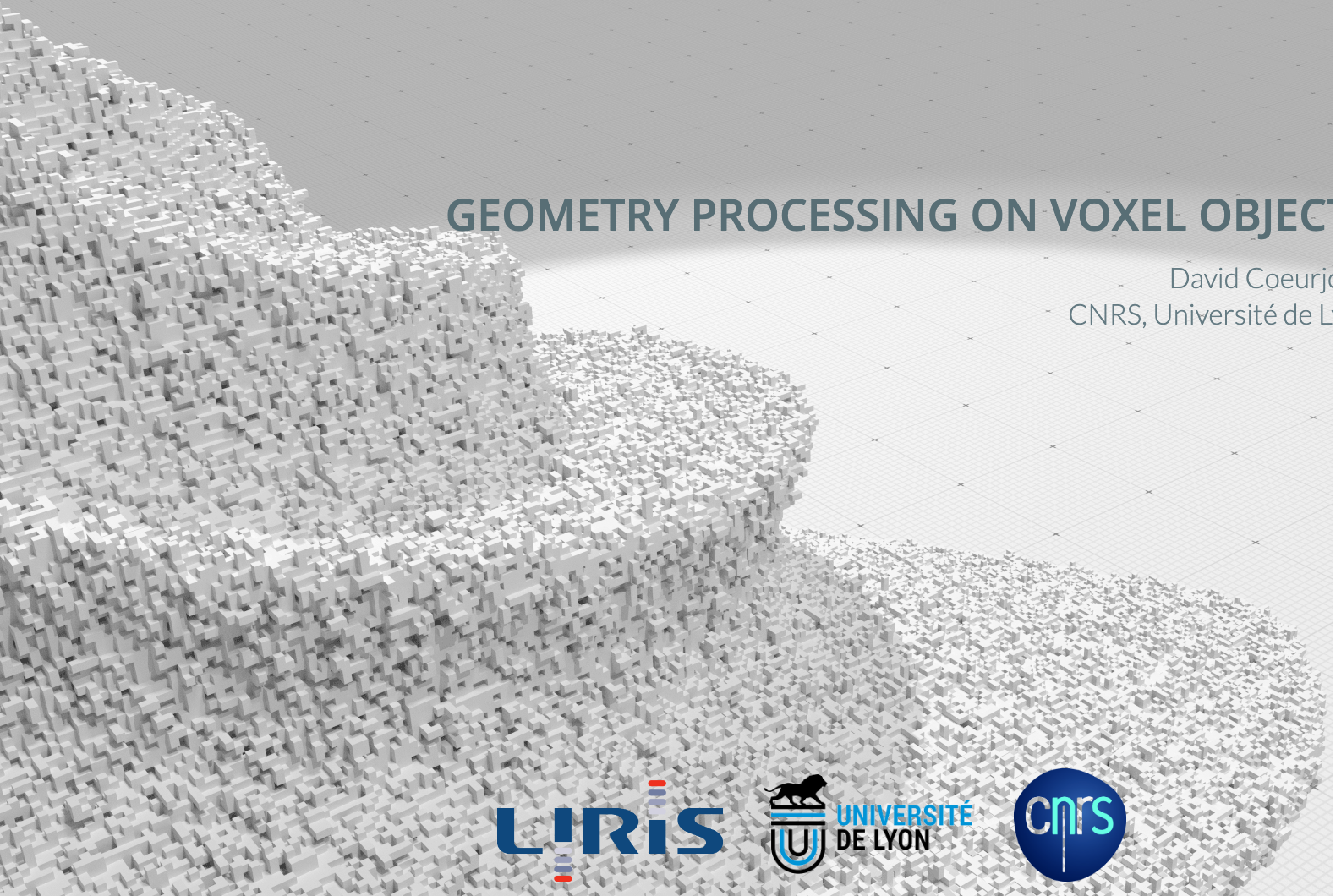
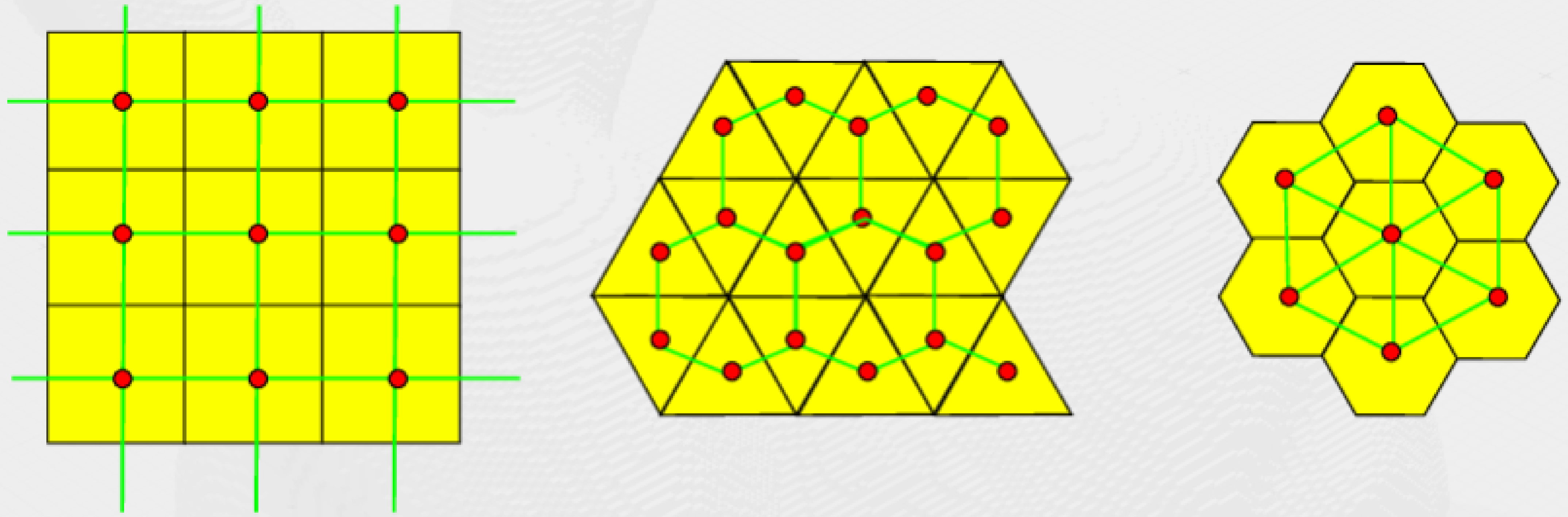


GEOMETRY PROCESSING ON VOXEL OBJECTS

David Coeurjolly
CNRS, Université de Lyon

Digital geometry model in one slide

Topology and geometry processing of objects defined on regular lattices



- *Digital Objects* = subsets of $\mathbb{Z}^d$
- *Digital Topology* = adjacency relationship induced by the lattice
- *Digital surfaces* = cells of a Cartesian cubical complex

- *Geometrical predicates* = integer only computations
  $\Rightarrow$ strong arithmetical results when doing geometry on grids

Why voxel objects?

- Widely used in geometric modeling and rendering to represent complex and interactive scenes
- Nice mathematical modeling framework

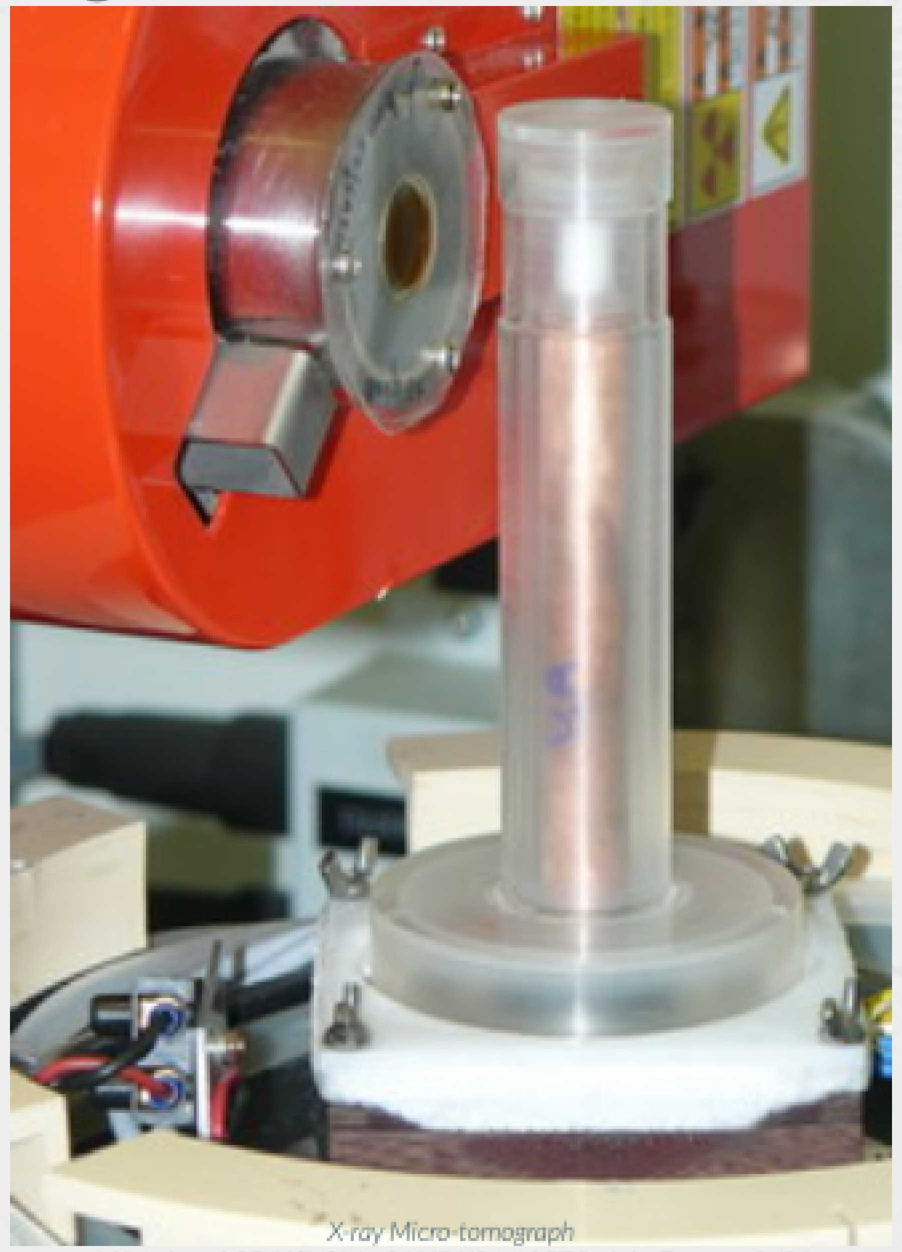
$64K^3$ voxel grid!

Villanueva, Alberto Jaspe, Fabio Marton, and Enrico Gobbetti. "SSVDAGs: symmetry-aware sparse voxel DAGs." In Proceedings of the 20th ACM SIGGRAPH Symposium on Interactive 3D Graphics and Games, pp. 7-14. ACM, 2016.


Curvature Tensor Estimat
Laplace-Beltrami d
Piecewise smooth reconstruction
DGtal

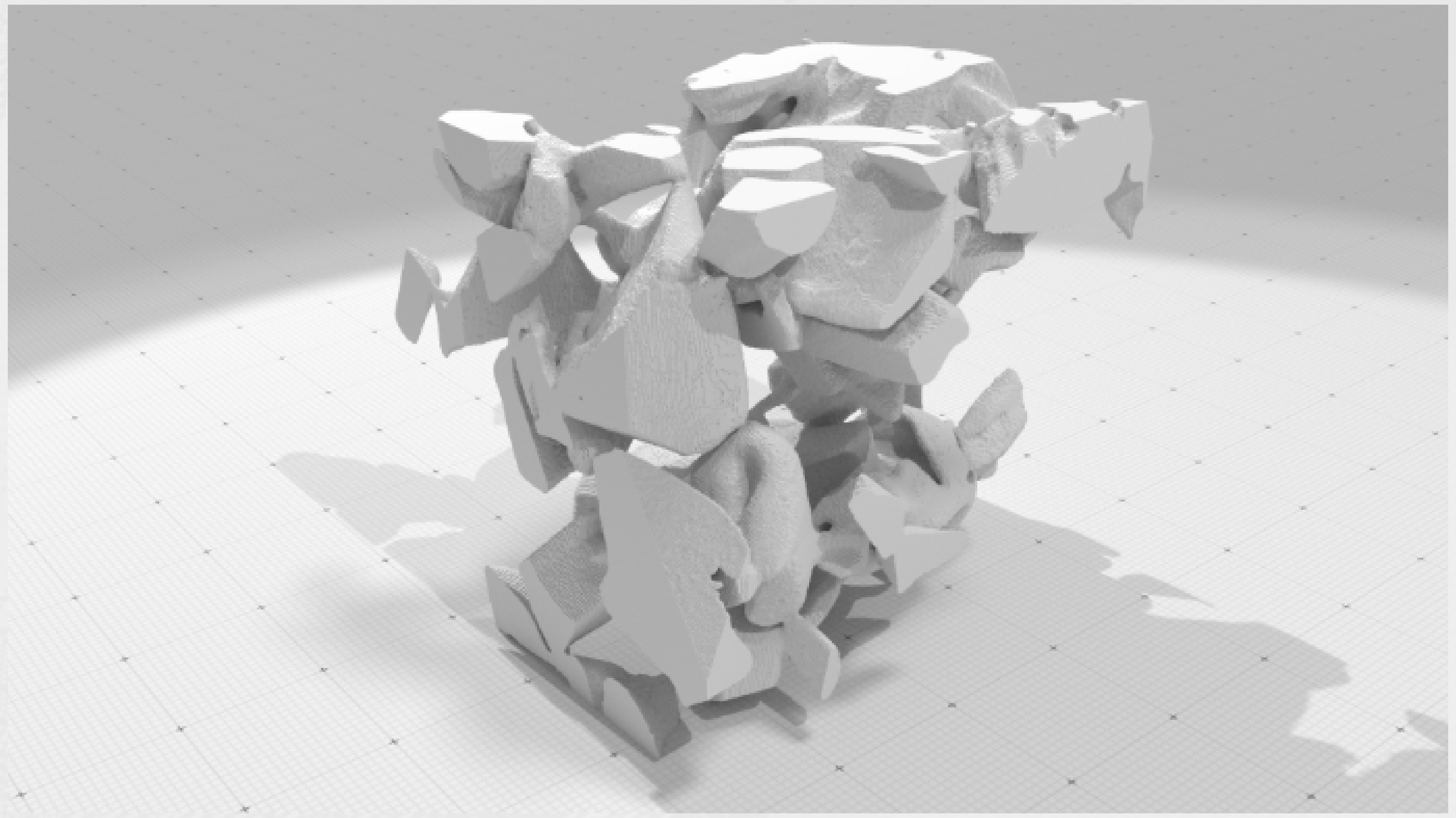
Material Sciences Applications

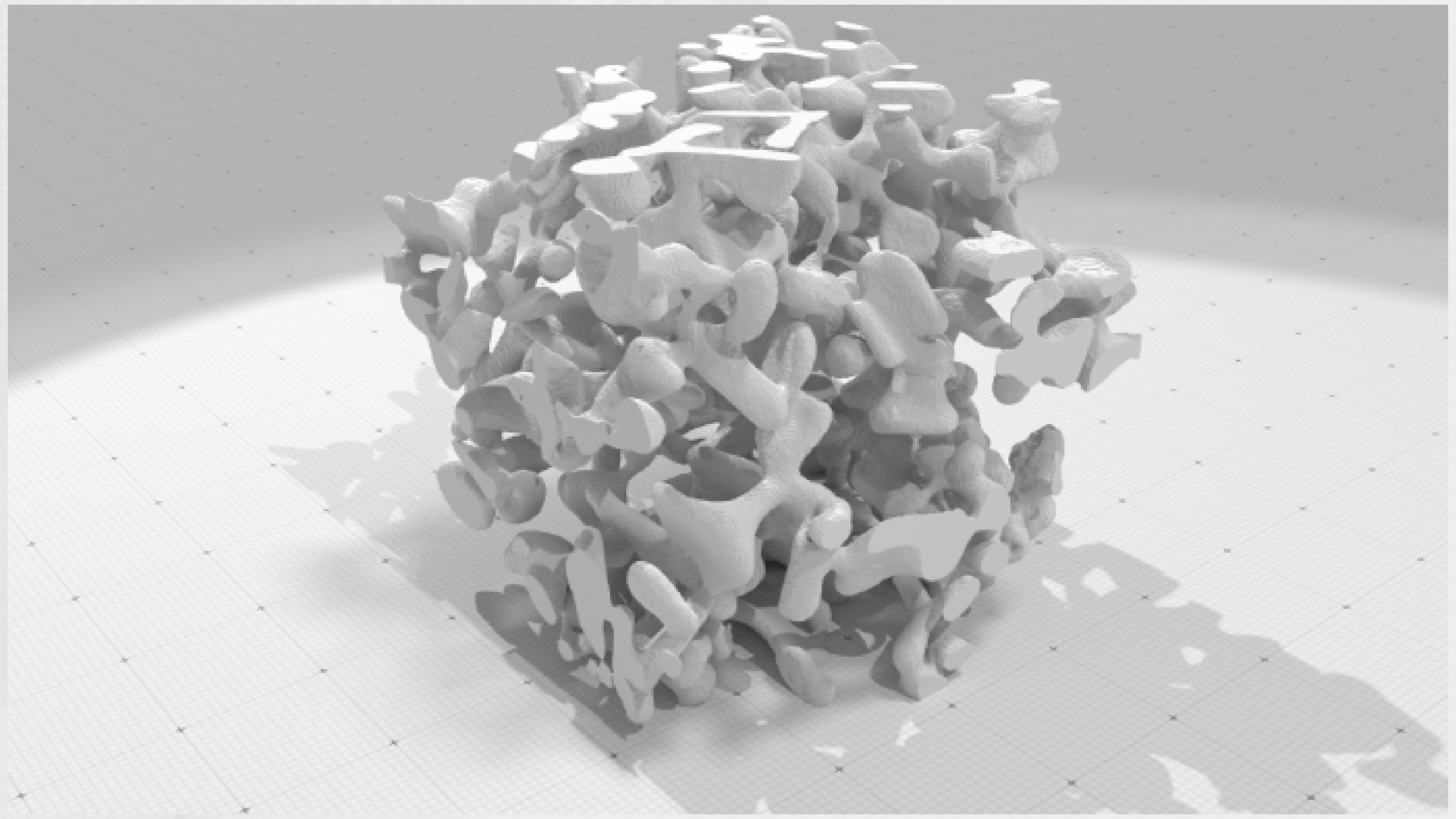
Non-invasive snow micro-tomographic images



X-ray Micro-tomograph
3SR Lab and CEN/CNRM - GAME URA 1357/Météo-France - CNRS

Up to 2048^3





Outline

- Curvature tensor estimation
- Laplace-Beltrami operator on digital surfaces
- Variational geometry processing: Ambrosio-Tortorelli functional

Collaborators: Jacques-Olivier Lachaud (Chambéry), Tristan Roussillon (Lyon), Nicolas Bonneel (Lyon), Jérémy Levallois (Lyon), Thomas Caissard (Lyon), Marion Foare (Lyon)

C., Jacques-Olivier Lachaud, Jérémy Levallois. "Multigrid Convergent Principal Curvature Estimators in Digital Geometry". Computer Vision and Image Understanding, 129(1):27-41, June 2014.
Thomas Caissard, C., Jacques-Olivier Lachaud, Tristan Roussillon. "Laplace–Beltrami Operator on Digital Surfaces". Journal of Mathematical Imaging and Vision, 2018.
C., Marion Foare, Pierre Gueth, Jacques-Olivier Lachaud. "Piecewise smooth reconstruction of normal vector field on digital data". Computer Graphics Forum (Proceedings of Pacific Graphics), 35(7), September 2016.
Nicolas Bonneel, C., Pierre Gueth, Jacques-Olivier Lachaud. "Mumford-Shah Mesh Processing using the Ambrosio-Tortorelli Functional". Computer Graphics Forum (Proceedings of Pacific Graphics), 37(7), October 2018.

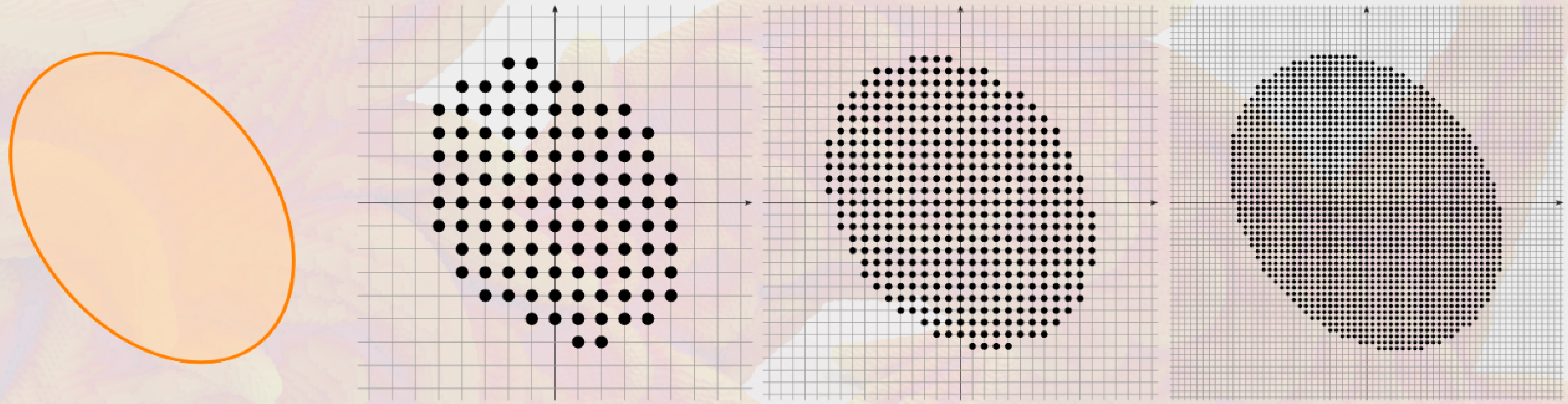
CURVATURE TENSOR ESTIMATION

Digitization model

Gauss Digitization

Given $M \subset \mathbb{R}^d$, its digitization at gridstep h is

$$\mathtt{G}_h(M) = \left(\frac{1}{h} \cdot M\right) \cap \mathbb{Z}^d$$



$M \in \mathbb{X}$ | $\mathtt{G}_1(M)$ | $\mathtt{G}_{0.5}(M)$ | $\mathtt{G}_{0.25}(M)$

Example: $h^2|\mathtt{G}_h(M)|$ converges to the measure of M as $h \to 0$

[Gauss, Dirichlet, Huxley...]

Multigrid convergence of a local estimator

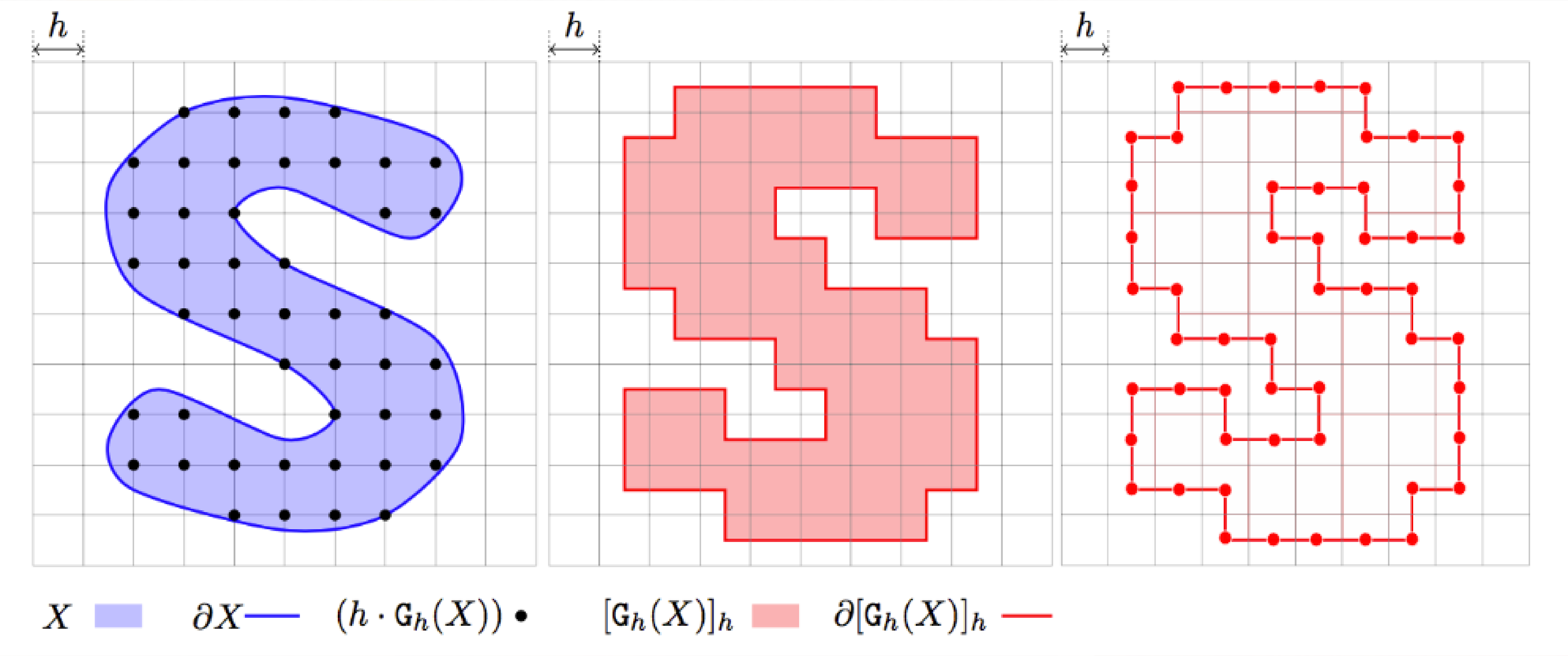
Multigrid convergence

Given a *digitization process* $\mathtt{D}$, a local discrete geometric estimator $\hat{E}$ of some geometric quantity E is multigrid convergent for the *family of shapes* $\mathbb{X}$ if and only if, for any $M \in \mathbb{X}$, there exists a grid step $h_M > 0$ such that the estimate $\hat{E}(\mathtt{D}_M(h), \hat{\mathbf{x}}, h)$ is defined for all $\hat{\mathbf{x}} \in \partial[\mathtt{D}_M(h)]_h$ with $0 < h < h_M$, and for any $\mathbf{x} \in \partial X$,

$$\forall \hat{\mathbf{x}} \in \partial[\mathtt{D}_M(h)]_h \text{ with } \|\hat{\mathbf{x}} - \mathbf{x}\|_\infty \leq h, \qquad |\hat{E}(\mathtt{D}_M(h), \hat{\mathbf{x}}, h) - E(M, \mathbf{x})| \leq \tau_{M,\mathbf{x}}(h),$$

where $\tau_{M,\mathbf{x}} : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+$ has null limit at 0. The convergence is uniform for M when every $\tau_{M,\mathbf{x}}$ is bounded from above by a function τ_M independent of $\mathbf{x} \in \partial M$ with null limit at 0.

Digital/Continuous mapping

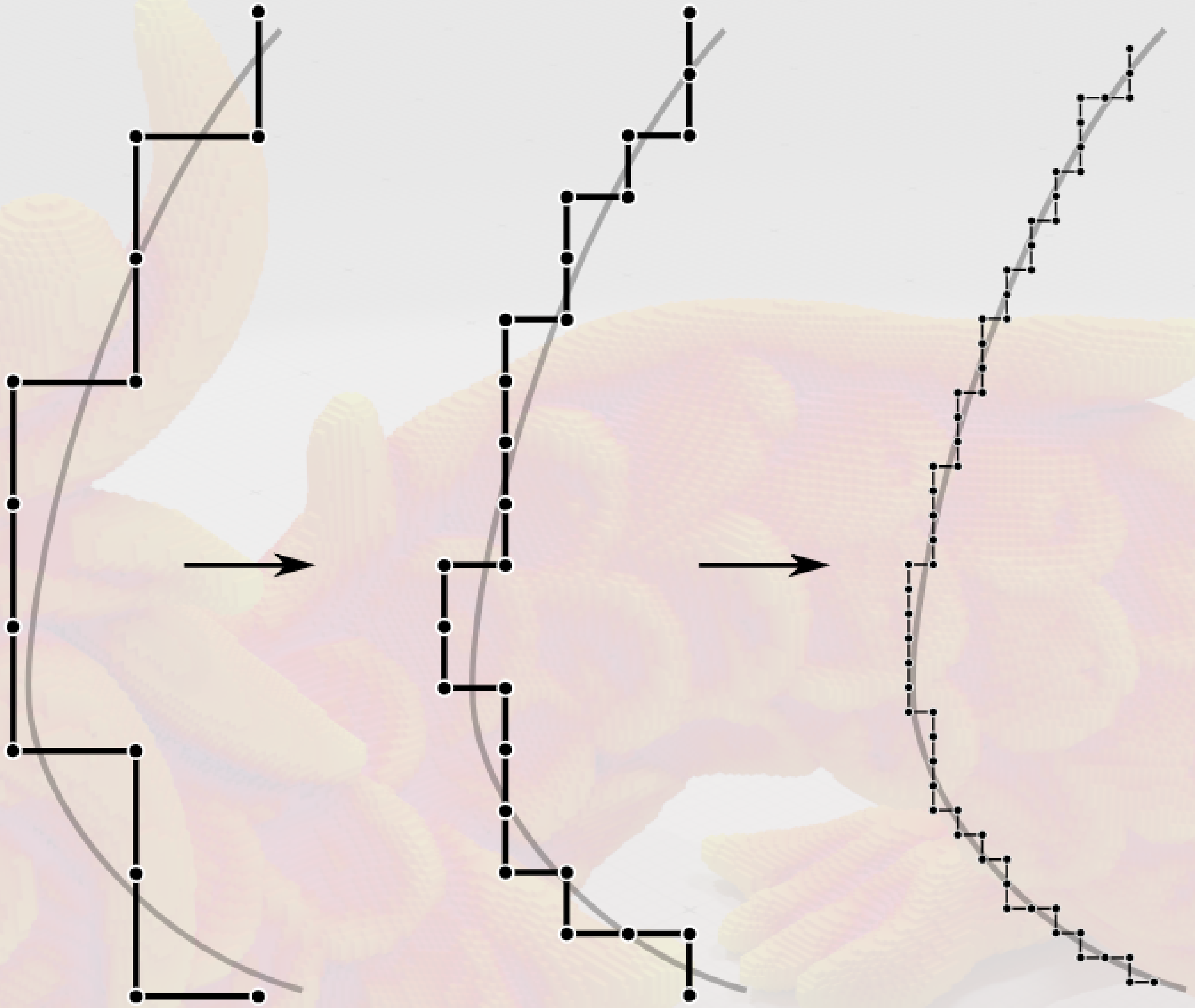


[Lachaud, Thibert]

Let M be a compact domain of $\mathbb{R}^d$ such that ∂X has positive *reach* greater that R. Let $\partial_h M := \partial[\mathtt{G}_h(M))]_h$. Then for any $0 < h < 2R/\sqrt{2}$,

- The Hausdorff between ∂M and $\partial_h M$ is bounded by $\sqrt{d}h/2$
- For $d = 2$, there exists an homeomorphism between ∂X and $\partial_h X$
- For $d \geq 3$, no homeomorphism ☹, but
  - *Projection operator* $\xi : \partial_h M \to \partial M$ is surjective
  - Area of non-injective parts of ξ the tends to zero

Digitization as an Hausdorff sampling of the continuous object



Can we estimate the curvature tensor on digital surfaces with multigrid convergence properties ?

Huge literature on differential quantity estimators

- Meshes
  - Local estimators (1- ou 2-rings) [Surazhsky et al. 2003][Gatzke, Grimm 2006]
  - Gauss-Bonnet formula based estimators [Xu 2006]
  - Normal cycles [Cohen-Steiner, Morvan 2006]
- Point Clouds
  - Jet-Fitting approaches [Cazals, Pouget 2005]
  - Voronoi cell covariance measure (VCM) [Alliez et al. 2007][Merigot et al. 2011][Cuel et al. 2014]
- Generic framework
  - Varifold approaches [Buet 2014] [Buet et al. 2015]

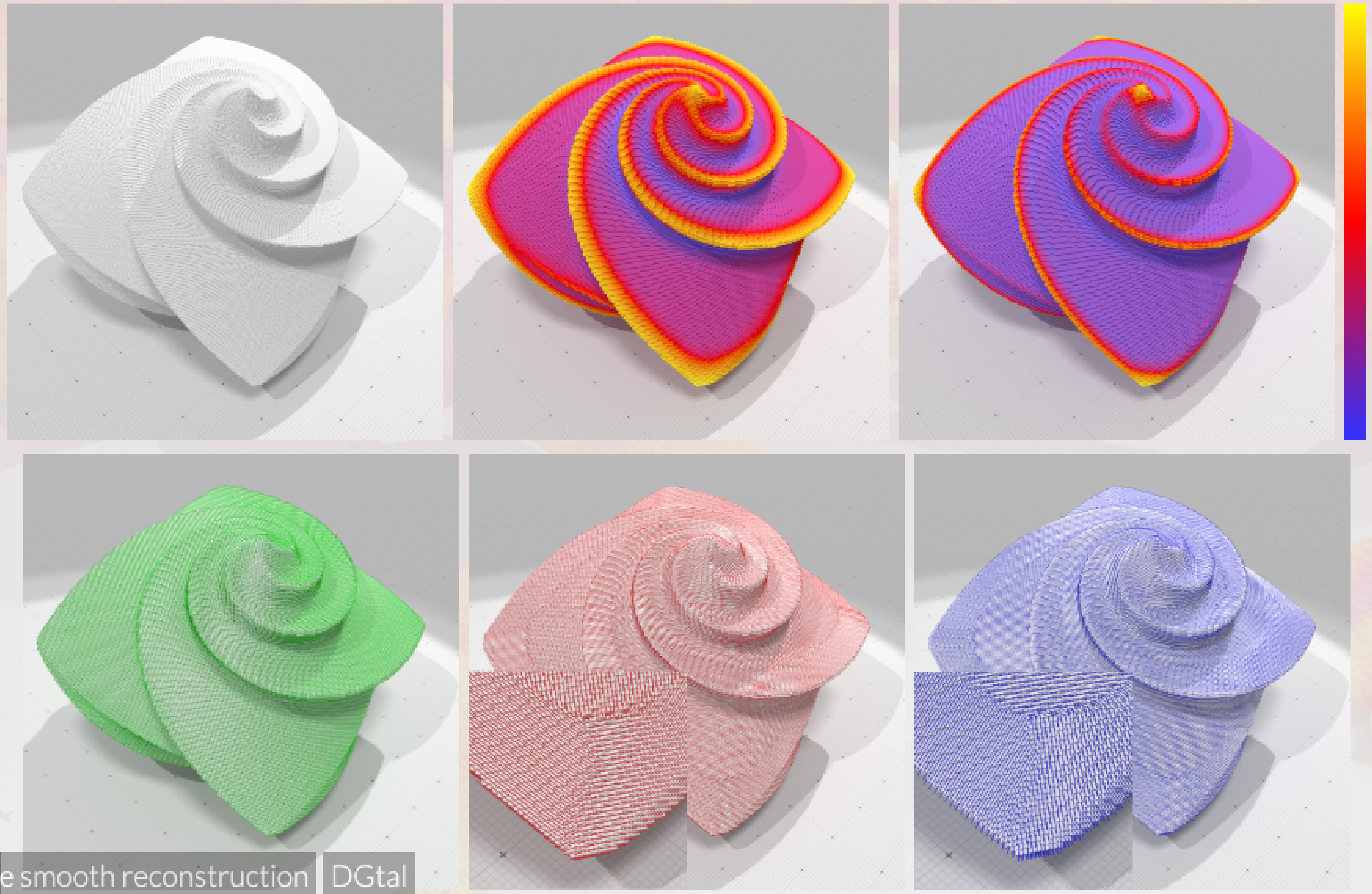
→ *accuracy depends on the mesh/point cloud quality*
→ *incompatible constraints on convergence theorem w.r.t. digital surface*

Main contributions: Integral Invariant approach

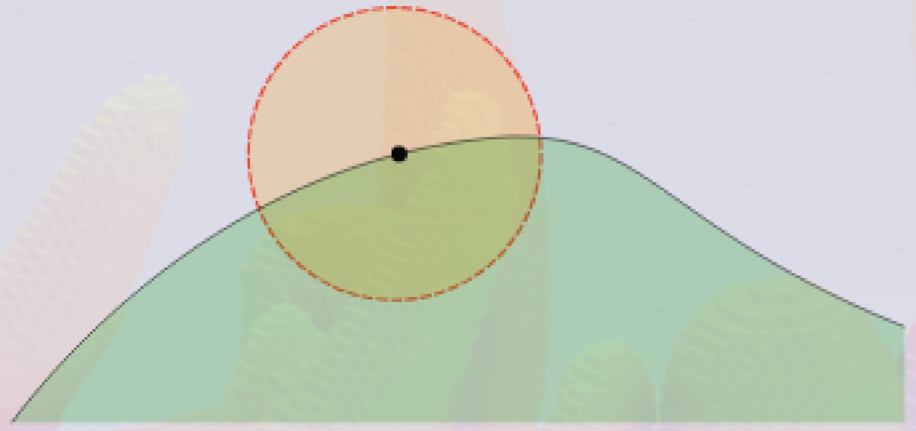
If M is compact and ∂M has positive reach ρ and C^3-continuity then

[C., Levallois, Lachaud]

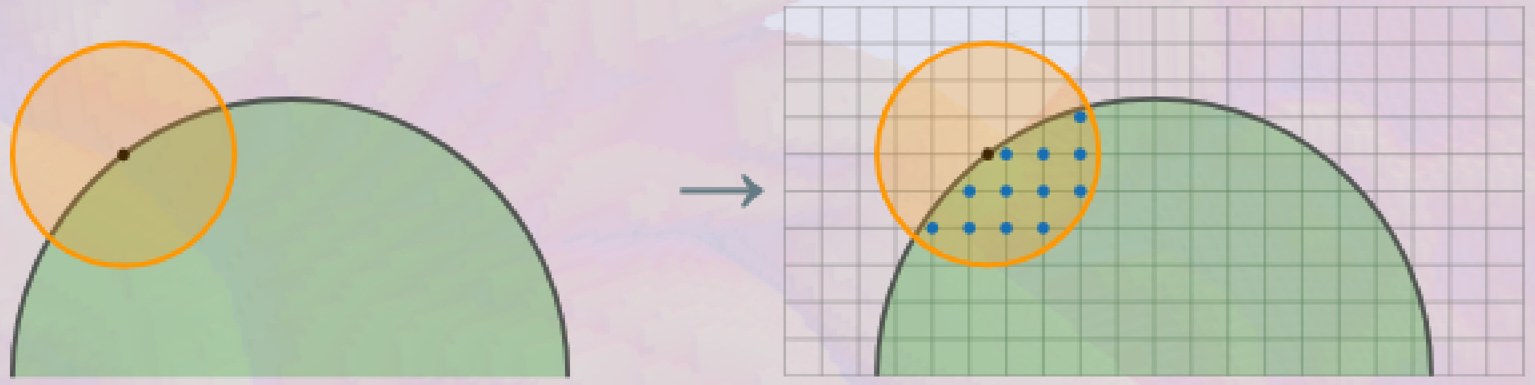
- Mean curvature and principal curvatures estimations converge in $O(h^{\frac{1}{3}})$
- Normal vector field estimation converges in $O(h^{\frac{2}{3}})$
- Principal Curvature directions converge in $\frac{1}{|\kappa_1(M,x)-\kappa_2(M,x)|}O(h^{\frac{1}{3}})$



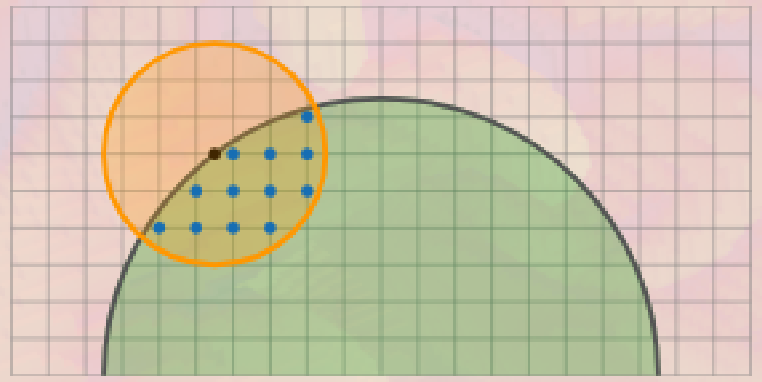
Overall proof scheme



$$\kappa(M,\mathbf{x}) := \underbrace{\frac{3\pi}{2R} - \frac{3\cdot A_R(M,\mathbf{x})}{R^3}}_{\tilde{\kappa}^R(M,\mathbf{x})} + O(R)$$ [Pottmann et al. 2007]

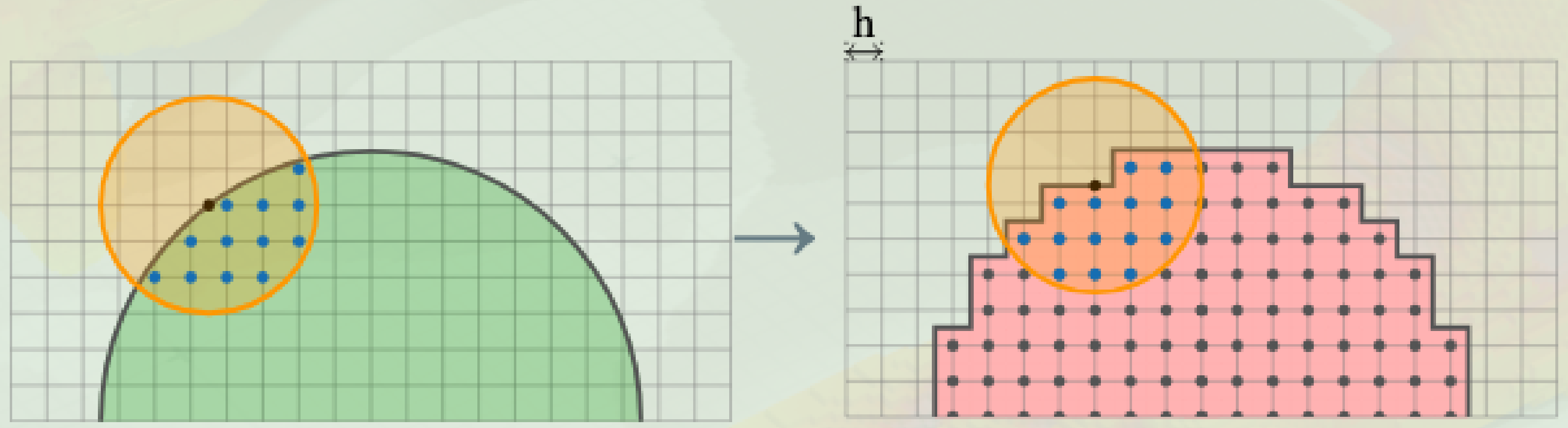


$$A_R(M,\mathbf{x}) \to \widehat{\mathrm{Area}}\,(B_{R/h}(\mathbf{x}/h) \cap \mathbb{G}_h(M))$$



+ [Pottmann et al. 2007]

$$\hat{\kappa}^R(\mathbb{G}_h(M), \mathbf{x}, h)$$



$$\hat{\kappa}^R(\mathbb{G}_h(M), \hat{\mathbf{x}}, h) \to \kappa(M,\mathbf{x})$$

Multigrid convergence of the digital curvature estimator

[C., Levallois, Lachaud]

Let M be a convex shape in $\mathbb{R}^2$ with C^3 bounded positive curvature boundary.

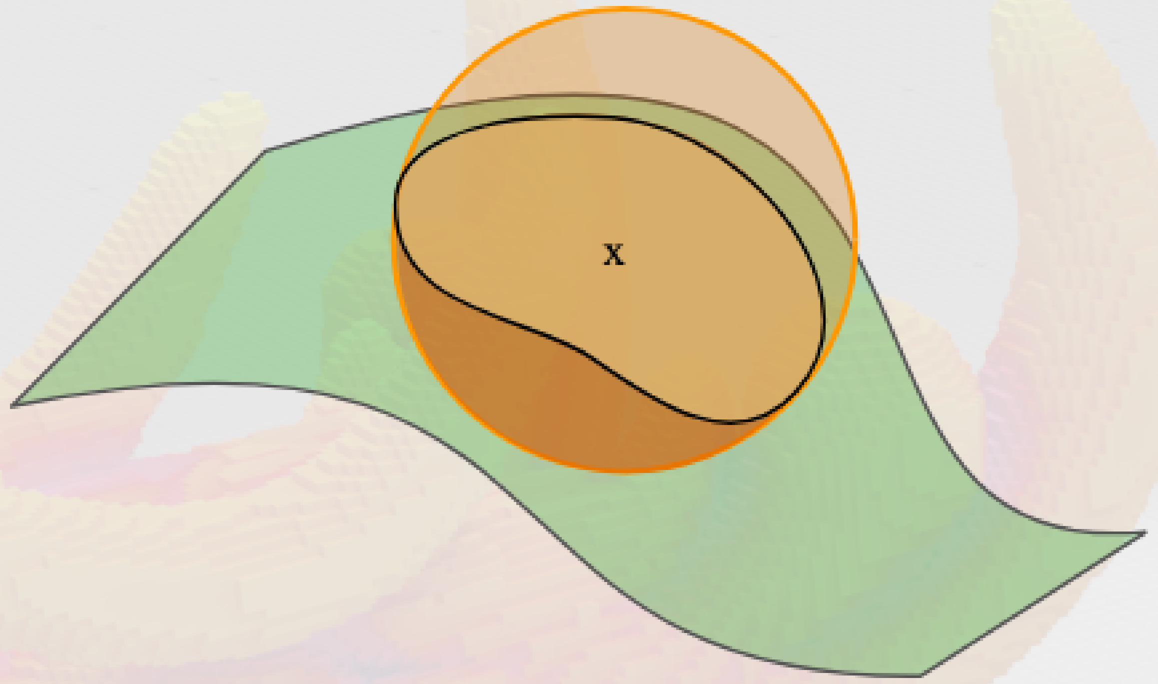
$$\forall \mathbf{x} \in \partial M, \forall \hat{\mathbf{x}} \in \partial[\mathbb{G}_h(M)]_h, \|\hat{x} - x\|_\infty \leq h \Rightarrow$$

$$|\hat{\kappa}^R(\mathbb{G}_h(M), \hat{\mathbf{x}}, h) - \kappa(M, \mathbf{x})| = O(R) + O\left(\frac{h^\beta}{R^{1+\beta}}\right) + O\left(\frac{h^{\alpha'}}{R^2}\right) + O\left(h^{\alpha'}\right) + O\left(\frac{h^{2\alpha'}}{R^2}\right)$$

$\rightarrow$ Setting $R = kh^\alpha$, we select α to minimize all errors.

$$\left|\hat{\kappa}^R(\mathbb{G}_h(M), \hat{\mathbf{x}}, h) - \kappa(M, \mathbf{x})\right| \leq O\left(h^{\frac{1}{3}}\right) \quad \text{setting} \quad R = kh^{\frac{1}{3}}$$

Curvature tensor on digital surfaces



$$\hat{\kappa}_1^R(\mathrm{G}_h(M), \mathbf{x}, h) := \frac{6}{\pi R^6}\left(\hat{\lambda}_2 - 3\hat{\lambda}_1\right) + \frac{8}{5R}$$
$$\hat{\kappa}_2^R(\mathrm{G}_h(M), \mathbf{x}, h) := \frac{6}{\pi R^6}\left(\hat{\lambda}_1 - 3\hat{\lambda}_2\right) + \frac{8}{5R}$$
$$\hat{\mathbf{w}}_1^R(\mathrm{G}_h(M), \mathbf{x}, h) := \hat{\nu}_1$$
$$\hat{\mathbf{w}}_2^R(\mathrm{G}_h(M), \mathbf{x}, h) := \hat{\nu}_2$$
$$\hat{\mathbf{n}}^R(\mathrm{G}_h(M), \mathbf{x}, h) := \hat{\nu}_3$$

$\{\hat{\nu}_i, \hat{\lambda}_i\}$ are the eigenvalues/eigenvectors of the covariance matrix of $B_r(\mathbf{x}) \cap M$

$$\left|\hat{\kappa}_i^R(\mathrm{G}_h(M), \hat{\mathbf{x}}, h) - \kappa_i(M, \mathbf{x})\right| \leq O\left(h^{\frac{1}{3}}\right) \quad \text{setting} \quad R = kh^{\frac{1}{3}}$$

$$\exists h_M \in \mathbb{R}^+,\ \forall h \in \mathbb{R},\ 0 < h < h_M,$$
$$\forall \mathbf{x} \in \partial M, \quad \forall \hat{\mathbf{x}} \in \partial[\mathrm{G}_h(M)]_h \text{ avec } \|\hat{\mathbf{x}} - \mathbf{x}\|_\infty \leq h,$$
$$\|\hat{\mathbf{w}}_1^R(\mathrm{G}_h(M), \hat{\mathbf{x}}, h) - \mathbf{w}_1(M, \mathbf{x})\| \leq \frac{1}{|\kappa_1(M,\mathbf{x}) - \kappa_2(M,\mathbf{x})|} O(h^{\frac{1}{3}}),$$
$$\|\hat{\mathbf{w}}_2^R(\mathrm{G}_h(M), \hat{\mathbf{x}}, h) - \mathbf{w}_2(M, \mathbf{x})\| \leq \frac{1}{|\kappa_1(M,\mathbf{x}) - \kappa_2(M,\mathbf{x})|} O(h^{\frac{1}{3}}),$$
$$\|\hat{\mathbf{n}}^R(\mathrm{G}_h(M), \hat{\mathbf{x}}, h) - \mathbf{n}(M, \mathbf{x})\| \leq O(h^{\frac{2}{3}}).$$

In summary

Multigrid convergent curvature tensor estimation

- Robust, Efficient implementation (convolutions)
- Parametrized by a integration radius R or a grid step h
- Proof relies on digital/continuous relationships and geometrical moment estimation

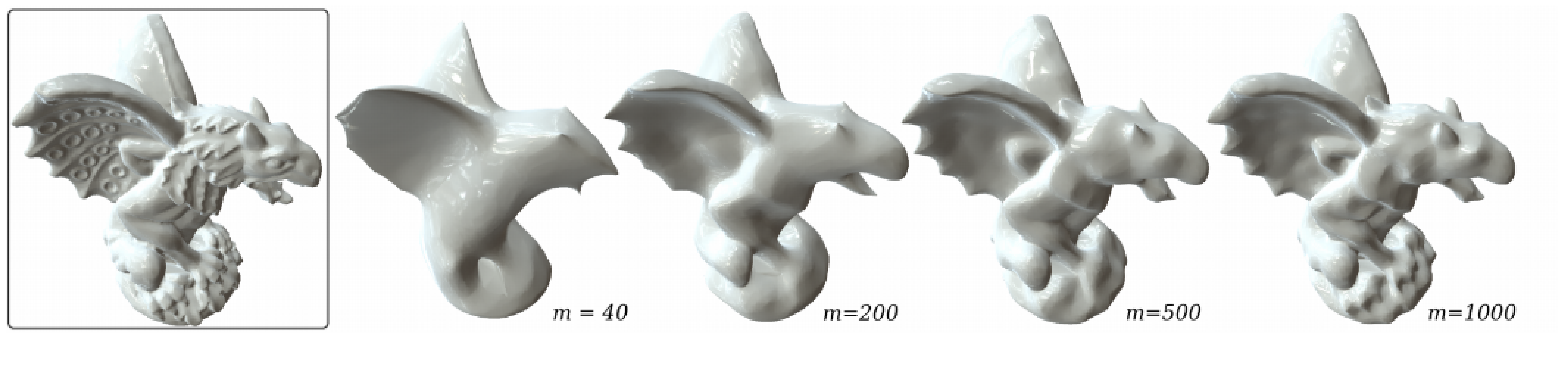
LAPLACE-BELTRAMI OPERATOR

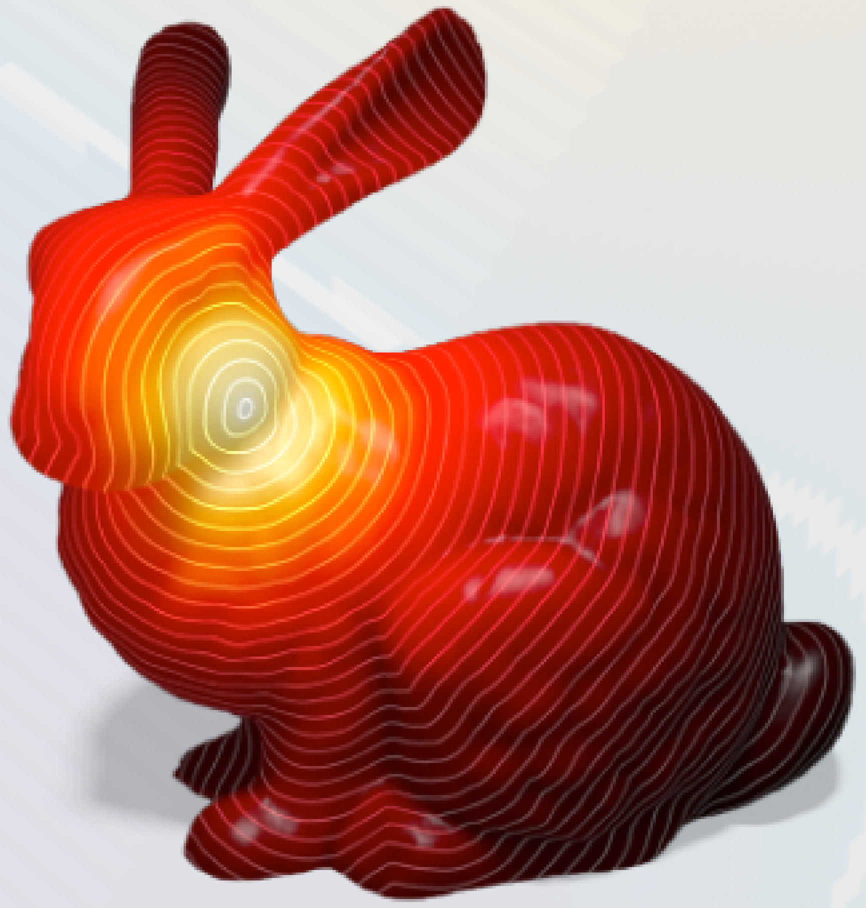
Motivations

Def.

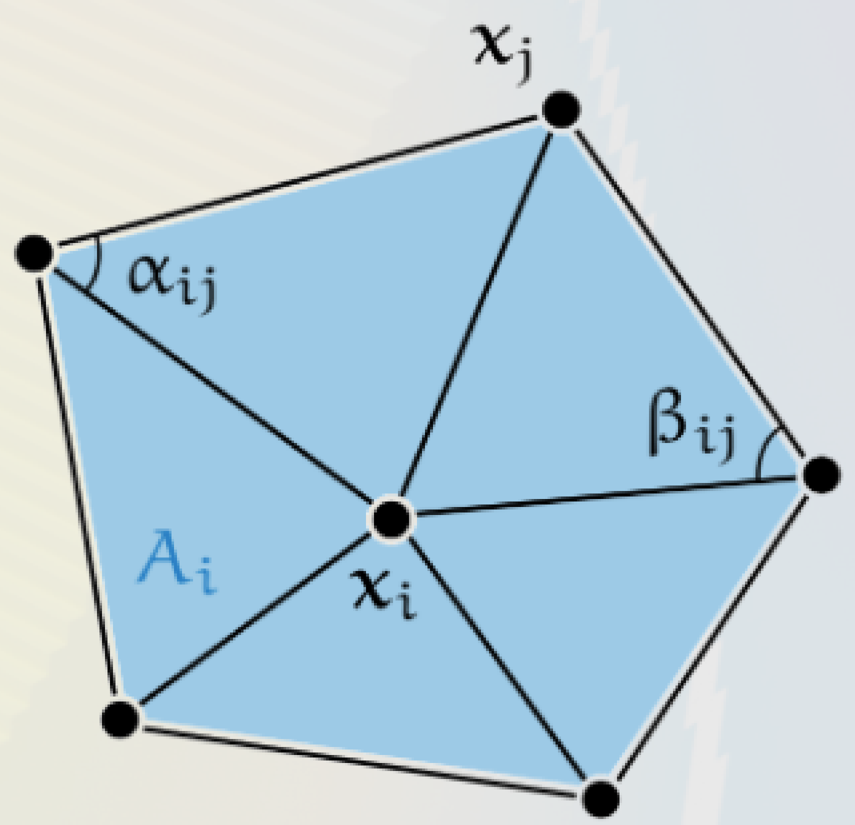
$$\Delta u = \nabla \cdot \nabla u$$

Key operator for many geometry processing tasks





Discretization of the Laplace-Beltrami operator



Many discretization scheme for triangular meshes, polygonal meshes, point clouds...

	SYM	LOC	LIN	POS	PSD	C^2-CON
Mean Value	x	✓	✓	✓	x	x
Intrinsic Del	✓	x	✓	✓	✓	x
Combinatorial	✓	✓	x	✓	✓	x
Cotan	x	✓	✓	x	✓	x
Polygonal Lap.	x	✓	✓	x	✓	x
Convolutional	x	x	?	✓	?	✓
r−local	✓	x	?	✓	?	✓

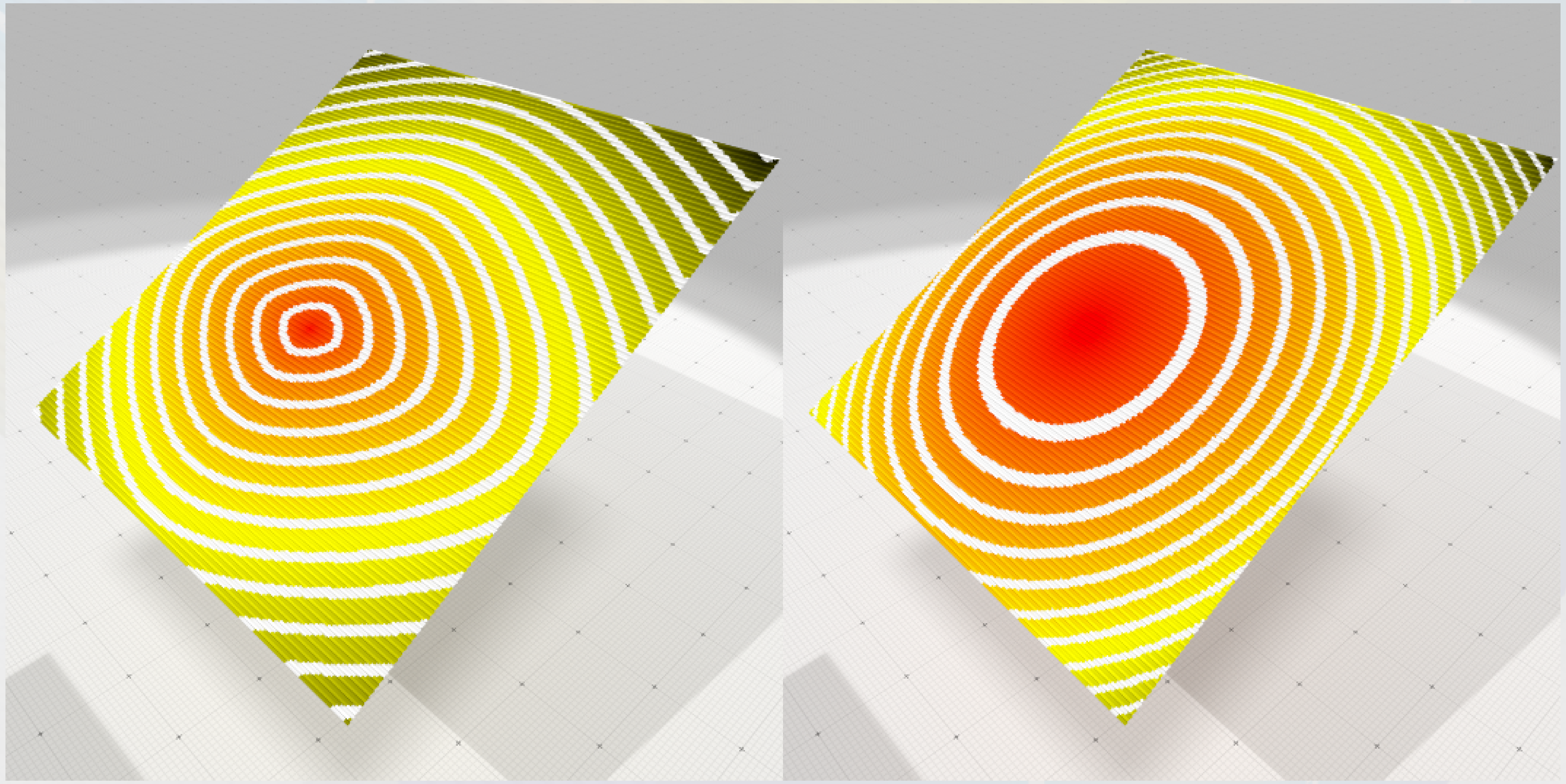
(update of "Discrete Laplace operators: No free lunch" [Wardetzky et al., 2007])

Strong consistency of the operator:

$$\lim_{\varepsilon\to 0} ||\Delta_\varepsilon v - \Delta v||_{L^\infty} = \lim_{\varepsilon\to 0}\ \sup_{x\in\partial M} |(\Delta_\varepsilon v)(x) - (\Delta v)(x)| = 0, \quad \forall v \in C^2(\partial M).$$

What about Digital Surfaces ?

- No Laplace-Beltrami operator with strong consistency property exists on digital surfaces
- Anisotropic nature of digital surfaces may lead to geometrical inconsistencies



Heat equation based Laplace-Beltrami operator on meshes

$\Delta g(x,t) = \frac{\partial}{\partial t} g(x,t), u = g(\bullet, 0) \quad \rightarrow \quad \Delta g(x,t) = \lim_{t\to 0} \frac{1}{t} \int_{\partial M} p(t,x,y)(u(y)-u(x))dy$

Functional Laplace-Beltrami [Belkin et al]

$$(\mathcal{L}_t\, u)(x) := \frac{1}{t(4\pi t)^{\frac{d}{2}}} \int_{y\in\partial M} e^{-\frac{||y-x||^2}{4t}} (u(y)-u(x))dy,$$

Thm.

As $t \rightarrow 0$, $(\mathcal{L}_t\, u)$ converges to $\Delta\, u$.

$$(\mathcal{L}_{MESH}\, u)(p) := \frac{1}{4\pi t^2} \sum_{f\in F} \frac{A_f}{3} \sum_{q\in V(f)} e^{-\frac{||p-q||^2}{4t}} (u(q)-u(p))$$

Thm.

If the mesh is a *nice triangulation* of a smooth manifold, $(\mathcal{L}_{MESH}\, u)$ converges to $(\mathcal{L}_t\, u)$ ($t \approx 1$/density).

(multigrid) Digital Surfaces are not *nice triangulations*!

Convolution based Laplace-Beltrami operator on Digital Surfaces

Def.

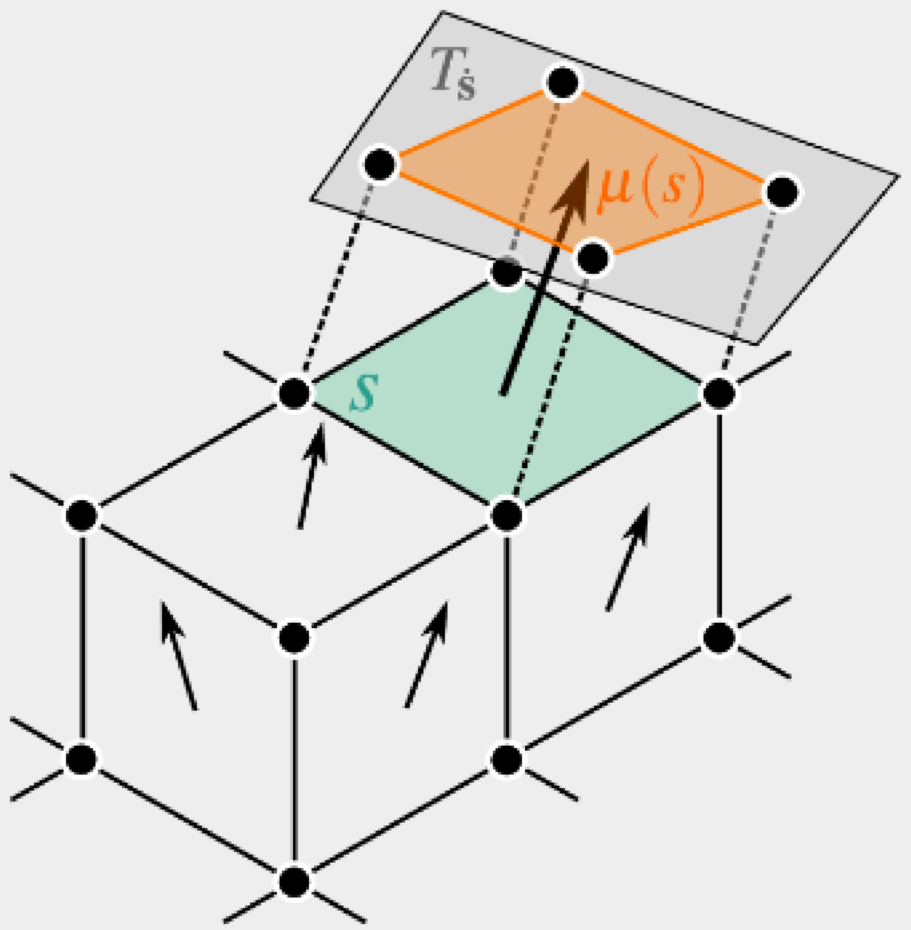
$$(L_h\,\tilde{u})(\mathbf{s}) := \frac{1}{t_h(4\pi t_h)^{\frac{d}{2}}} \sum_{\mathbf{r}\in S} e^{-\frac{||\mathbf{r}-\mathbf{s}||^2}{4t_h}} [\tilde{u}(\mathbf{r}) - \tilde{u}(\mathbf{s})]\mu(\mathbf{r})$$

Main result [Caissard, C., Lachaud 18]

As $h \to 0$, $(L_h\,\tilde{u})$ converges to $\Delta\, u$.

($\tilde{u}$ is the extension of u from M to $\mathbb{R}^3$ along the normal vectors)

Measure of a surface element $\mu(s) := \mathbf{n}_s \cdot \mathbf{n}_s^e$



Convolution based Laplace-Beltrami operator on Digital Surfaces

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Main result [Caissard, C., Lachaud 18]

As $h \to 0$, $(L_h\,\tilde{u})$ converges to $\Delta\, u$.

($\tilde{u}$ is the extension of u from M to $\mathbb{R}^3$ along the normal vectors)

Sketch of the proof

$$\Big|(\Delta u)(\xi(s)) - (L_h\tilde{u})(s)\Big| \leq \Big|(\Delta u)(\xi(s)) - (\mathcal{L}_t u)(\xi(s))\Big| + \Big|(\mathcal{L}_t u)(\xi(s)) - (\mathcal{L}\tilde{u})(s)\Big| + \Big|(\mathcal{L}_t\tilde{u})(s) - (L_h\tilde{u})(s)\Big|$$

Sketch of the proof

$$\left|(\Delta u)(\xi(s)) - (L_h\tilde{u})(s)\right| \leq \underbrace{\left|(\Delta u)(\xi(s)) - (\mathcal{L}_t u)(\xi(s))\right|}_{\text{[Belkin et al]}} + \underbrace{\left|(\mathcal{L}_t u)(\xi(s)) - (\mathcal{L}\tilde{u})(s)\right|}_{\text{Projection error}} + \underbrace{\left|(\mathcal{L}_t\tilde{u})(s) - (L_h\tilde{u})(s)\right|}_{\text{Digital integration error}}$$

Sketch of the proof

$$\underbrace{\left|(\Delta u)(\xi(s)) - (L_h\tilde{u})(s)\right|}\leq \underbrace{\left|(\Delta u)(\xi(s)) - (\mathcal{L}_t u)(\xi(s))\right|}_{\text{[Belkin et al]}} + \underbrace{\left|(\mathcal{L}_t u)(\xi(s)) - (\mathcal{L}_t\tilde{u})(s)\right|}_{\text{Projection error}} + \underbrace{\left|(\mathcal{L}_t\tilde{u})(s) - (L_h\tilde{u})(s)\right|}_{\text{Digital integration error}}$$

Lemma

Let $\mathbf{s} \in \partial_h M$, a function $u \in C^2(\partial M)$ and its extension $\tilde{u}$. For $t_h = h^\alpha, 0 < \alpha \leq \frac{2}{2+d}$ and $h \leq h_{max}$ with h_{max} the minimum between $\mathrm{Diam}(\partial M)$, $K_3(d, \alpha, \mathrm{Diam}(\partial M))$ and $R/\sqrt{d+1}$, we have

$$|(\mathcal{L}_t u)(\xi(s)) - (\mathcal{L}_t\tilde{u})(s)| \leq \mathrm{Area}(\partial M)\,||\nabla u||_\infty \left[K_1(d)h^{1-\alpha(1+\frac{d}{2})} + K_2(d)h^{2-\alpha\frac{3+d}{2}}\right]$$

with

$$K_1(d) := \frac{\sqrt{d+1}}{2^{d-1}e\pi^{\frac{d}{2}}} \text{ and } K_2(d) := \frac{3(d+1)}{2^{d+\frac{5}{2}}\sqrt{e}\pi^{\frac{d}{2}}}.$$

Technical proof using the regularity of u and Hausdorff distance between ∂M and $\partial_h M$.

Sketch of the proof

$$\left|(\Delta u)(\xi(s)) - (L_h\tilde{u})(s)\right| \leq \underbrace{\left|(\Delta u)(\xi(s)) - (\mathcal{L}_t u)(\xi(s))\right|}_{\text{[Belkin et al]}} + \underbrace{\left|(\mathcal{L}_t u)(\xi(s)) - (\mathcal{L}\tilde{u})(s)\right|}_{\text{Projection error}} + \underbrace{\left|(\mathcal{L}_t\tilde{u})(s) - (L_h\tilde{u})(s)\right|}_{\text{Digital integration error}}$$

Sketch of the proof

$$\left|(\Delta u)(\xi(s)) - (L_h\tilde{u})(s)\right| \leq \underbrace{\left|(\Delta u)(\xi(s)) - (\mathcal{L}_t u)(\xi(s))\right|}_{\text{[Belkin et al]}} + \underbrace{\left|(\mathcal{L}_t u)(\xi(s)) - (\mathcal{L}\tilde{u})(s)\right|}_{\text{Projection error}} + \underbrace{\left|(\mathcal{L}_t\tilde{u})(s) - (L_h\tilde{u})(s)\right|}_{\text{Digital integration error}}$$

[Lachaud, Thibert]

Let M be a compact domain whose boundary has positive reach R. For $h \leq \frac{R}{\sqrt{d+1}}$, the digital integral is multigrid convergent toward the integral over ∂M. More precisely, for any measurable function $f : \mathbb{R}^{d+1} \to \mathbb{R}$, one gets

$$\left|\int_{\partial M} f(x)dx - \mathrm{DI}_h(f, M_h, \hat{\mathbf{n}})\right| \leq 2^{d+3}(d+1)^{\frac{3}{2}}\, Area(\partial M)\Big(\mathrm{Lip}(f)\sqrt{d+1}\, h + ||f||_\infty \cdot ||\hat{\mathbf{n}} - \mathbf{n}||_{est}\Big),$$

($\mathrm{DI}_h(f, M_h, \hat{\mathbf{n}}) \approx$ summation of f evaluated at each surfel s and weighted by $\mu(s)$)

⇒ We need a multigrid convergent normal vector estimation

Remaining steps: We set $f(x) := \frac{1}{t_h(4\pi t_h)^{\frac{d}{2}}} e^{-\frac{||x-s||^2}{4t_h}} (\tilde{u}(x) - \tilde{u}(s))$ and we derive bounds for $\mathrm{Lip}(f)$ and $||f||_\infty$.

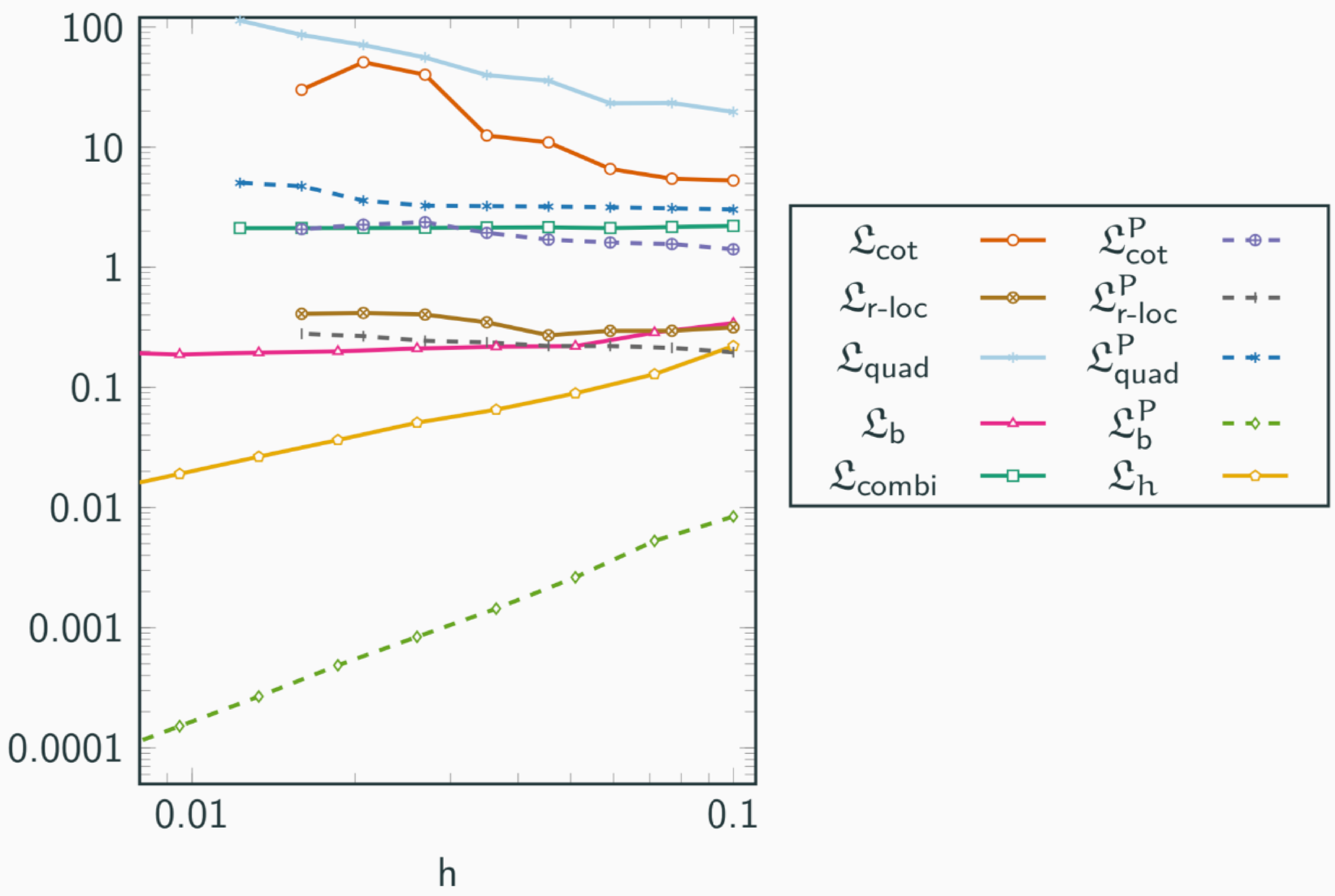
Main result

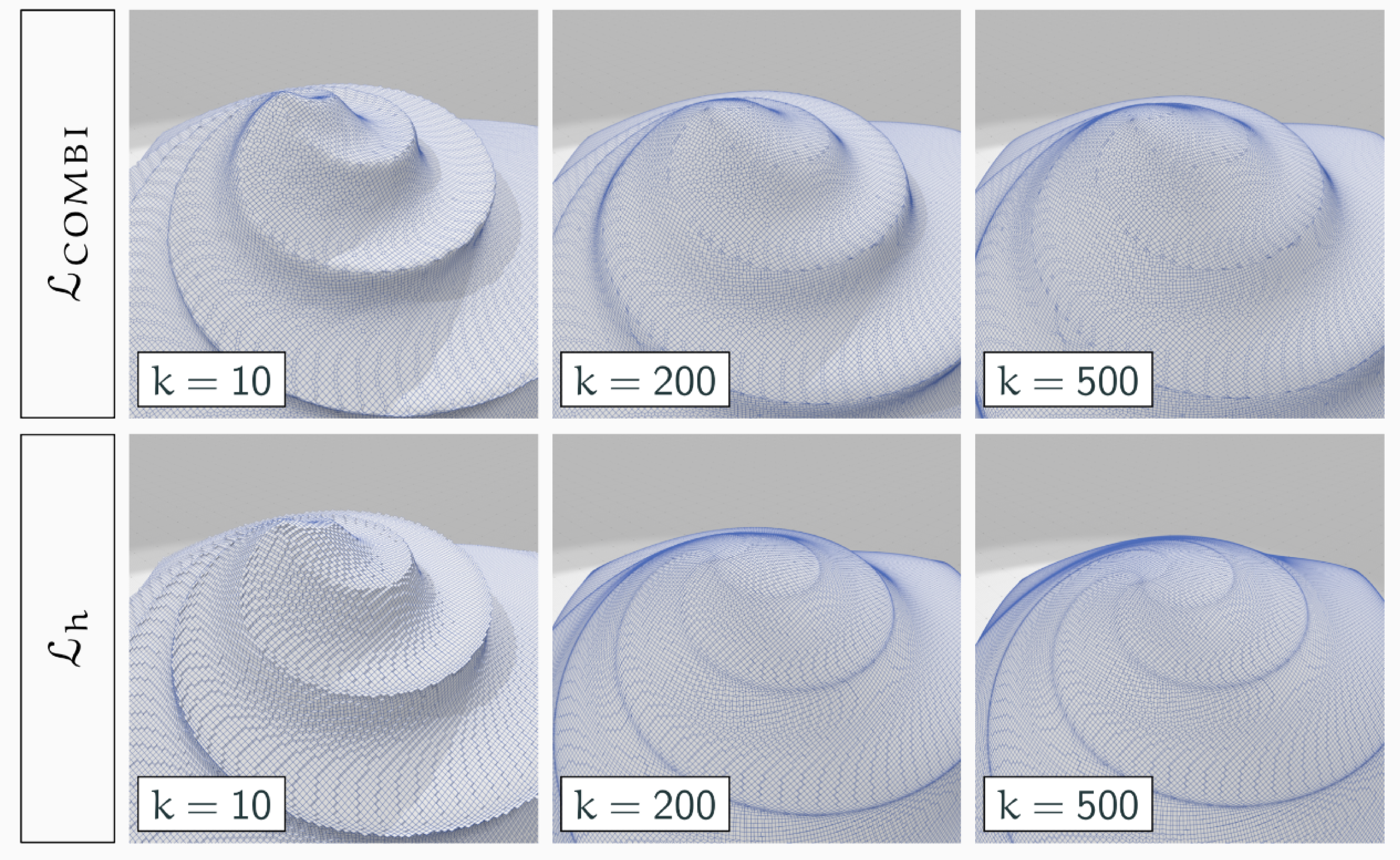
[Caissard, C., Lachaud, Roussillon]

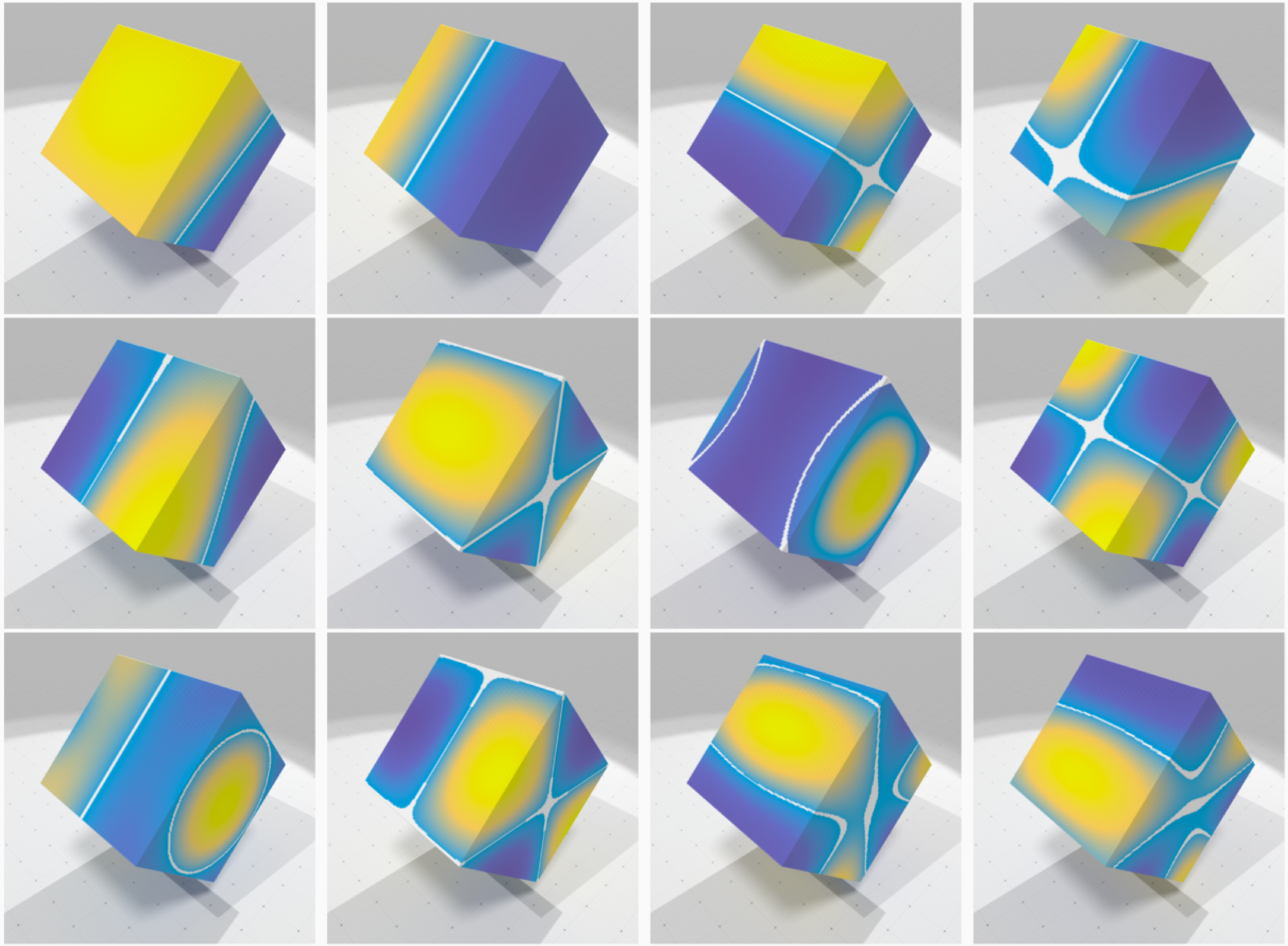
Let $\mathbf{s}$ be a surfel in $\partial_h M$, a function $u \in C^2(\partial M)$ and its extension $\tilde{u}$. Let $t_h = h^\alpha$ and let the convergence speed of the normal estimator be in $O(h^\beta)$. Let h_0 be the minimum between $\mathrm{Diam}(\partial M), R/\sqrt{d+1}$ and $K_3(d, \alpha, \mathrm{Diam}(\partial M))$. For $0 < h \leq h_0$ we have

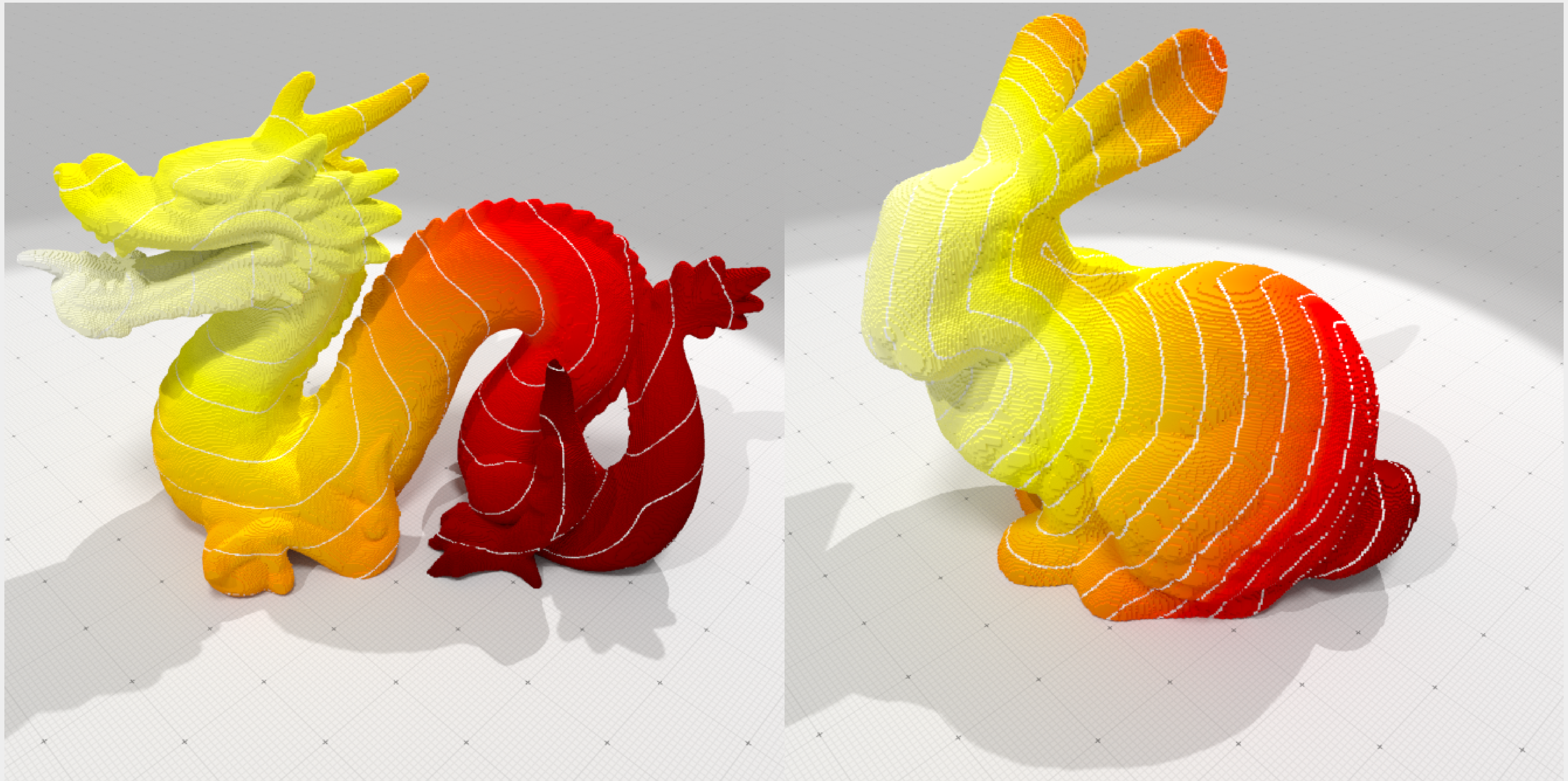
$$\lim_{h \to 0} |(\Delta u)(\xi(s)) - (L\tilde{u})(s)| = 0$$

if $0 < \alpha < \min\left(\frac{2}{d+2}, \frac{2\beta}{d+1}\right)$.


100
10
1
0.1
0.01
0.001
0.0001
0.01
0.1
h
$\mathfrak{L}_{cot}$
$\mathfrak{L}^P_{cot}$
$\mathfrak{L}_{r\text{-}loc}$
$\mathfrak{L}^P_{r\text{-}loc}$
$\mathfrak{L}_{quad}$
$\mathfrak{L}^P_{quad}$
$\mathfrak{L}_b$
$\mathfrak{L}^P_b$
$\mathfrak{L}_{combi}$
$\mathfrak{L}_h$


$\mathcal{L}_{\mathrm{COMBI}}$
k = 10
k = 200
k = 500
$\mathcal{L}_h$
k = 10
k = 200
k = 500



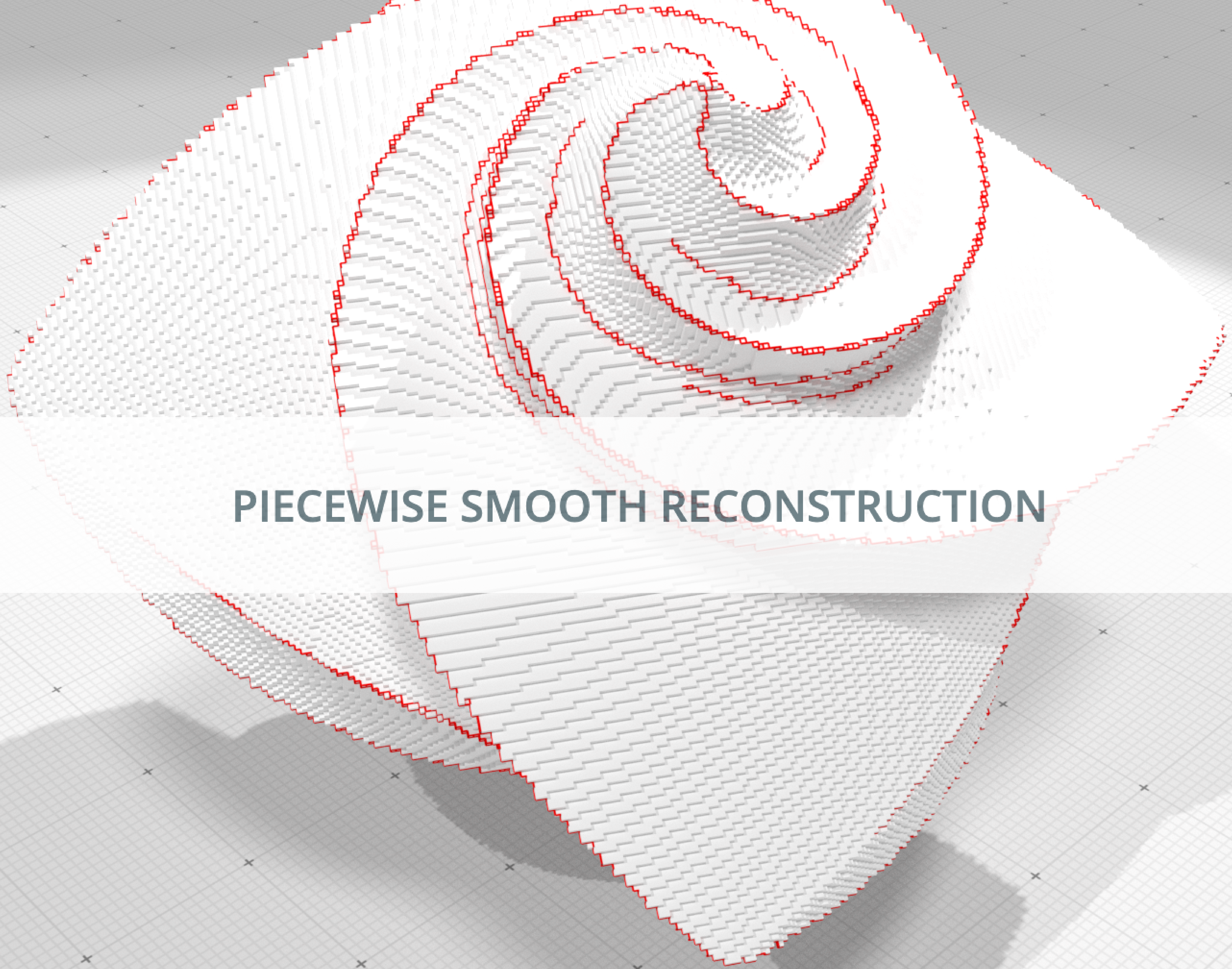


In summary

Laplace-Beltrami operator on Digital Surfaces

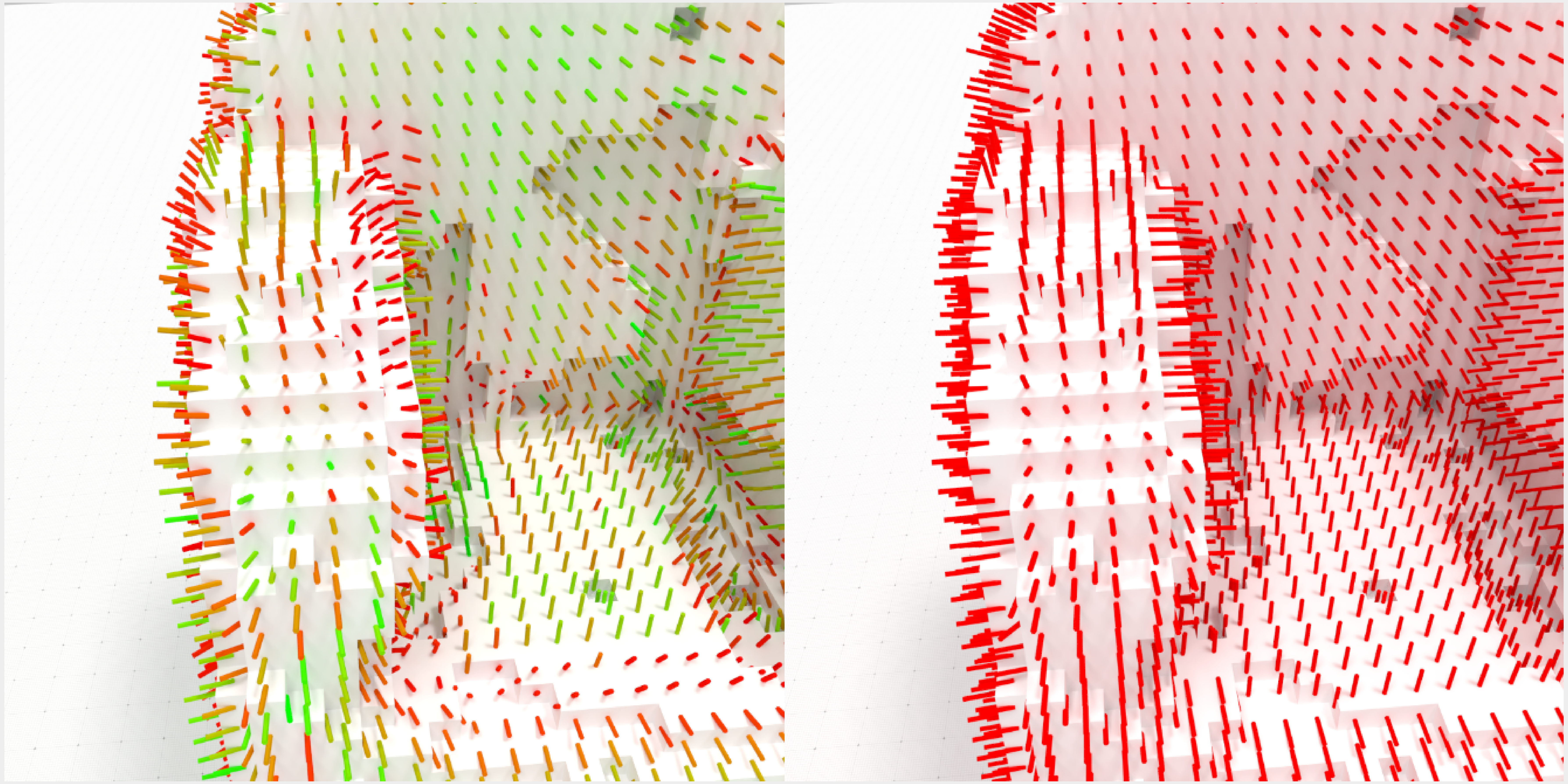
- *Strong consistency* thanks to a multigrid convergent normal vector field

- Discrete operator is not as sparse as the cotangent one
- Efficient implementation (convolutions on compact support)



PIECEWISE SMOOTH RECONSTRUCTION

Problem statement



⇒ Piecewise smooth reconstruction of the normal vector field

Ambrosio-Tortorelli functional

[Ambrosio and Tortorelli, 1990]

$$\mathcal{AT}_\epsilon(u,v) = \underbrace{\alpha \int_M |u-g|^2 dx}_{\text{attachment term}} + \underbrace{\int_M |v\nabla u|^2 dx}_{\text{smoothness term}} + \underbrace{\lambda \int_M \epsilon|\nabla v|^2 + \frac{1}{4\epsilon}|1-v|^2 dx}_{\text{discontinuities length}}$$

- Two functions to optimize: u (scalar map) and v (feature scalar map, $\mathcal{M} \to [0,1]$)
- $v \approx 1$ on smooth parts, $v \approx 0$ near features
- Quadratic terms
- the AT functional $\Gamma-$converges to Mumford-Shah's functional $\mathcal{AT}_\epsilon \xrightarrow[\epsilon\to 0]{\Gamma} \mathcal{MS}$
- (integration domain does not change, no Hausdorff measure)

Ambrosio-Tortorelli functional

[Ambrosio and Tortorelli, 1990]

$$\mathcal{AT}_\epsilon(u,v) = \alpha \underbrace{\int_M |u-g|^2 dx}_{\text{attachment term}} + \underbrace{\int_M |v\nabla u|^2 dx}_{\text{smoothness term}} + \lambda \underbrace{\int_M \epsilon|\nabla v|^2 + \frac{1}{4\epsilon}|1-v|^2 dx}_{\text{discontinuities length}}$$

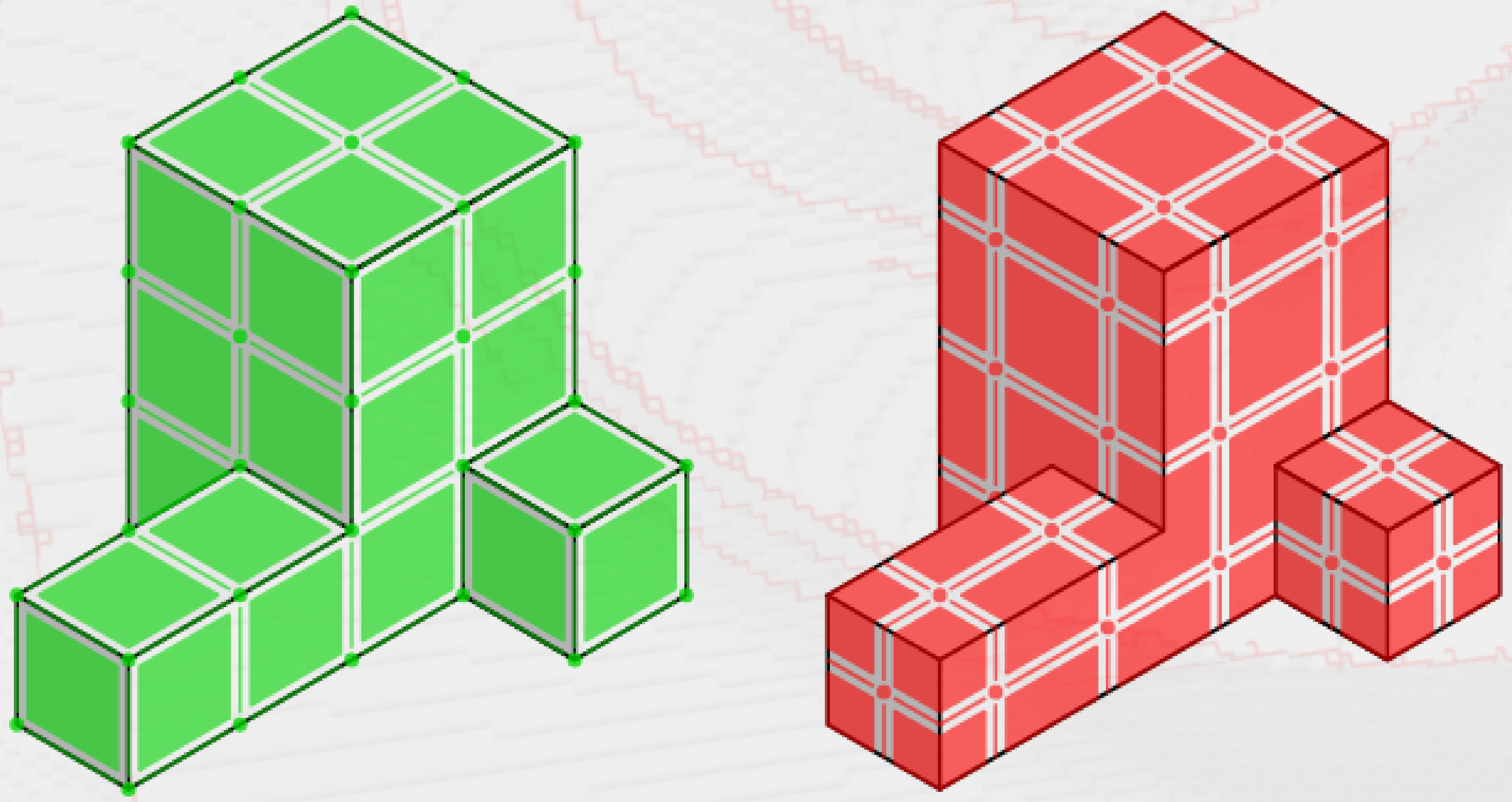
Parameters

- ϵ -- *thickness* of the feature set ($[distance]$)
- α -- attachment coefficient to control the *smoothing* strength ($[area^{-1}]$)
- λ -- proportional to the *length* of discontinuities ($[distance^{-1}]$)

Discretization

"à la" Discrete Exterior Calculus *[Hirani, Desbrun, Grady...]:*

$\mathcal{M}$ is a cellular complex ($\mathcal{M}'$ its dual), σ^k are $k-$cells of $\mathcal{M}$ (resp. σ'^k of $\mathcal{M}'$)



- *$k-$forms* are vectors of $|\{\sigma^k\}|$ scalars
- *Linear operators* are matrices
- e.g.
  - d_k (exterior derivative) maps primal $k-$forms to primal $(k+1)-$forms
  - *wedge product* $\alpha \wedge \beta$ maps $k-$forms and $l-$forms to $(k+l)-$forms
  - *Hodge-star* $\star_k$ operator to maps primal $k-$forms to dual $k-$forms
  - ...

Discretization (bis)

from [Ambrosio and Tortorelli, 1990]

$$\mathcal{AT}_\epsilon(u,v) = \alpha \underbrace{\int_M |u-g|^2 dx}_{\text{attachment term}} + \underbrace{\int_M |v \nabla u|^2 dx}_{\text{smoothness term}} + \lambda \underbrace{\int_M \epsilon |\nabla v|^2 + \frac{1}{4\epsilon}|1-v|^2 dx}_{\text{discontinuities length}}$$

v is a primal $\mathbf{0}$-form, u is a triple of dual $\mathbf{0}$-forms (u_1, u_2, u_3) and we discretize $\mathcal{AT}_\epsilon$

Discretization (ter)

[C., Foare, Gueth, Lachaud]

$$\mathcal{AT}_\epsilon^d(u,v) := \alpha \sum_{i=1}^{3} \langle u_i - g_i, u_i - g_i \rangle_{\bar{0}} + \sum_{i=1}^{3} \langle v \wedge d_{\bar{0}} u_i, v \wedge d_{\bar{0}} u_{i\bar{1}} \rangle_{\bar{1}}$$

$$+ \lambda\epsilon \langle d_0 v, d_0 v \rangle_1 + \frac{\lambda}{4\epsilon} \langle 1 - v, 1 - v \rangle_0$$

- if γ is a primal $0-$form and β a dual $1-$form, then $\gamma \wedge \beta = diag(\beta) M \gamma$ with $M := \frac{1}{2}|d_0|$
- A is the matrix form of d_0
- B is the matrix form of $d_{\bar{0}}$
- $\mathbf{u}_i,\ \mathbf{v},\ \mathbf{g}$ are column vectors containing associated $k-$form scalars
- S_i is a diagonal matrix encoding the Hodge star $\star_i$

$$\mathcal{AT}_\epsilon^d(\mathbf{u},\mathbf{v}) = \alpha (u-g)^T S_{\bar{0}} (u-g) + u^T B^T \operatorname{Diag}(Mv) S_{\bar{1}} \operatorname{Diag}(Mv) B u$$

$$+ \lambda\, \epsilon\, v^T A^T S_1 A v + \frac{\lambda}{4\epsilon} (1-v)^T S_0 (1-v)$$

Optimization

$\min_{u,v} \mathcal{AT}_\epsilon \Leftrightarrow (\nabla_u \mathcal{AT}_\epsilon = 0) \wedge (\nabla_v \mathcal{AT}_\epsilon = 0)$

For a given ϵ:

$$\nabla_u AT_\epsilon[u,v] = 0 \Leftrightarrow \left[\alpha\, S_{\bar{0}} - B^T\, \mathrm{Diag}(Mv) S_{\bar{1}}\, \mathrm{Diag}(Mv) B\right] u = \alpha\, S_0\, g$$

$$\nabla_v AT_\epsilon[u,v] = 0 \Leftrightarrow \left[\frac{\lambda}{4\epsilon} S_0 + \lambda\, \epsilon\, A^T S_1 A + M^T\, \mathrm{Diag}(Bu) S_{\bar{1}}\, \mathrm{Diag}(Bu) M\right] v = \frac{\lambda}{4\,\epsilon} S_0$$

$\Rightarrow$ only linear system solves on sparse matrices !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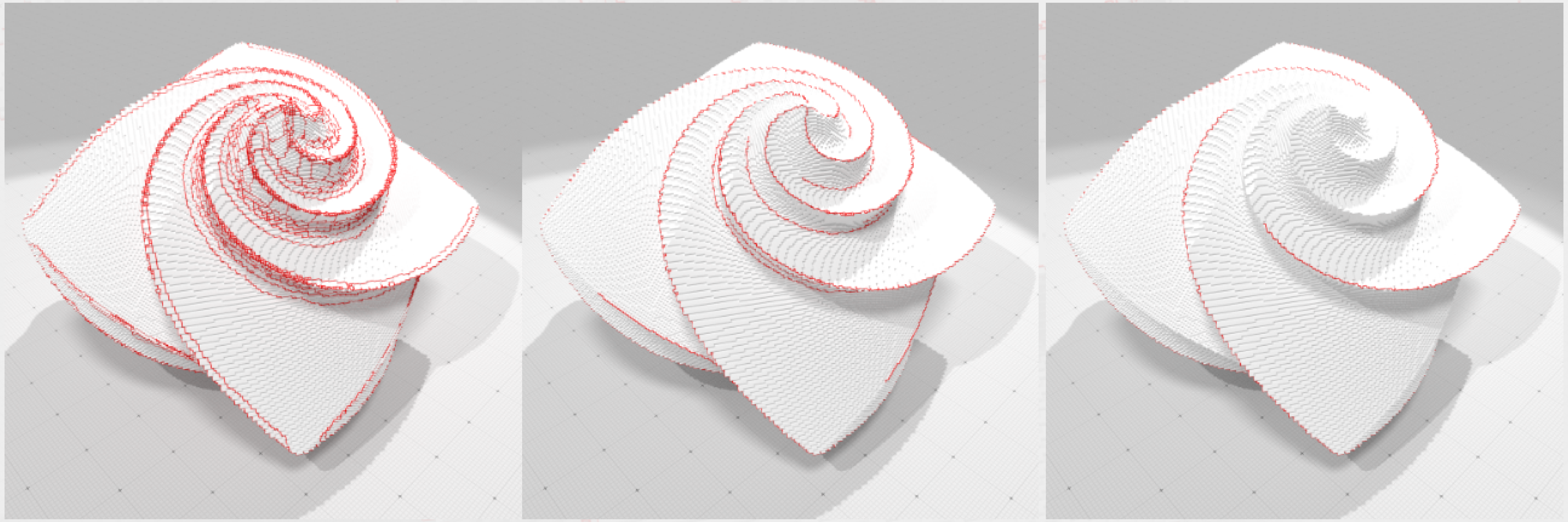
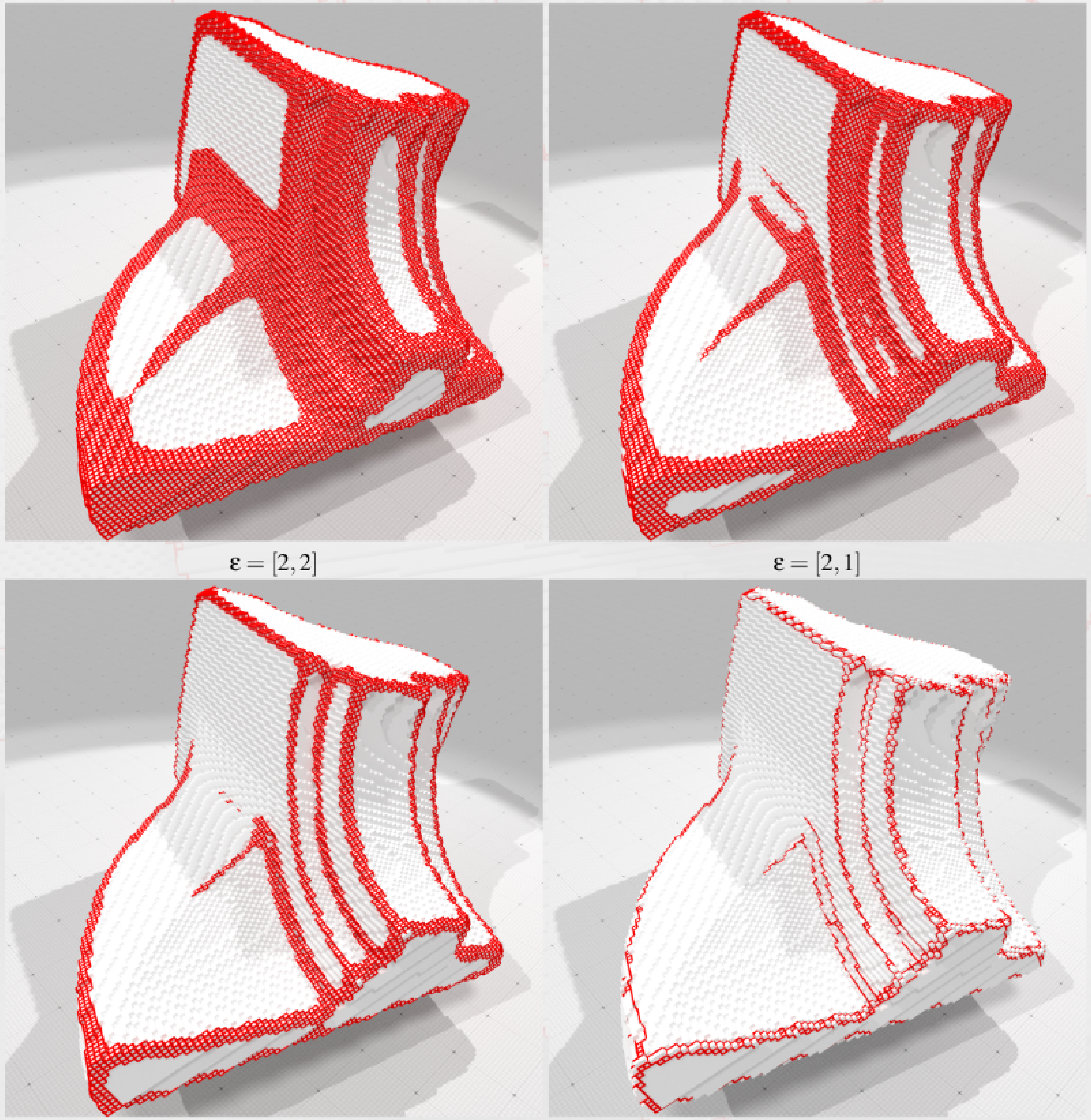
λ parameter



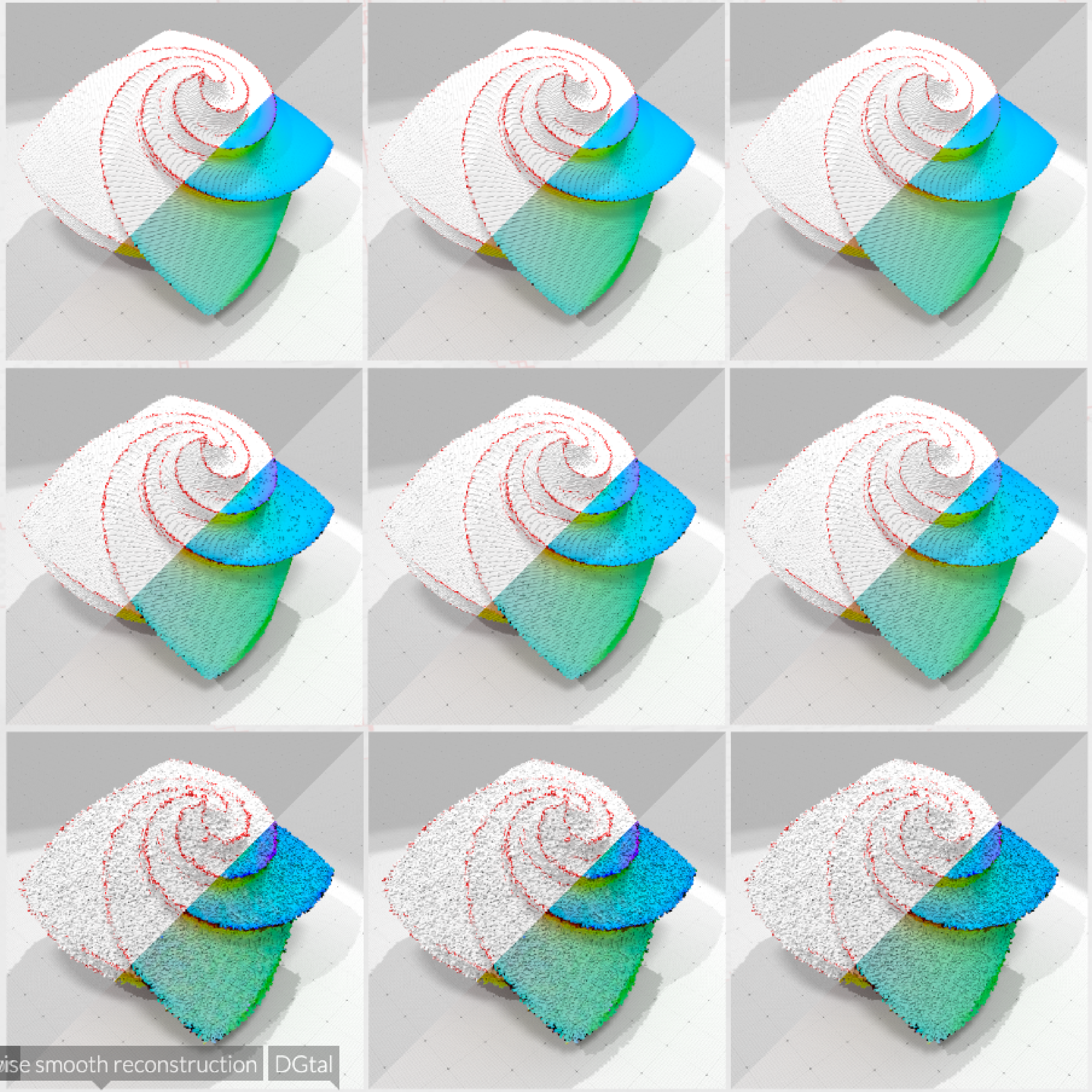
ϵ parameter



$\varepsilon = [2, 2]$ $\varepsilon = [2, 1]$

$\varepsilon = [2, 0.5]$ $\varepsilon = [2, 0.25]$

Noise level w.r.t. α parameter



In summary

Piecewise smooth Reconstruction

- Anisotropic normal vector field regularization with feature selection
- Sharp features
- Parameters make sense :)
- Variational problem discretization using a combinatorial representation of the digital surface

Chicken/egg problem: measure of quads $\mu(s)$ used in the DEC operators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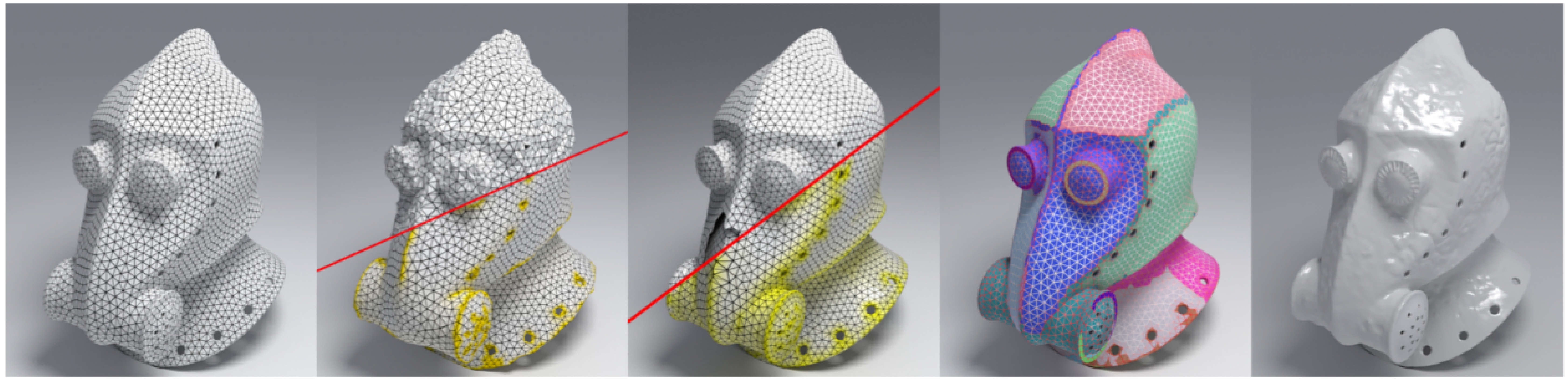
Ambrosio-Tortorelli on meshes



(a) Original (b) Denoising (c) Inpainting (d) Segmentation (e) Embossing

Nicolas Bonneel, C., Pierre Gueth, Jacques-Olivier Lachaud. Mumford-Shah Mesh Processing using the Ambrosio-Tortorelli Functional. Computer Graphics Forum (Proceedings of Pacific Graphics), 37(7), October 2018.

DGtal / dgtal.org

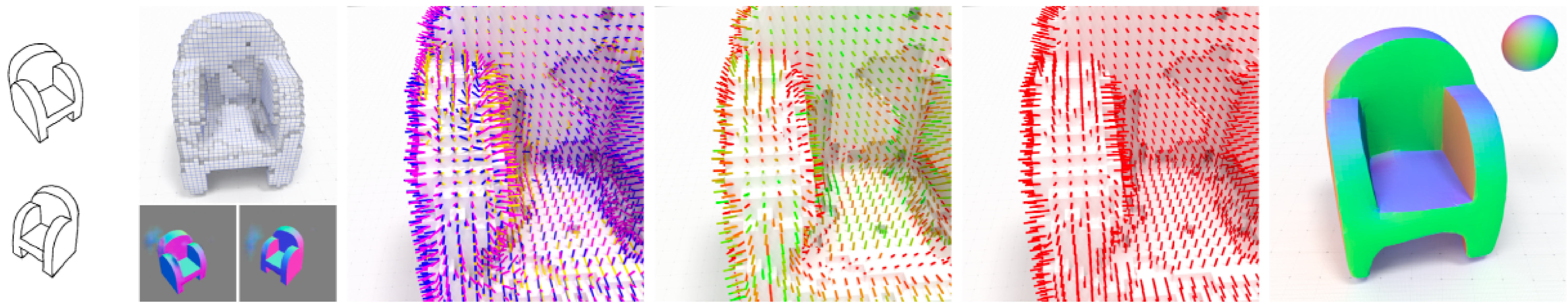
github.com/DGtal-team

@libdgtal

Conclusion

Digital Geometry

- Nice geometrical model with many interactions (arithmetic's, theory of words, computational geometry, discrete mathematics...)
- Very specific discrete/continuous properties
- Related to various areas (image processing, material sciences, geometrical modeling, rendering...) data



(a) Input (b) CNNs predictions (c) Candidate normals (d) Aggregated normals (e) Piecewise-smooth normals (f) Final surface