Fast marching methods for anisotropic eikonal equations.

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Geometric processing workshop, IPAM, UCLA

In collaboration with : L. Cohen, R. Duits, Da Chen.
Introduction: anisotropy and cartesian grids

Problems addressed

Finslerian eikonal equations, and the Stern-Brocot tree
  Semi-Lagrangian schemes
  Adaptive stencil refinement
  Application to image segmentation

Riemannian eikonal equations, and Voronoi’s reduction
  Monotone and causal schemes
  Grid-adapted tensor decomposition

Global optimization of curvature dependent energies
  The Reeds-Shepp models
  Euler-Mumford elastica curves, and others

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Anisotropy in Partial Differential Equations (PDEs)

Anisotropy is the existence of preferred directions, locally, in a domain. The phenomenon is generic and ubiquitous, and may have a variety of causes, such as:

▶ Micro-structure, either biological, geologic, synthetic.
▶ Different nature of the domain dimensions, e.g. $\mathbb{R}^2 \times S^1$.
▶ Proximity of the domain boundary, or of discontinuities.
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In the numerical analysis of PDEs, (strong) anisotropy is a source of difficulties.

▶ Increased numerical cost, accuracy loss, instabilities or failure of the numerical methods.

Several approaches can be envisioned to address these.
First approach: adapt the domain representation

Figure: Adaptive interpolation of a function with a sharp transition.
First approach: adapt the domain representation

- Encode the problem anisotropy in a Riemannian metric.

**Figure:** Adaptive interpolation of a function with a sharp transition.
First approach: adapt the domain representation

- Encode the problem anisotropy in a Riemannian metric.
- Create an anisotropic mesh of the domain.

Figure: Adaptive interpolation of a function with a sharp transition.
Second approach: adapt the numerical scheme

- A basic cartesian grid is used throughout this work.

- Easily address asymmetric or three dimensional anisotropy.
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Comparative advantages of the two approaches

Adaptive meshes

- Locally adjust the sampling density.
- Domains of arbitrary shape, and topology.

Different tools for different applications.
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Adaptive stencils on cartesian grids

- Simplicity of implementation.
- Numerical cost.
- Tools of lattice geometry.

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Different tools for different applications.
Lattice geometry
Is the simultaneous study of a positive quadratic form, and of a discrete subgroup of a vector space.
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Sample applications

▶ What is the densest periodic packing of spheres?
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- *What is the densest periodic packing of spheres?*
- *Which integers are sums of three squares?*
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Legendre’s theorem:

\[ \mathbb{N} = \{i^2 + j^2 + k^2; (i, j, k) \in \mathbb{Z}^3\} \cup \{4^a(8b + 7); a, b \in \mathbb{N}\}. \]
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Sample applications

- What is the densest periodic packing of spheres?
- Which integers are sums of three squares?
- Message coding: error correction, cryptography.

What are the prime factors of RSA-768?

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63657518745202199786469389956474942774063845925192557326303453731548268507917
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- Few applications to PDE discretization. Bonnans et al, 04.
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Conclusion
The eikonal equation (Riemannian case)

The distance map $u : \Omega \to \mathbb{R}$ to a domain's boundary obeys

$$\|\nabla u(x)\|_{D(x)} = 1$$

for a.e. $x \in \Omega$, in viscosity sense, and $u = 0$ on $\partial \Omega$. Where $D(x)$ is the inverse metric tensor and $\|v\|_D := \sqrt{\langle v, Dv \rangle}$. 
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Contribution: Single pass methods for (strongly) anisotropic pb.
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Contribution: Single pass methods for (strongly) anisotropic pb.

Applications

- Image segmentation, with L. Cohen, R. Duits, et al
- Motion planning, with J. Dreo (Thales).
- Perspectives: Seismology, with L. Metivier.
Anisotropic diffusion

Given a diffusion tensor field $D : \Omega \subseteq \mathbb{R}^d \to S_+^{++}$, takes the form

$$\partial_t u = \text{div}(D \nabla u),$$

with suitable initial and boundary conditions.
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Given a diffusion tensor field $D : \Omega \subseteq \mathbb{R}^d \rightarrow S_{d}^{++}$, takes the form

$$\partial_t u = \text{Tr}(D \nabla^2 u),$$

with suitable initial and boundary conditions.

Contribution: Monotone schemes for anisotropic problems.

Applications:

- Image processing, with J. Fehrenbach. ($D = D(u)$)
- Non-divergence form anisotropic diffusion.
- Monge-Ampere operator, with Benamou, Collino.
- Perspectives: HJB PDEs of stochastic models, F. Bonnans.
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**Context:** Domain $\Omega$, local metric $F : T\Omega \to \mathbb{R}^+$ defining

$$d_F(x, y) := \inf_{\gamma} \int_0^1 F_\gamma(t)(\gamma'(t)) \, dt \quad \text{s.t.} \quad \begin{cases} \gamma \in \text{Lip}_{\text{loc}}([0, 1], \overline{\Omega}) \\ \gamma(0) = x, \gamma(1) = y. \end{cases}$$
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**Objective:** Compute numerically the exit time to the boundary

$$u(x) = \min_{y \in \partial \Omega} d_F(x, y).$$
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**Bellman’s optimality principle**

For any neighborhood $V$ of $x$ contained in $\Omega$.

$$u(x) = \min_{y \in \partial V} \left( d_{\mathcal{F}}(x, y) + u(y) \right).$$
**Context:** Finite sets $X, \partial X$ approximating $\Omega, \partial \Omega$. Polyhedral neighborhood $V(x)$, of each $x \in X$, with vertices in $X \cup \partial X$. 
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**Semi-Lagrangian discretization**

Find $U : X \cup \partial X \to \mathbb{R}$, vanishing on $\partial X$, and such that $\forall x \in X$

$$U(x) = \min_{y \in \partial V(x)} \left( F_x(y - x) + I_{V(x)} U(y) \right).$$
Discretization yields a coupled system of non-linear equations.

\[ \forall x \in X, U(x) = \Lambda U(x), \quad \forall x \in \partial X, U(x) = 0. \quad (1) \]
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Analogous problem: computing shortest paths on graphs
Neighbors \( S(x) \) of \( x \in X \), edge lengths \( w(x, y) \), operator

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System (1) is solvable in a single pass by Dijkstra’s algorithm iff

\[ w(x, y) \geq 0 \text{ for all } x \in X, y \in S(x). \]
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Acuteness implies causality (Sethian, Kimmel, Vladimirsky, 96)

System (1) is solvable by the fast marching method, in a single pass, iff

\( (u, v) \) make an \( F_x \)-acute angle,

whenever \( x + u, \ x + v \) lie in a common facet of \( V(x) \).
What is an acute angle? (Sethian, Kimmel, Vladimirsky, 96)

Vectors \( u, v \in E = \mathbb{R}^d \), make an \( F \)-acute angle, where \( F : E \to \mathbb{R}_+ \) an asymmetric norm, iff

- (Euclidean case) Assuming \( F(x) = m \|x\| \), where \( m > 0 \),

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\langle u, v \rangle \geq 0.
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- (Differentiable case) $\langle \nabla F(u), v \rangle \geq 0$, and $\langle \nabla F(v), u \rangle \geq 0$. 

\[\begin{align*}
\text{Euclidean case:} & \quad \langle u, v \rangle \geq 0 \\
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Figure: Stencil constructions proposed for isotropic (left), or midly anisotropic metrics, due to Sethian, Kimmel, Alton, ...
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Fast-Marching using Adaptive Stencil Refinement

- Iterative refinement of the stencil until the acuteness property is met. (FM-ASR scheme, on 2D cartesian grids)
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- The splitting procedure exploits the grid additivity.
Fast-Marching using Adaptive Stencil Refinement

- Iterative refinement of the stencil until the acuteness property is met. (FM-ASR scheme, on 2D cartesian grids)

- The splitting procedure is exploits the grid additivity.
Obtain the \((n + 1)\)-th line by inserting \(\frac{a + a'}{b + b'}\) between consecutive elements \(\frac{a}{b}\) and \(\frac{a'}{b'}\) of the \(n\)-th line.

- Each positive rational number appears exactly once in the tree, in its irreducible form.
- Well studied arithmetic object, used for rational approximation.
**Context:** $F$ asymmetric norm on $\mathbb{R}^2$, $T(F)$ the FM-ASR stencil,

$$
\mu(F) := \max_{|u|=|v|=1} \frac{F(u)}{F(v)}.
$$
Context: $F$ asymmetric norm on $\mathbb{R}^2$, $T(F)$ the FM-ASR stencil,

$$\mu(F) := \max_{|u|=|v|=1} \frac{F(u)}{F(v)}.$$

Theorem (Worst and average stencil size)

For any asymmetric norm $F$ on $\mathbb{R}^2$, denoting by $R_\theta$ the rotation of angle $\theta$, one has $\#(T(F)) \leq C\mu \ln \mu$, and

$$\int_0^{2\pi} \#(T(F \circ R_\theta)) \, d\theta \leq C \ln^3 \mu$$

where $\mu := \max\{2, \mu(F)\}$, and $C$ is an absolute constant.
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Region segmentation using Rander geodesics

Define for any region $\Omega \subseteq \mathbb{R}^2$ the energy

$$\mathcal{E}(\Omega) := \int_{\Omega} f(x)dx + \int_{\partial \Omega} g(x)dx.$$
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$$E(\Omega) := \int_{\Omega} f(x)dx + \int_{\partial \Omega} g(x)dx.$$  

Assuming $f = \text{div } w$, this rewrites as

$$E(\Omega) = \int_{\partial \Omega} (\langle w(x), n(x) \rangle + g(x))dx = \int_0^1 \mathcal{F}_\gamma(t)(\gamma'(t)) dt,$$

where $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ parametrizes $\partial \Omega$ counter clockwise, and

$$\mathcal{F}_x(v) := g(x)\|v\| + \langle w(x)^\perp, v \rangle.$$ 

Joint work with Laurent Cohen, Da Chen.
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- A *Rander* metric, provided $|w(x)| < g(x)$ for all $x$.
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- A Rander metric, provided $|w(x)| < g(x)$ for all $x$.
- $\Omega$ is (locally) optimal iff $\partial \Omega$ is a geodesic.
- Joint work with Laurent Cohen, Da Chen.
Compute the field $w$ by solving an elliptic equation:

$$\Delta p = f \quad \Rightarrow \quad \text{div}(w) = f \quad \text{where} \quad w = \nabla p.$$  

Extract the minimizing Rander geodesics between some known boundary points.
Anisotropic Fast Marching

Jean-Marie Mirebeau

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Panorama

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Other models

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- Compute the field $w$ by solving an elliptic equation:
  \[ \Delta p = f \quad \Rightarrow \quad \text{div}(w) = f \text{ where } w = \nabla p. \]
- Extract the minimizing Rander geodesics between some known boundary points.
- The method is actually iterative, due to constraint $|w| < g$, and so as to accept approximate boundary points.

Figure: Image segmentation examples, with L. Cohen, Da Chen.
Compute the field $w$ by solving an elliptic equation:

$$\Delta p = f \text{ on } U \supseteq \partial \Omega \Rightarrow \text{div}(w) = f \text{ where } w = \nabla p.$$ 

Extract the minimizing Rander geodesics between some known boundary points.

The method is actually iterative, due to constraint $|w| < g$, and so as to accept approximate boundary points.
Introduction: anisotropy and cartesian grids

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Conclusion
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Conclusion
A generalized finite differences scheme takes the form:

\[ FU(x) := F(x, U(x), (U(x) - U(y))_{y \in X}) \]

where \( X \) is a finite set, and \( U : X \to \mathbb{R} \). Desirable properties:

\[
\| \nabla u(x) \|^2 \approx h^{-2} \sum_{1 \leq i \leq d} \max\{0, u(x) - u(x - he_i), u(x) - u(x + he_i)\}^2
\]
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where \(X\) is a finite set, and \(U : X \rightarrow \mathbb{R}\). Desirable properties:

**Monotony**

\(F\) is non-decreasing in its second and (each) third variable.

- Yields comparison principles, used for convergence analysis.

**Scheme for isotropic eikonal eqns (Rouy 92, Sethian 96)**

At first order, denoting \((e_i)_{1 \leq i \leq d}\) the canonical basis of \(\mathbb{R}^d\),

\[
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**Monotony**

$F$ is non-decreasing in its second and (each) third variable.

- Yields comparison principles, used for convergence analysis.

**Causality**

$F$ only depends on the positive part of (each) third variable.

- Enables solving $FU \equiv 0$ in a single pass, using the fast-marching method ($\sim$ Dijkstra’s algorithm).

**Scheme for isotropic eikonal eqns (Rouy 92, Sethian 96)**

At first order, denoting $(e_i)_{1 \leq i \leq d}$ the canonical basis of $\mathbb{R}^d$,

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Conclusion
We generalize the isotropic scheme using tensor decompositions

\[ D = \sum_{1 \leq i \leq l} \lambda_i e_i e_i^T, \]

with non-negative weights \( \lambda_i \geq 0 \), and integer offsets \( e_i \in \mathbb{Z}^d \).

- We select an admissible decomposition maximizing

\[ \sum_{1 \leq i \leq l} \lambda_i. \]

Optimal solution has \( l = d(d + 1)/2 \).

- The resulting linear program is known as Voronoi’s first reduction (dual), and widely studied.

- Symmetries enable extremely fast resolution (one per discretization point).
Fast Marching using Voronoi’s First Reduction

At first order, assuming a decomposition \( D = \sum_{i=1}^{l} \lambda_i e_i e_i^T \),

\[
\| \nabla u(x) \|_D^2 \approx \frac{1}{h^2} \sum_{1 \leq i \leq l} \lambda_i \max\{0, u(x) - u(x - he_i), u(x) - u(x + he_i)\}^2
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▶ Number of terms: $l = d(d + 1)/2$. 
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- Reduces to the original scheme if $D = \lambda I d$. 
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$$\|\nabla u(x)\|^2_D \approx \frac{1}{h^2} \sum_{1 \leq i \leq l} \lambda_i \max\{0, u(x) - u(x - he_i), u(x) - u(x + he_i)\}^2$$

- Number of terms: $l = d(d + 1)/2$.
- Reduces to the original scheme if $D = \lambda I d$.
- Implemented in dimensions 2 to 5.

**Figure:** Unit ball defined by $D^{-1}$, and offsets $e_i$ appearing in the decomposition of $D$ associated with Voronoi’s first reduction.
Fast Marching using Voronoi’s First Reduction

At first order, assuming a decomposition $D = \sum_{i=1}^{l} \lambda_i e_i e_i^T$,

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$$\| \nabla u(x) \|^2_D \approx \frac{1}{h^2} \sum_{1 \leq i \leq l} \lambda_i \max\{0, u(x) - u(x - he_i), u(x) - u(x + he_i)\}^2$$

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\]

- Number of terms: \( l = d(d + 1)/2 \).
- Reduces to the original scheme if \( D = \lambda I_d \).
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Theorem

For each \( \mu \geq 1 \) and \( \theta \in [0, 2\pi] \), denote \( r_\mu(\theta) := \max_{1 \leq i \leq l} \| e_i \| \) where \((\lambda_i, e_i)_{1 \leq i \leq l}\) comes from the decomposition of

\[
\mu^{-1} e(\theta) \otimes e(\theta) + \mu e(\theta) \perp \otimes e(\theta) \perp.
\]

Then, as \( \mu \to \infty \)

\[
\| r_\mu \|_{L^p([0,2\pi])} \ll \begin{cases} \mu^{1-1/p} & \text{if } 2 < p \leq \infty \\ \sqrt{\mu \ln \mu} & \text{if } p = 2 \\ \sqrt{\mu} & \text{if } 1 \leq p < 2. \end{cases}
\]
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Conclusion
Introduction: anisotropy and cartesian grids

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  Euler-Mumford elastica curves, and others

Conclusion
The Reeds-Shepp model

Configuration space $\mathbb{M}_2 := \mathbb{R}^2 \times S^1$, positions and orientations.

$$\| (\dot{x}, \dot{n}) \|^2_{\mathcal{M}(x,n)} := c(x, n)^2 (\langle n, \dot{x} \rangle^2 + \varepsilon^{-2} \langle n^\perp, \dot{x} \rangle^2 + \| \dot{n} \|^2).$$

- Theoretically, $\varepsilon \to 0$ and the model is sub-Riemannian.
- Numerically, good results obtained with $\varepsilon = 0.1$.

Parametrization $n = (\cos \theta, \sin \theta)$, $\theta \in [0, 2\pi]$, of $S^1$. 
Application to tubular structure segmentation

- Segmenting the retinal vascular tree using Reeds-Shepp model, with data driven $c(x, n)$. With R. Duits et al.
Application to tubular structure segmentation

- Segmenting the retinal vascular tree using Reeds-Shepp model, with data driven $c(x, n)$. With R. Duits et al.

**Figure**: Density plot of the cost function $c(x, y, \theta)$. Related: (radius lift) Li and Yezzi 07, ($\theta$ lift) Péchaud et al 09.
Higher dimensional Reeds-Shepp model

The model extends to $\mathbb{R}^3 \times S^2$,

$$\|(\dot{x}, \dot{n})\|^2_{M(x,n)} := c(x,n)^2 \left( \langle n, \dot{x} \rangle^2 + \varepsilon^{-2} \|P_n(\dot{x})\|^2 + \|\dot{n}\|^2 \right).$$

where $P_n := \text{Id} - n \otimes n$ is the orthogonal projection onto $(\mathbb{R}n)^\perp$.

Figure: Left: $P_n(\dot{x}) = 0$. 
Higher dimensional Reeds-Shepp model(s)

The model extends to $\mathbb{R}^3 \times S^2$, and has an unexpected variant.

$$\| (\dot{x}, \dot{n}) \|^2_M(x,n) := c(x,n)^2 \left( \langle n, \dot{x} \rangle^2 + \varepsilon^{-2} \| P_n(\dot{x}) \|^2 + \| \dot{n} \|^2 \right).$$

$$\| (\dot{x}, \dot{n}) \|^2_{\tilde{M}(x,n)} := c(x,n)^2 \left( \varepsilon^{-2} \langle n, \dot{x} \rangle^2 + \| P_n(\dot{x}) \|^2 + \| \dot{n} \|^2 \right).$$

where $P_n := \text{Id} - n \otimes n$ is the orthogonal projection onto $(\mathbb{R}^n)^\perp$.

Figure: Left: $P_n(\dot{x}) = 0$. Right: $\langle n, \dot{x} \rangle = 0.$
Determination of the connectivity of white matter fibers, on dMRI data, with R. Duits et al

- Difficulty: some fiber bundles cross each other.
- Our solution: minimal paths w.r.t. the Reeds-Shepp model, imposing directional consistency, with suitable $c(x, n)$. 

> Anisotropic Fast Marching
> Jean-Marie Mirebeau

Introduction
Panorama
Finslerian eikonal
Semi-Lagrangian schemes
Adaptive stencil refinement
Image segmentation
Riemannian eikonal
Monotony and causality
Grid-adapted tensor decomposition
Curvature penalization
Reeds-Shepp
Other models
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  Monotone and causal schemes
  Grid-adapted tensor decomposition

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  The Reeds-Shepp models
  Euler-Mumford elastica curves, and others

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Curvature penalized planar paths

The Reeds-Shepp model penalizes path curvature, but it:

- Allows for cusps (shift into reverse gear).
- Has specific cost dependency $\sqrt{1 + \kappa^2}$, w.r.t. curvature $\kappa$. 

Figure: Minimal paths, in free space.
Curvature penalized planar paths

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Reeds-Shepp forward: \( \sqrt{1 + \kappa^2} \)

Euler-Mumford: \( 1 + \kappa^2 \)

Dubins: \( 1 + \infty_{\kappa>1} \)

Figure: Control sets of the different models
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Figure: Stencils used for the eikonal PDE discretization.
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**Figure:** Level set of the distance function.
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Dubins: $1 + \infty$ for $\kappa > 1$

Figure: Backtracked geodesics,
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$1 + \infty \kappa > 1$

Figure: Backtracked geodesics, projected onto $\mathbb{R}^2$. 
An adversary starts at ●, then visits ●, finally returns to ●

Related work by Barbaresco, Strode.
Computation of threatening trajectories, and optimization of a surveillance system. With J. Dreo.

- An adversary starts at ●, then visits ●, finally returns to ●.
- His turning radius is bounded. *Admissible paths* $\Gamma$.

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- Probability of detection depends on the distance and orientation relative to the sensors. Cost function $c_\lambda(x, n)$.

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- His turning radius is bounded. *Admissible paths* $\Gamma$.
- Probability of detection depends on the distance and orientation relative to the sensors. *Cost function* $c_\lambda(x, n)$.
- Goal: optimize the sensor configuration $\lambda \in \Lambda$.

$$
\max_{\lambda \in \Lambda} \min_{\gamma \in \Gamma} \int_0^{L(\gamma)} c_\lambda(\gamma(s), \gamma'(s)) \, ds.
$$

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Thanks for your attention.

- Cartesian grids, which are natural for numerous applications, are not incompatible with anisotropic pbs.

Numerical codes, demo notebooks, available at github.com/mirebeau/
Thanks for your attention.

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- The tools of lattice geometry allow to build fast, robust, and accurate, adaptive numerical schemes.

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- Cartesian grids, which are natural for numerous applications, are not incompatible with anisotropic pbs.
- The tools of lattice geometry allow to build fast, robust, and accurate, adaptive numerical schemes.
- Handling (strongly) anisotropic PDEs allows to address new models and applications.

Numerical codes, demo notebooks, available at github.com/mirebeau/
References


References (continued)


[A14] M, Anisotropic fast-marching on cartesian grids using Voronoi’s first reduction of quadratic forms, (submitted) 2018
Peer reviewed conference proceedings


Reproducible research papers, with C++ codes


The Reeds-Shepp model in the visual system.

- Neurons of the first layer V1 of the visual cortex react to stimuli at a specific position $x$ and orientation $\theta \in [0, \pi]$. 

Figure: Pinwheel structure of V1. Orientations coded by color.
The Reeds-Shepp model in the visual system.

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**Poggendorff’s visual illusions**

According to Franceschiello et al, the visual system infers, between the endpoints of curves, a connection that:

- Has the correct tangents.
- Minimizes the Reeds-Shepp sub-Riemannian length.

**Figure:** First Poggendorff illusion (perceived misalignment of lines), and its interpretation based on the Reeds-Shepp model.
Poggendorff’s visual illusions

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Figure: Second Poggendorff illusion (perceived misalignment of circle arcs), and its interpretation based on the Reeds-Shepp model.
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