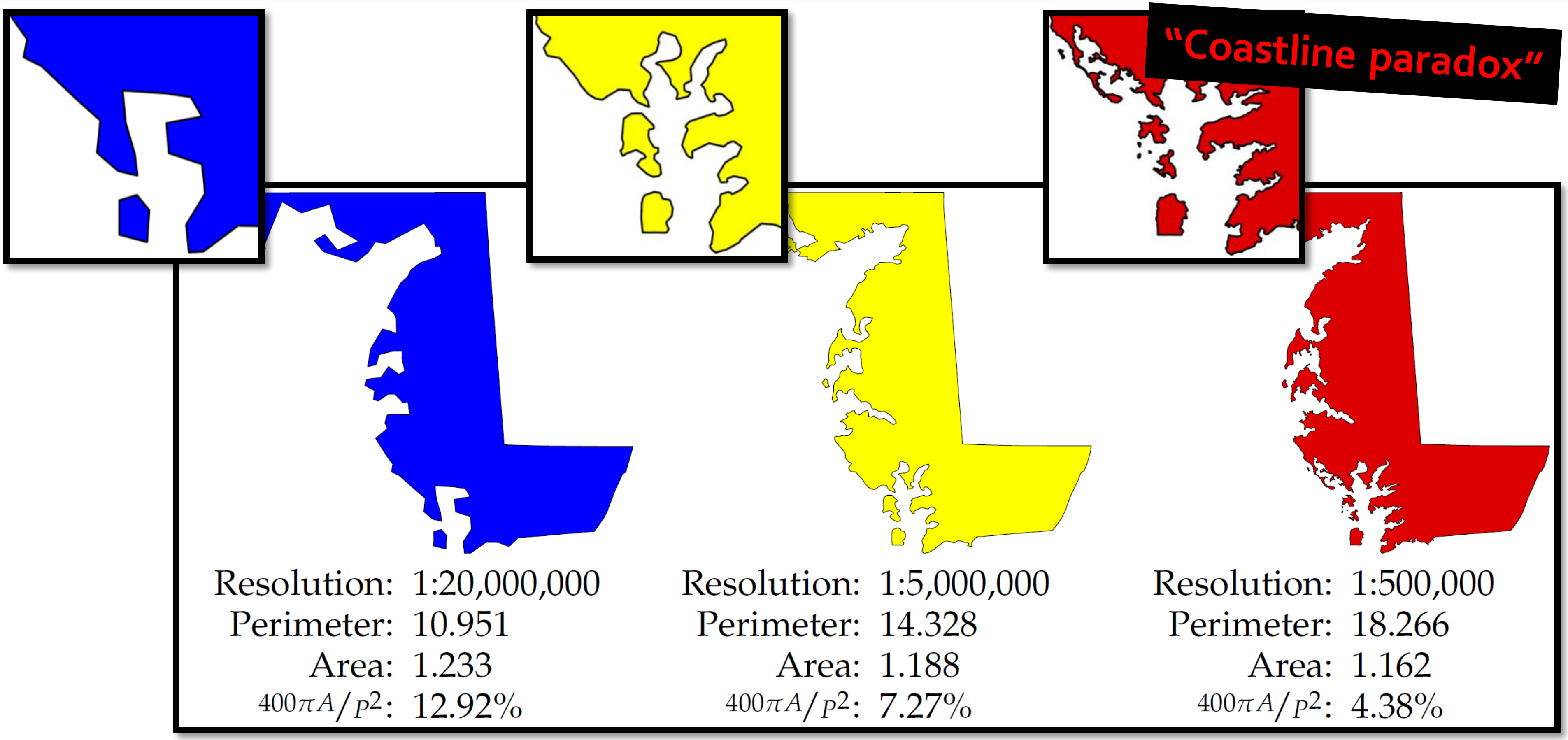


Multiscale analysis


AND NOW
FOR SOMETHING
COMPLETELY
DIFFERENT.


Greensboro
Winston-Salem
Durham
High Point
NC12
Charlotte
Election
Data
Services
Inc.

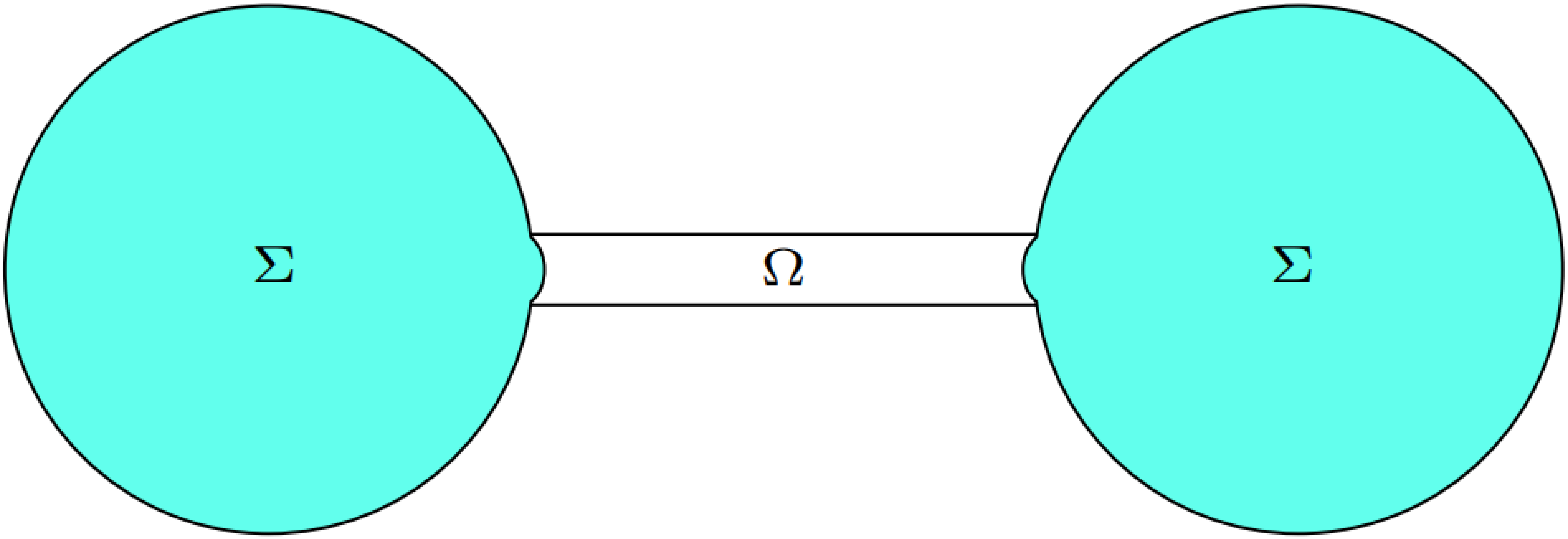
Compactness as a Proxy?



Example courtesy Mira Bernstein and Assaf Bar-Natan

Maryland district 1

Potentially Intractable Solution



$$I_\Omega(t) := \min\{\mathrm{area}(\partial\Sigma) : \Sigma \subseteq \Omega \text{ and } \mathrm{vol}(\Sigma) = t\}$$

Isoperimetric profile

Convex Relaxation: TV Profile

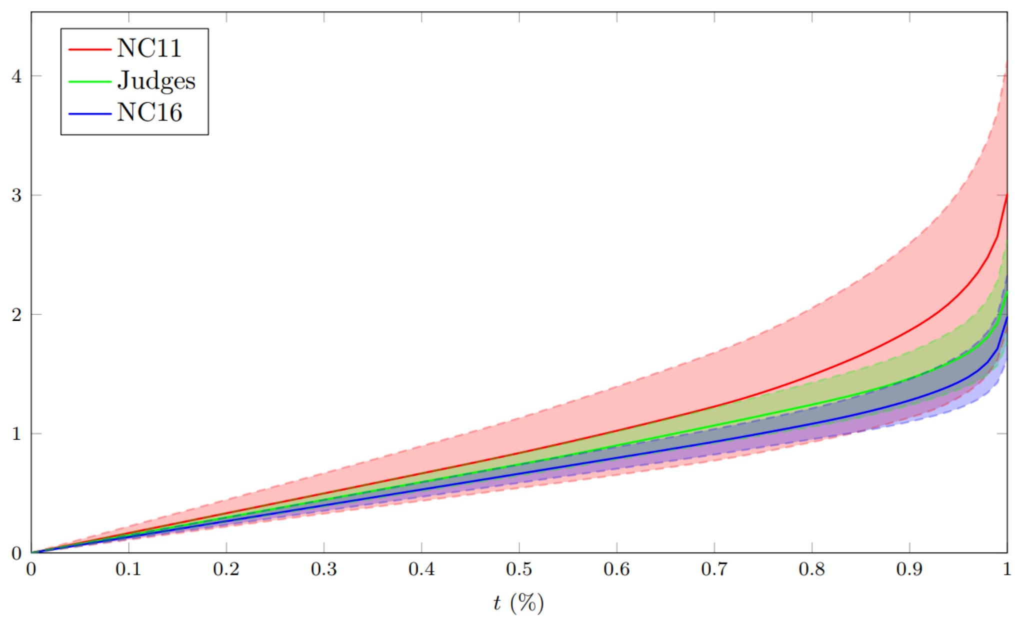
$$I_\Omega^{\mathrm{TV}}(t) := \begin{cases} \min_{f\in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\,dx = t \\ & 0 \le f \le \mathbb{1}_\Omega \end{cases}$$

Theoretical properties:

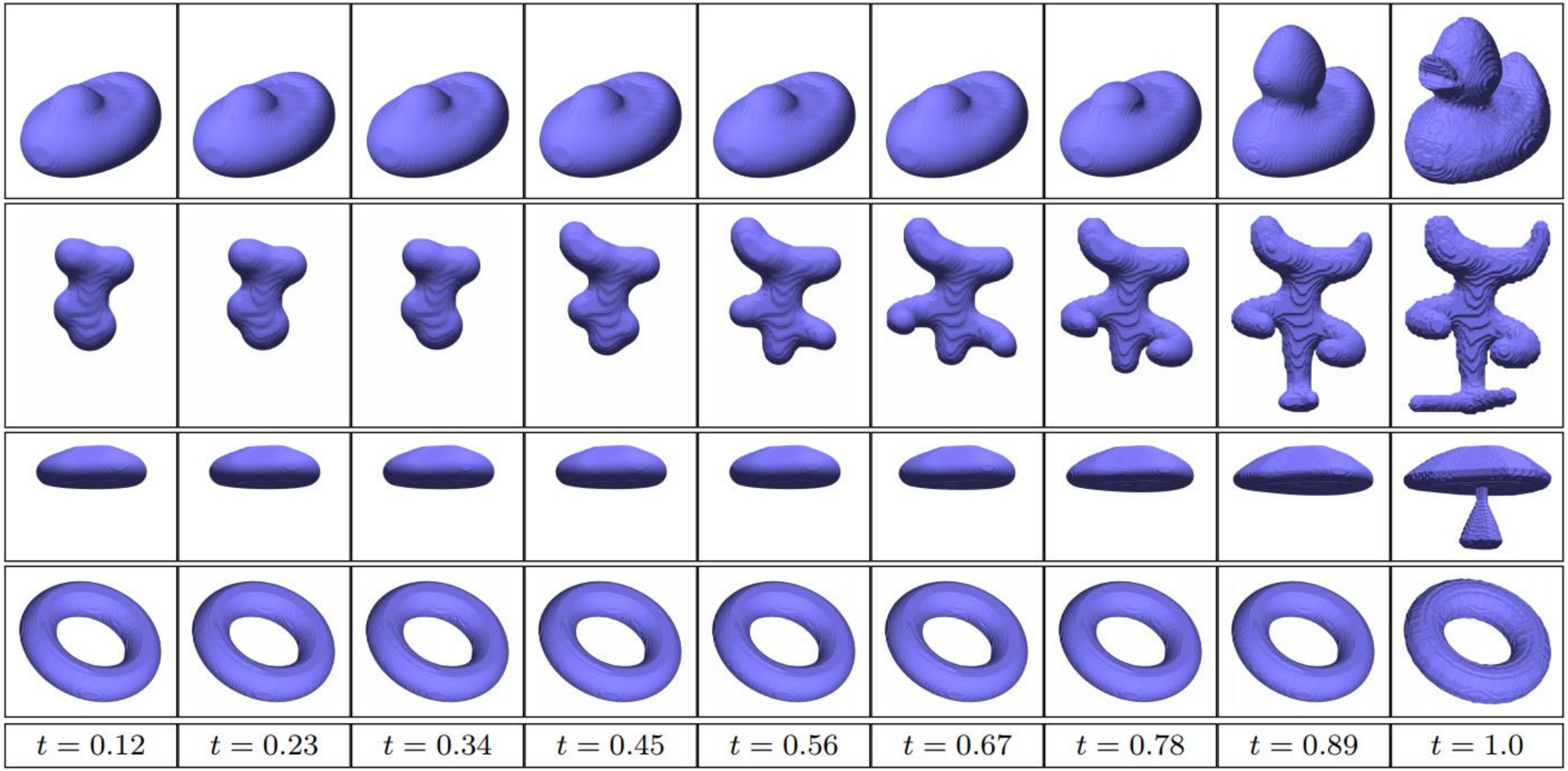
- Convex function of *t*
- Minimized at any *t* for a circle
- (Surprising) optimal *f* takes on at most 3 values: {0, *c*, 1}

DeFord et al. *Total Variation Isoperimetric Profiles*. SIAM SIAGA, pending revision.

Examples



In Case You're Wondering

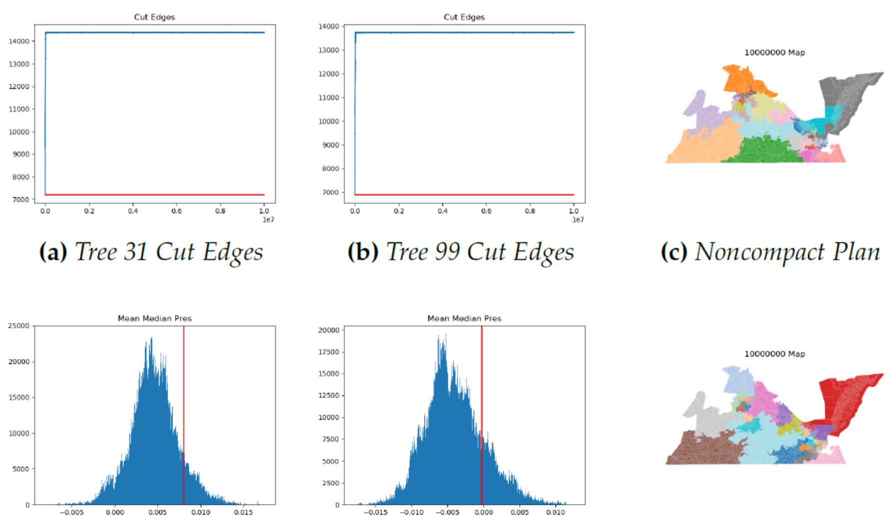


Works in 3D (Why bother? Why not!)

Aside: Only One Small Piece

Current focus:

Sampling in the space of districting plans



(a) *Tree 31 Cut Edges* **(b)** *Tree 99 Cut Edges* **(c)** *Noncompact Plan*

(d) *Tree 31 MM* **(e)** *Tree 99 MM* **(f)** *Noncompact Plan*

Figure 6: *The behavior of the single edge flip ensembles is also poor under other measures. Plots (a) and (b) show the number of cut edges found by the single edge flip proposal. Note that the plans immediately proceed to the upper bound and never leaves over the 10,000,000 steps. Figures (c) and (f) show examples of these non-compact plans. Plots (d) and (e) show the mean median scores for these ensembles. Note that each forms a distribution around the starting value and that in these cases the bulk of the distributions are on opposite sides of 0.*

Interesting confluence:

Extension: 2D Field Design

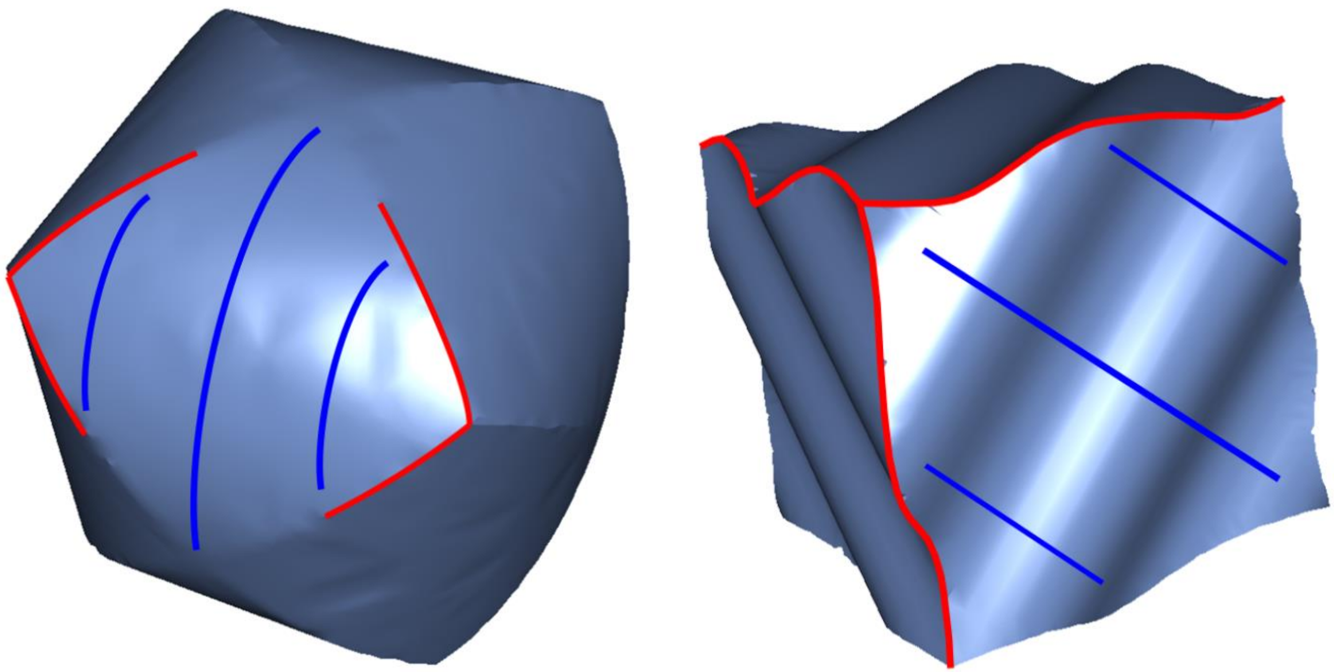


Fig. 1. The three-cylinder-intersection, and wavey-box meshes respectively. These geometries have maximal curvature directions (Blue lines) that contradict its feature curves (Red lines).

Features vs. curvature directions

New Objective Function

$$\mathrm{VTV}[h] = \sup_{\|\phi\|_F \leq 1, \phi \in C_c^1} \sum_i \int_\Omega h_i \nabla \cdot \phi_i \cong \int_\Omega \|\nabla h\|_F$$

Vector of frame coefficients

Surface
(not volume!)

Vectorial analog of L_1 norm
(convex)

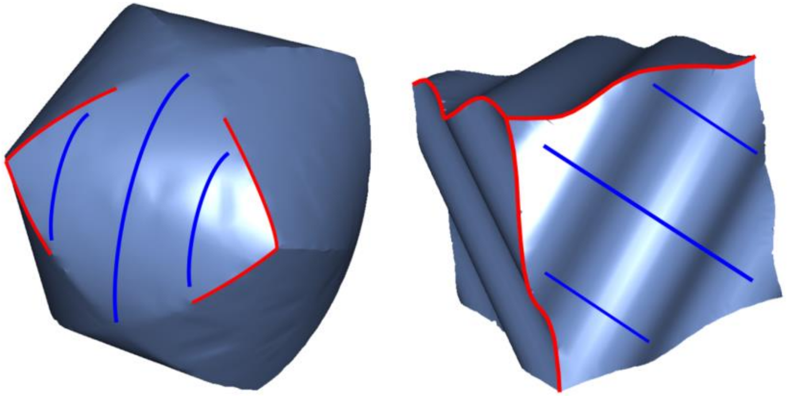
"Spherical Harmonic Frames for Feature-Aligned Cross-Fields."
Zhang, Vekhter, Bommes, Vouga, & Solomon; in preparation.

Vectorial total variation

Key Theoretical Property

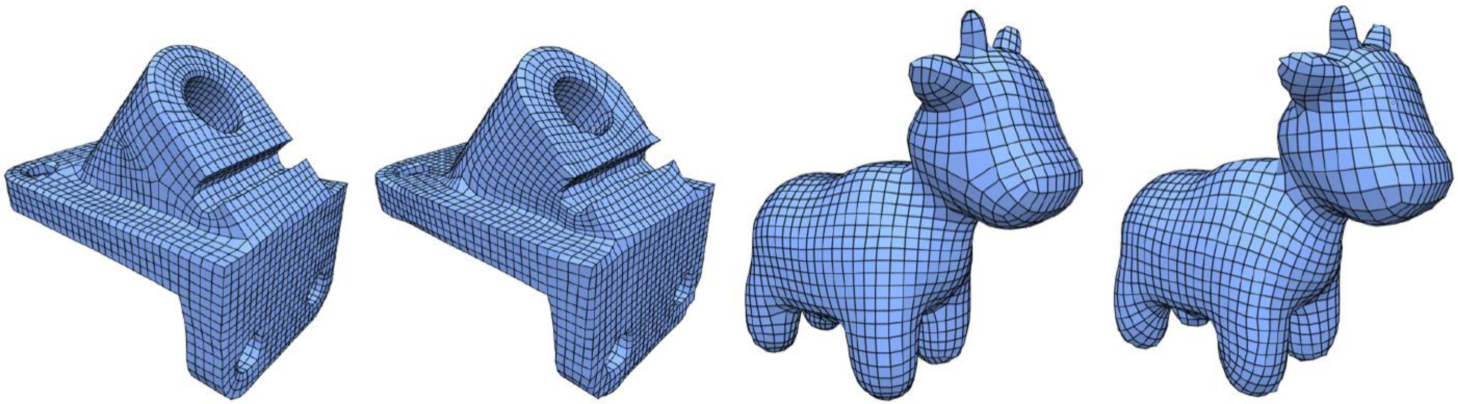
$$\mathrm{VTV}[f] = \sum_{j=1}^{n} \int_{\mathring{\Omega}_j} \|\nabla f\|_F \, dA + \sum_{k=1}^{s} \int_{\gamma_k} \|f^+ - f^-\|_2 \, d\ell$$

Intrinsic smoothness   Crease alignment



Separates features from smoothness

Application to Quad Meshing

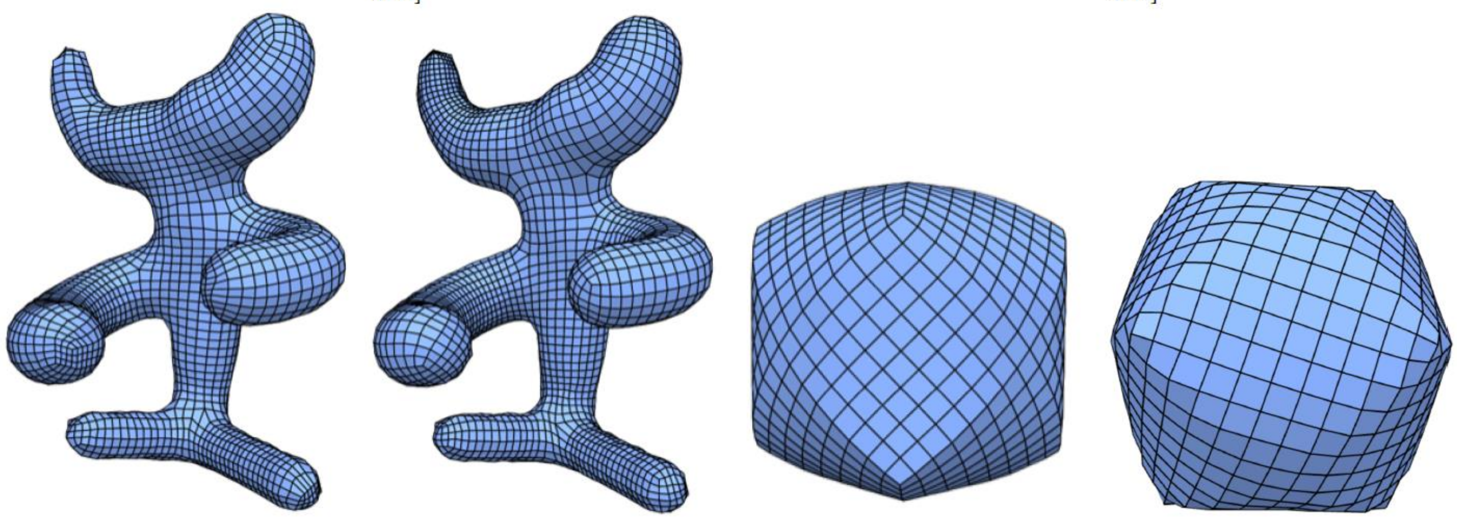


(a) Anchor Mesh with E_2

(b) Anchor Mesh with [Campen et al. 2015]

(c) Spot Mesh with E_2

(d) Spot Mesh with [Campen et al. 2015]



(e) Moomoo Mesh with E_2

(f) Moomoo Mesh with [Campen et al. 2015]

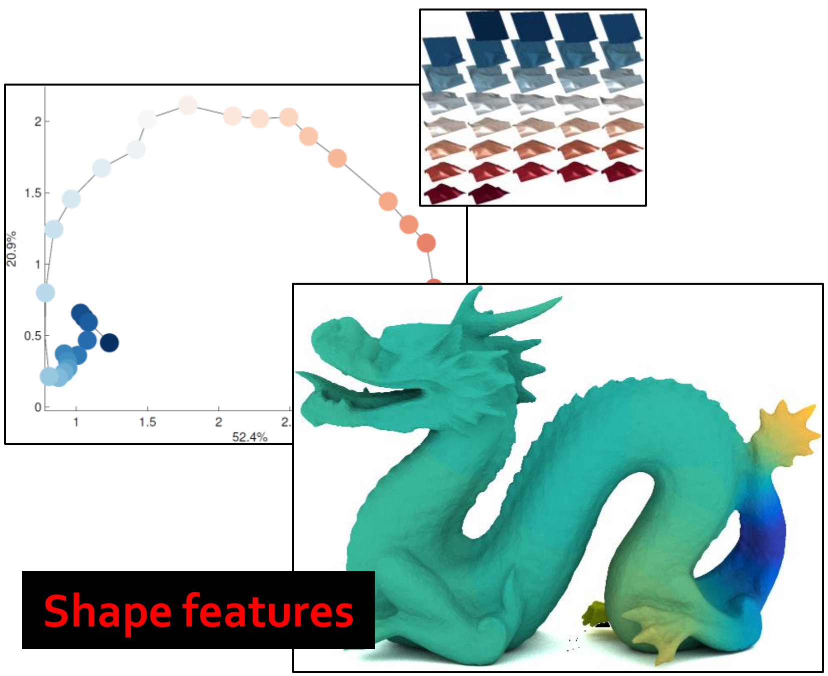
(g) 3 Cylinder Intersection with E_2

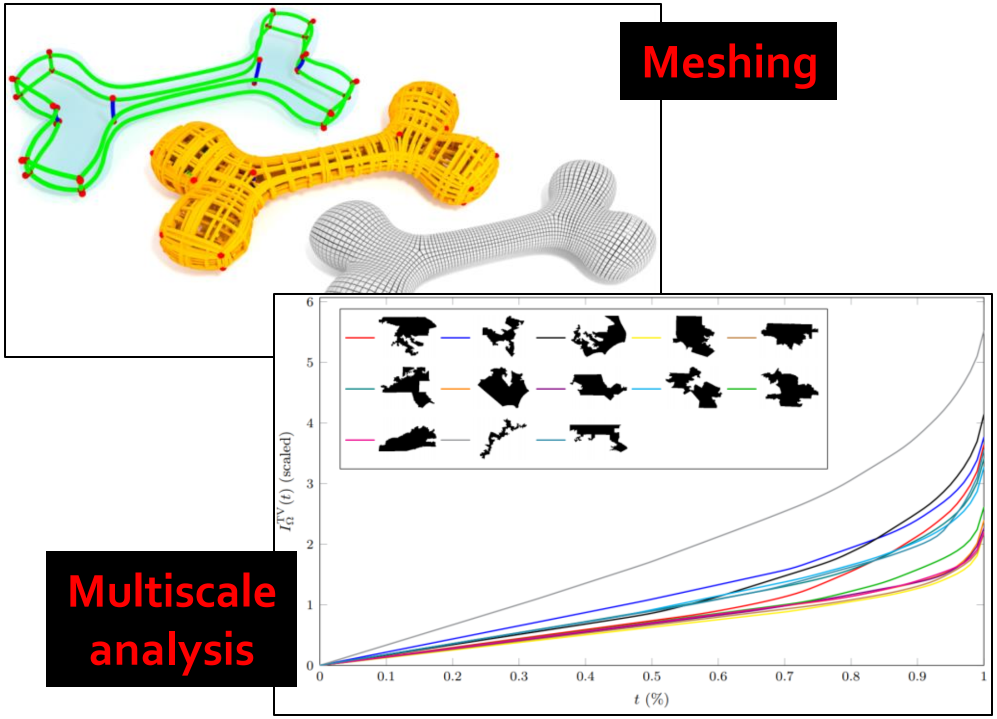
(h) 3 Cylinder Intersection with [Campen et al. 2015]

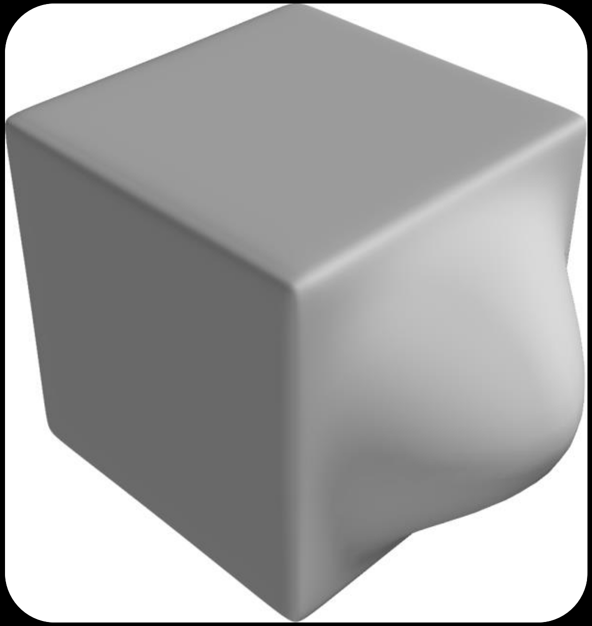
Theme

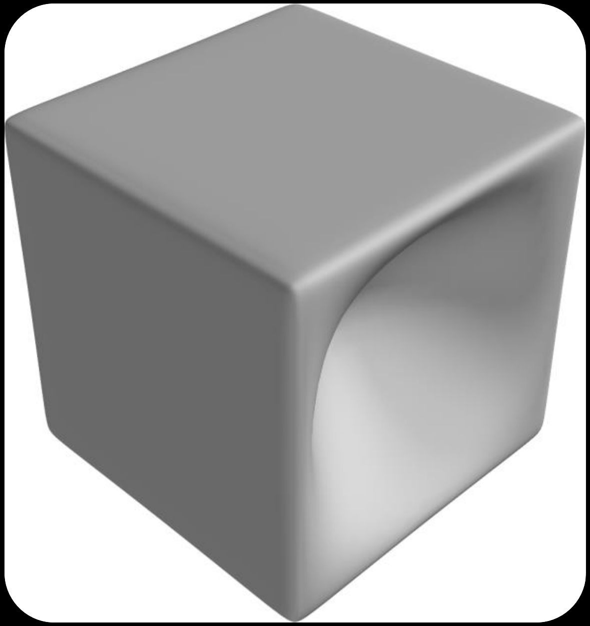
Processing volumetric data requires unique

algorithms & representations.









Volumetric Challenges in Shape Analysis

Questions?