A New Operator Splitting Method for Euler's Elastica Model

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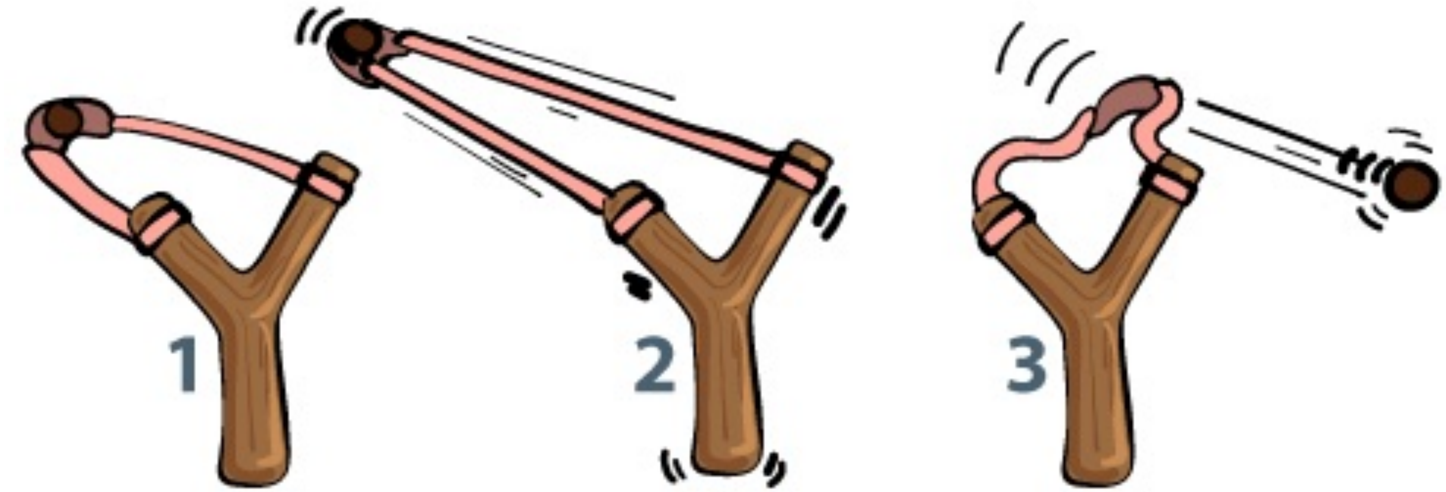
Euler's Elastica energy

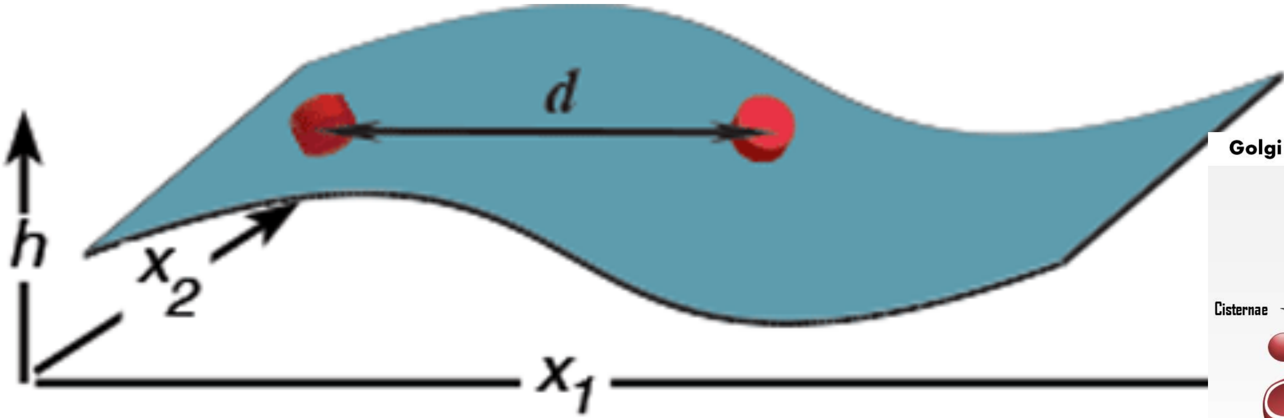


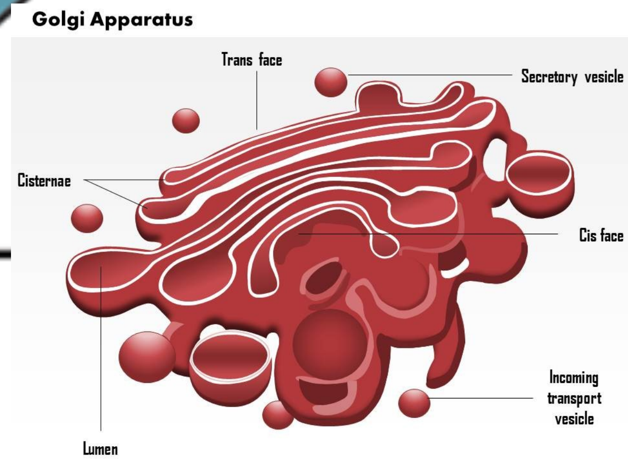
Euler's elastica:

$$E(\Gamma) = \int_{\Gamma} (a + b\kappa^2) ds, \ a,b \geq 0.$$

κ is the curvature of the curve.

Willmore energy





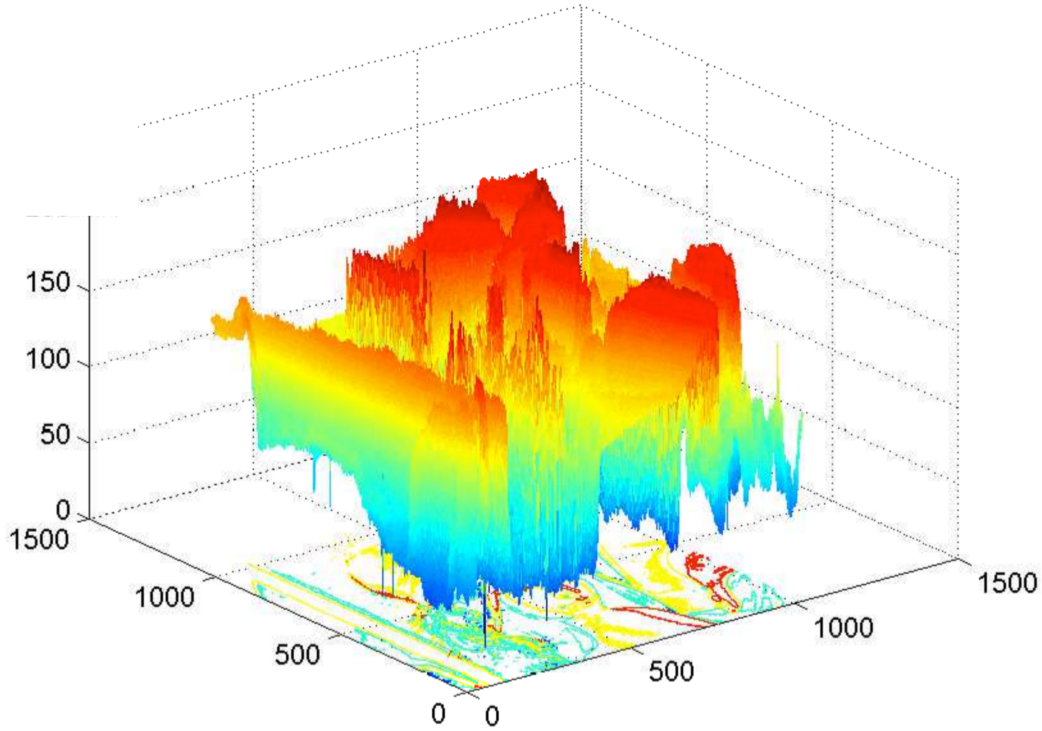
Willmore energy

$$E(\Gamma) = \int_\Gamma (H^2 - K)\,ds,$$

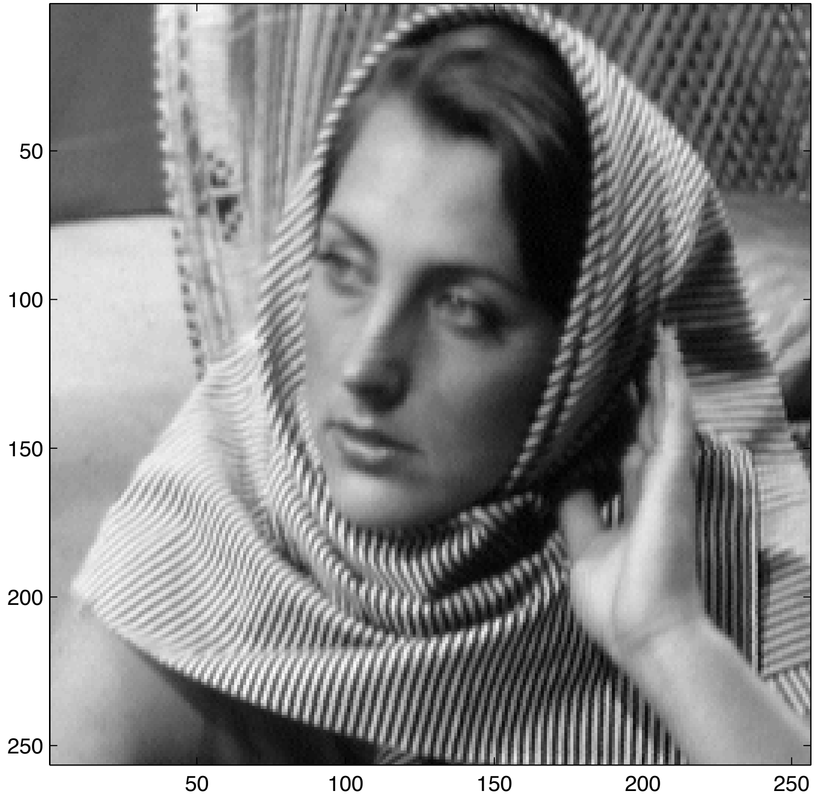
where H is the mean curvature and K is the Gauss curvature.

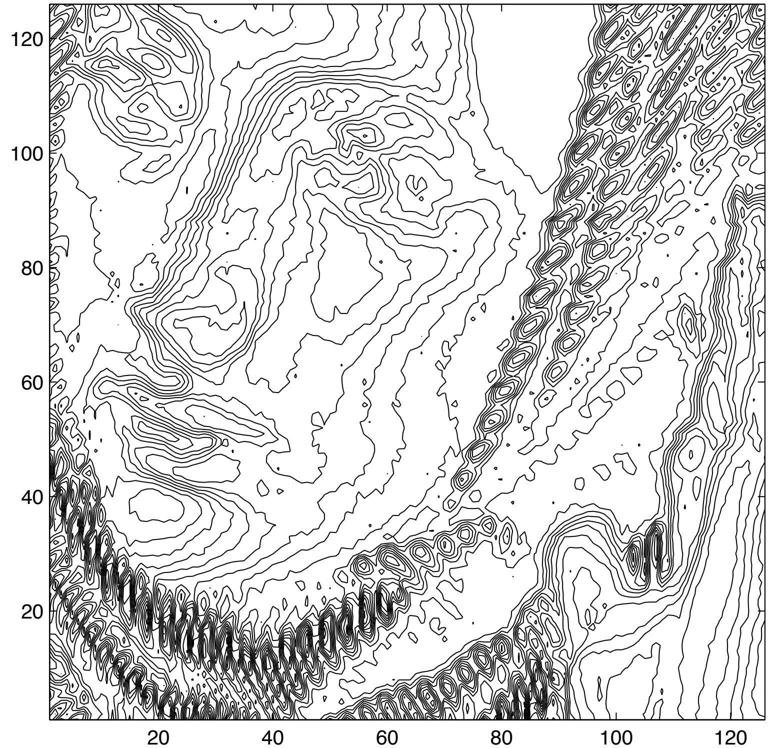
Image surface

An image is regarded as
a surface. Its level contours
are important.

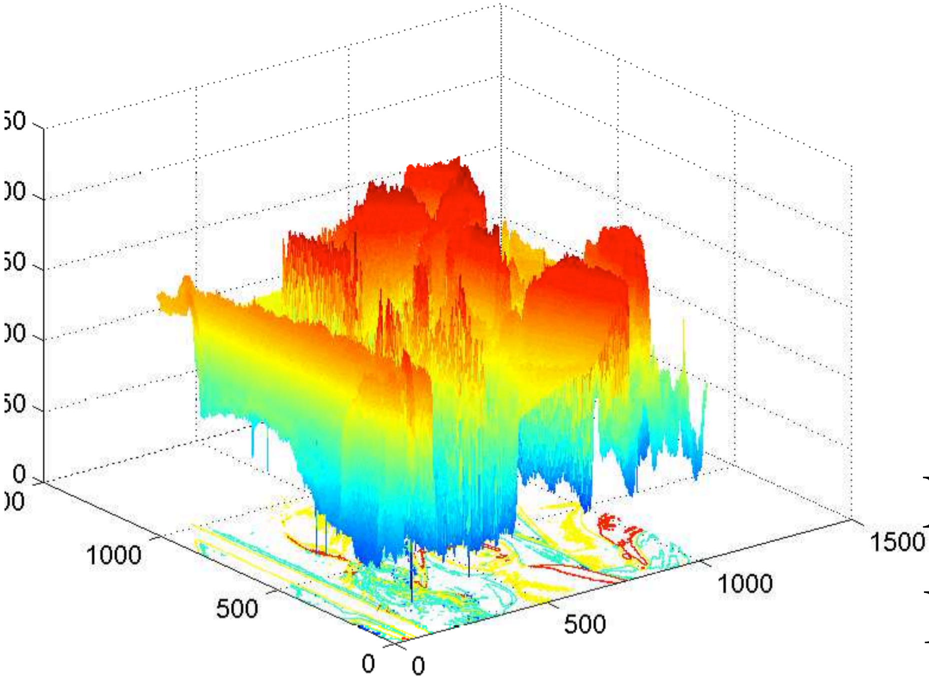


Images and level contours





Regularity of An Image



Ref: ROF model (92)

Level Curves:

$$\Gamma_c : \; u(x) = c, \quad c \in (0, \infty).$$

Total length of all contours:

$$R(u) = \int_0^\infty |\Gamma_c| \, dc \doteq \int_\Omega |\nabla u| \, dx.$$

Regularity: Elastica energy

Contour Curves:

$\Gamma_c: \ u(x) = c, \quad c \in (0, \infty).$

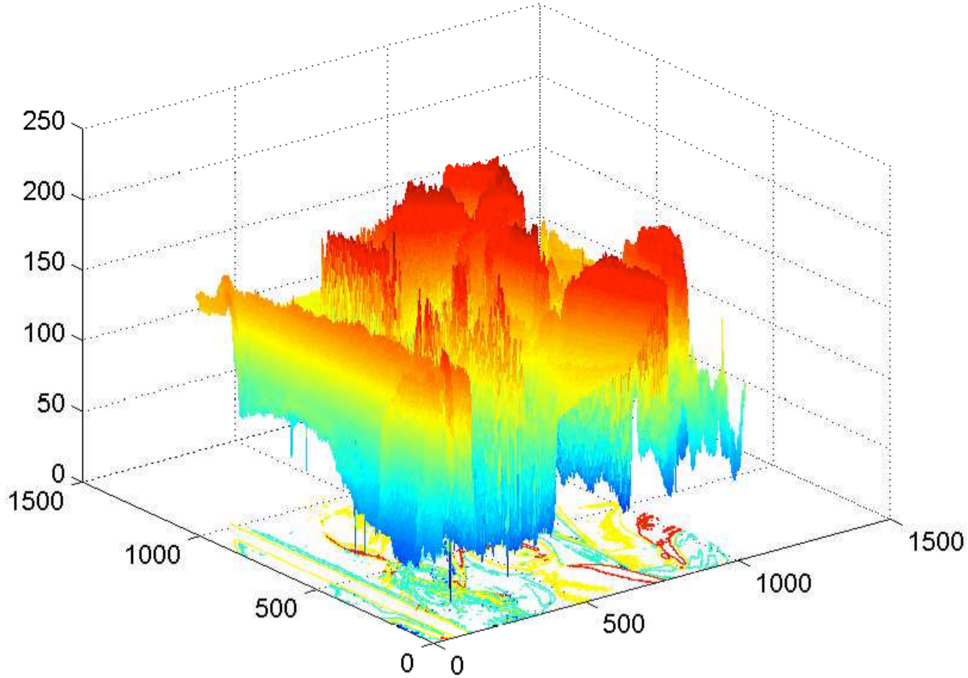
Euler's elastica:

$E(\Gamma) = \int_\Gamma (a + b\kappa^2) ds, a, b \geq 0.$

κ is the curvature of the curve.

On a level curve:

$\kappa = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right).$



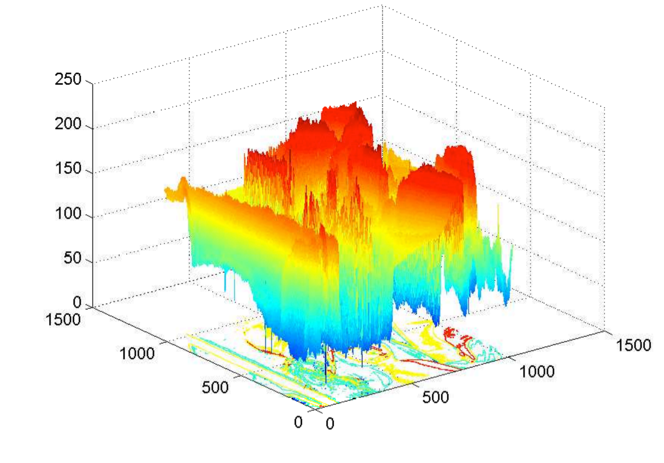
Total Elastica of all contours:

$$R(u) = \int_0^\infty E(\Gamma_c) dc \doteq \int_\Omega \left(a + b \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right)^2 \right) |\nabla u| dx.$$

Regularity: Mean Curvature

The image surface:

$u(x) - z = 0$.

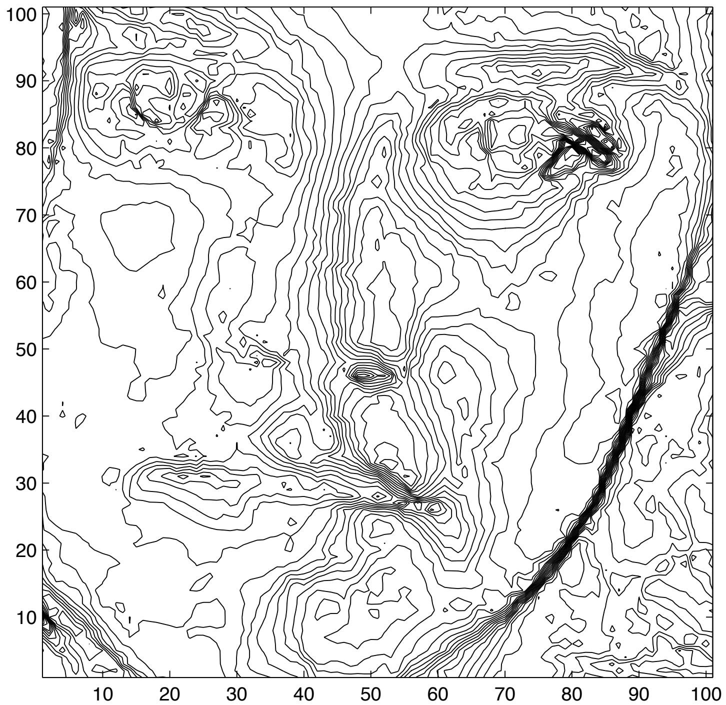


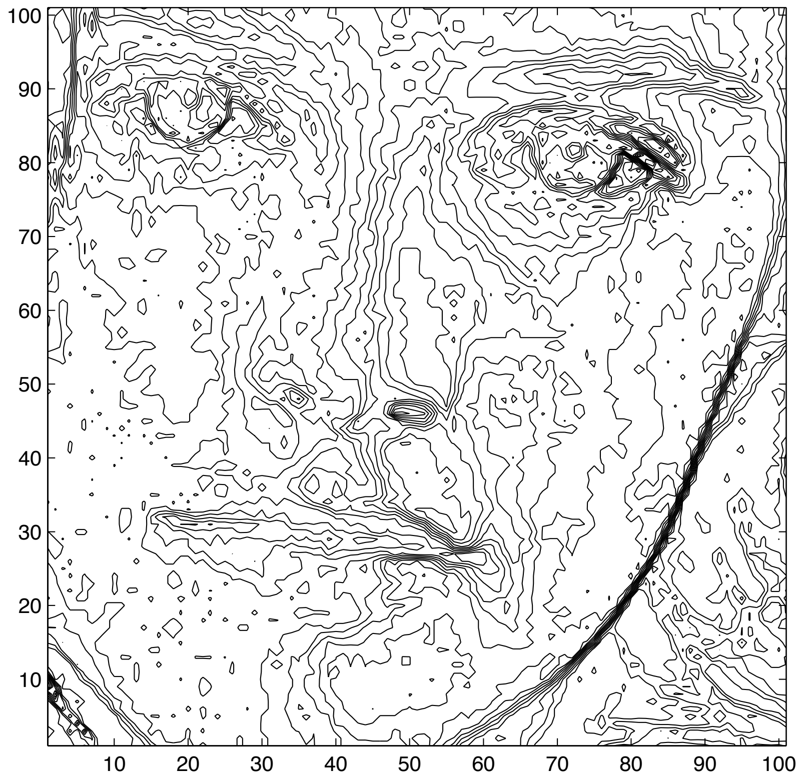
Its Gauss mean curvature is:

$$\frac{1}{2}\nabla_{(x,z)}\cdot\left(\frac{\nabla_{(x,z)}u}{\left|\nabla_{(x,z)}u\right|}\right)=\frac{1}{2}\nabla_{(x,z)}\cdot\left(\frac{(\nabla_x u,-1)}{\left|(\nabla_x u,-1)\right|}\right)=\frac{1}{2}\nabla_x\cdot\left(\frac{\nabla_x u}{\sqrt{1+\left|\nabla_x u\right|^2}}\right)=H_u$$

c.f. Kimmel-Malladi-Sochen 97, Zhu-Chan 12, Schoenemann-Masnou-Cremers 12.

Images and edges





Elastica energy has a long history

Elastica Model:

$$\min_{v} \int_{\Omega} \left(a + b\left(\nabla \cdot \frac{\nabla v}{|\nabla v|}\right)^2 \right) |\nabla v| + \frac{\lambda}{2}(v - f)^2.$$

The story about elastica energy for elasto-plastic numerical partial differential equations is even longer and earlier.

References for imaging:

- Nitzberg, Mumford and Shiota 1993
- Masnou and Morel 1998
- Ballester, Bertalmio, Caselles, Sapiro and Verdera 2001
- Chan-Kang-Shen, 2002
- Ambrosio and Masnou 2004

Literature- Algorithms for Euler's Elastica

- Discrete curvature:
  - ❖ Schoenemann-Kahl-Cremers (2009), El-Zehiry-Grady (2010,,2016), Boykov-et-al, Veksler-et-al …

- Continuous curvature:
  - ❖ Masnou-Morel (1998), Nitzberg-Mumford-Shiota (93), Ballester-Bertalmio-Caselles-Sapiro-Verdera (2001), Chan-Kang-Shen(2002), Bruckstein-et-al (01), Bae-Shi-T. (2011), T.-Hahn-Chung(2011), Bredies-Pock-Wirth(2014), Papafitsoros-Schonlieb(2014), Kimmel-Malladi-Sochen (97) .

- Recent algorithms (continuous curvature):
  - ❖ Sun-Chen BIT 14, Myllykoski-Glowinski-Karkkainen-Ross (SIIMS 15), Zhang-Chen-Deng-Wang (NMTMA17), Chen-Mirebeau-Cohen (IJCV 16), Yashtini-Kang (SIIMS 16), Duan-et-al.
  - ❖ Bredies-Pock-With (SIAM J. Math Anal 13), Chambolle-Pock (2018).

Reformulation I

- The minimization of Euler's Elastica

$$E(u) = \int_\Omega \left[a + b \left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right] |\nabla u| + \frac{\lambda}{2} \int_\Omega (f-u)^2$$

- Introducing new variables for the gradient and the unit normal vector

$$p = \nabla u, \quad n = \frac{p}{|p|},$$

- The problem can be casted as a constrained minimization problem with new variables

$$\min_{u,p,n} \int_\Omega \left[a + b(\nabla \cdot n)^2 \right] |p| + \frac{1}{2} \int_\Omega (f-u)^2$$

$$\text{subject to } p = \nabla u, \ |p| n = p.$$

Ref: (Duan- Wang-T.-Hahn, SSVM2012)

Reformulation II

- In (Duvaut and J. Lions1976, Dean-Glowinski-Guidoboni2007, Bertalmio-et-al 2000,)

$$\text{If } n \neq 0, p \neq 0, \textit{ and } |n| \leq 1, \textit{ then}$$

$$|p| = n \cdot p \longleftrightarrow n = p / |p|$$

- Equivalent formulation:

$$\min_{u,p,n} \int_\Omega \left[a + b \left(\nabla \cdot n \right)^2 \right] |p| + \frac{1}{2} \int_\Omega (f - u)^2$$

$$\text{subject to } p = \nabla u, |p| = p \cdot n, |n| \leq 1$$

- The minimization variables are: u, p, n. When two of them are fixed and we just need to minimize with one of them, each problem is convex

Approach of Chambolle-Pock

Total Roto-Translational Variation, In arXiv:1709.09953, (2017) ▪

For E $\subset \Omega \subset R^2$, we have for $f(\kappa) = 1 + |\kappa|^p$ that

$$\int_{\partial E} (1 + |\kappa_{\partial E}|^p) d\mathcal{H}^1(x)$$

$$= \int_{\Omega \times \mathcal{S}^1} f(\tau^\theta / |\tau^x|) |\tau^x| d\mathcal{H}^1 \llcorner \Gamma_E = F(\chi_E).$$

where F is a convex functional. Moreover

$$\int_\Omega f(\kappa(x)) |\nabla u(x)| dx = \int_R \int_{\partial\{u \geq s\}} f(\kappa(x)) d\mathcal{H}^1(x) = F(u).$$

Related References: Bredies-Pock-Wirth (2013,2015),

Our New Reformulation

- Euler's Elastica problem reads as:

$$\min_{v\in\mathcal{V}}\left[\int_\Omega\left(a+b\left|\nabla\cdot\frac{\nabla v}{|\nabla v|}\right|^2\right)|\nabla v|dx+\frac{1}{2}\int_\Omega|f-v|^2dx\right] \quad (3)$$

$\mathcal{V}$: a functional space of the Sobolev's type, typically.

- A popular way to overcome the singularity: replace $|\nabla v|$ by $\sqrt{\epsilon^2+|\nabla v|^2}$, ϵ being a small parameter.
- A more sophisticated way is to replace $\frac{\nabla v}{|\nabla v|}$ by a vector-valued function $\boldsymbol{\mu}$ (borrow from *viscoplasticity*, used in this work):

$$\boldsymbol{\mu}\cdot\nabla v=|\nabla v|,\quad |\boldsymbol{\mu}|\leq 1, \quad (4)$$

with $|\boldsymbol{\mu}|=\sqrt{\mu_1^2+\mu_2^2}$, $\forall\boldsymbol{\mu}=(\mu_1,\mu_2)$.

Our New Reformulation

- The elastica problem by (4) becomes $(v, \boldsymbol{\mu})$-problem:

$$\min_{(v,\boldsymbol{\mu})\in\mathcal{W}} \left[\int_\Omega \left(a + b\,|\nabla \cdot \boldsymbol{\mu}|^2 \right) |\nabla v| dx + \frac{1}{2} \int_\Omega |f - v|^2 dx \right] \qquad (5)$$

with

$$\mathcal{W} = \{(v, \boldsymbol{\mu}) \in \mathcal{H}^1(\Omega) \times \mathcal{H}(\Omega, \mathrm{div}), \quad \boldsymbol{\mu} \cdot \nabla v = |\nabla v|, \quad |\boldsymbol{\mu}| \leq 1\},$$

and

$$\mathcal{H}(\Omega, \mathrm{div}) = \{\boldsymbol{\mu} \in (\mathcal{L}^2(\Omega))^2, \nabla \cdot \boldsymbol{\mu} \in \mathcal{L}^2(\Omega)\}.$$

- **Proposition 1:** Suppose that $(u, \boldsymbol{\lambda})$ is solution of problem (5), we have then

$$\int_\Omega u dx = \int_\Omega f dx. \qquad (6)$$

Our New Reformulation

- Let us define the sets Σ_f and S by

$$\Sigma_f = \{\mathbf{q} \in (\mathcal{L}^2(\Omega))^2, \exists\, v \in \mathcal{H}^1(\Omega), \text{s.t. } \mathbf{q} = \nabla v \text{ and } \int_\Omega (v - f)dx = 0\}$$

and

$$S = \{(\mathbf{q}, \boldsymbol{\mu}) \in (\mathcal{L}^2(\Omega))^2 \times (\mathcal{L}^2(\Omega))^2,\ \mathbf{q} \cdot \boldsymbol{\mu} = |\mathbf{q}|,\ |\boldsymbol{\mu}| \leq 1\}.$$

Reformulation III

- Problem (5) is equivalent to the following one:

$$\min_{(\mathbf{q},\boldsymbol{\mu})}\left[\int_\Omega \left(a+b|\nabla\cdot\boldsymbol{\mu}|^2\right)|\mathbf{q}|dx+\frac{1}{2}\int_\Omega |v_\mathbf{q}-f|^2dx+I_{\Sigma_f}(\mathbf{q})+I_S(\mathbf{q},\boldsymbol{\mu})\right] \tag{7}$$

where $(\mathbf{q},\boldsymbol{\mu})\in(\mathcal{L}^2(\Omega))^2\times(\mathcal{L}^2(\Omega))^2$, and I_{Σ_f} and I_S are indicator functionals:

$$I_{\Sigma_f}(\mathbf{q})=\begin{cases}0, & \text{if } \mathbf{q}\in\Sigma_f,\\ +\infty, & \text{if } \mathbf{q}\in(\mathcal{L}^2(\Omega))^2\backslash\Sigma_f.\end{cases}\quad I_S(\mathbf{q},\boldsymbol{\mu})=\begin{cases}0, & \text{if } (\mathbf{q},\boldsymbol{\mu})\in S,\\ +\infty, & \text{if } (\mathbf{q},\boldsymbol{\mu})\in(\mathcal{L}^2(\Omega))^2\times(\mathcal{L}^2(\Omega))^2\backslash S.\end{cases} \tag{8}$$

- $v_\mathbf{q}$ is the unique solution of the following problem

$$\begin{cases}\nabla^2 v_\mathbf{q}=\nabla\cdot\mathbf{q}, & \text{in } \Omega,\\ \int_\Omega v_\mathbf{q}dx=\int_\Omega f dx.\end{cases} \tag{9}$$

Part II: New Operator-Splitting Method For Model (7)

- Denote by J_1 and J_2 the functionals defined by

$$\begin{cases} J_1(\mathbf{q},\boldsymbol{\mu}) = \displaystyle\int_\Omega \left(a + b|\nabla\cdot\boldsymbol{\mu}|^2\right)|\mathbf{q}|dx, \\ J_2(\mathbf{q}) = \dfrac{1}{2}\displaystyle\int_\Omega |v_{\mathbf{q}} - f|^2 dx, \end{cases} \tag{10}$$

- Model (7) becomes:

$$\min_{(\mathbf{q},\boldsymbol{\mu})} \left[J_1(\mathbf{q},\boldsymbol{\mu}) + J_2(\mathbf{q}) + I_{\Sigma_f}(\mathbf{q}) + I_S(\mathbf{q},\boldsymbol{\mu})\right] \tag{11}$$

where $(\mathbf{q},\boldsymbol{\mu}) \in (\mathcal{L}^2(\Omega))^2 \times (\mathcal{L}^2(\Omega))^2$.
Suppose that $(\mathbf{p},\boldsymbol{\lambda})$ is a minimizer of the functional in (11).

- We have then $v = v_{\mathbf{p}}$ and the following system of (necessary) optimality conditions holds:

$$\begin{cases} \partial_{\mathbf{q}} J_1(\mathbf{p}, \boldsymbol{\lambda}) + DJ_2(\mathbf{p}) + \partial I_{\Sigma_f}(\mathbf{p}) + \partial_{\mathbf{q}} I_S(\mathbf{p}, \boldsymbol{\lambda}) \ni \mathbf{0}, \\ D_{\boldsymbol{\mu}} J_1(\mathbf{p}, \boldsymbol{\lambda}) + \partial_{\boldsymbol{\mu}} I_S(\mathbf{p}, \boldsymbol{\lambda}) \ni \mathbf{0}, \end{cases} \tag{12}$$

where the D_s (resp., the ∂s) denotes classical differentials (resp., generalized differentials, e.g., subdifferentials).

- Compute the solutions of (12) via computing the steady state solutions of the following initial value problem (dynamical flow):

$$\begin{cases} \dfrac{\partial \mathbf{p}}{\partial t} + \partial_{\mathbf{q}} J_1(\mathbf{p}, \boldsymbol{\lambda}) + DJ_2(\mathbf{p}) + \partial I_{\Sigma_f}(\mathbf{p}) + \partial_{\mathbf{q}} I_S(\mathbf{p}, \boldsymbol{\lambda}) \ni \mathbf{0} \text{ on } \Omega \times (0, +\infty), \\ \gamma \dfrac{\partial \boldsymbol{\lambda}}{\partial t} + D_{\boldsymbol{\mu}} J_1(\mathbf{p}, \boldsymbol{\lambda}) + \partial_{\boldsymbol{\mu}} I_S(\mathbf{p}, \boldsymbol{\lambda}) \ni \mathbf{0} \text{ on } \Omega \times (0, +\infty), \\ (\mathbf{p}(0), \boldsymbol{\lambda}(0)) = (\mathbf{p}_0, \boldsymbol{\lambda}_0), \end{cases} \tag{13}$$

with $\gamma > 0$ (the choice of γ will be discussed latter).

The Final Three Subproblems Need to Solve

$$\begin{cases} \dfrac{\mathbf{p}^{n+1/3} - \mathbf{p}^{n}}{\tau} + \partial_{\mathbf{q}} J_1(\mathbf{p}^{n+1/3}, \boldsymbol{\lambda}^{n}) \ni \mathbf{0} \\ \gamma \dfrac{\boldsymbol{\lambda}^{n+1/3} - \boldsymbol{\lambda}^{n}}{\tau} + D_{\boldsymbol{\mu}} J_1(\boldsymbol{\lambda}^{n+1/3}) = \mathbf{0} \end{cases} \quad \text{in } \Omega \Rightarrow (\mathbf{p}^{n+1/3}, \boldsymbol{\lambda}^{n+1/3}) \tag{22}$$

$$\begin{cases} \dfrac{\mathbf{p}^{n+2/3} - \mathbf{p}^{n+1/3}}{\tau} + \partial_{\mathbf{q}} I_S(\mathbf{p}^{n+2/3}, \boldsymbol{\lambda}^{n+2/3}) \ni \mathbf{0} \\ \gamma \dfrac{\boldsymbol{\lambda}^{n+2/3} - \boldsymbol{\lambda}^{n+1/3}}{\tau} + \partial_{\boldsymbol{\mu}} I_S(\mathbf{p}^{n+2/3}, \boldsymbol{\lambda}^{n+2/3}) \ni \mathbf{0} \end{cases} \quad \text{in } \Omega \Rightarrow (\mathbf{p}^{n+2/3}, \boldsymbol{\lambda}^{n+2/3}) \tag{23}$$

$$\begin{cases} \dfrac{\mathbf{p}^{n+1} - \mathbf{p}^{n+2/3}}{\tau} + DJ_2(\mathbf{p}^{n+1}) + \partial I_{\Sigma_f}(\mathbf{p}^{n+1}) \ni \mathbf{0} \\ \gamma \dfrac{\boldsymbol{\lambda}^{n+1} - \boldsymbol{\lambda}^{n+2/3}}{\tau} = \mathbf{0} \end{cases} \quad \text{in } \Omega \Rightarrow (\mathbf{p}^{n+1}, \boldsymbol{\lambda}^{n+1}) \tag{24}$$

In the following, we will discuss the solution of the above subproblems when applying scheme (22)-(24) to the solution of problem (7).

Computing $\mathbf{p}^{n+1/3}$ from (22)

- The multi-valued equation verified by $\mathbf{p}^{n+1/3}$ in the first equation of (22) is nothing but the (formal) Euler-Lagrange equation of the following minimization problem

$$\mathbf{p}^{n+1/3} = \arg \min_{\mathbf{q}\in(\mathcal{L}^2(\Omega))^2} \left[\frac{1}{2}\int_\Omega |\mathbf{q}-\mathbf{p}^n|^2 dx + \tau \int_\Omega \left(a + b|\nabla\cdot\boldsymbol{\lambda}^n|^2\right)|\mathbf{q}|dx\right] \tag{25}$$

- Problem (25) is very common and has a closed form solution by:

$$\mathbf{p}^{n+1/3} = \mathbf{max}\left\{0, 1-\frac{c}{|\mathbf{p}^n|}\right\}\mathbf{p}^n, \tag{26}$$

where $c = \tau a + \tau b|\nabla\cdot\boldsymbol{\lambda}^n|^2$.

Computing $\lambda^{n+1/3}$ from (22)

- The equation verified by $\boldsymbol{\lambda}^{n+1/3}$ in the second equation of (22) is the (formal) Euler-Lagrange equation of the following minimization problem

$$\boldsymbol{\lambda}^{n+1/3} = \arg \min_{\boldsymbol{\mu} \in (\mathcal{L}^2(\Omega))^2} \left[\gamma \int_\Omega \frac{|\boldsymbol{\mu} - \boldsymbol{\lambda}^n|^2}{2\tau} dx + \int_\Omega \left(a + b|\nabla \cdot \boldsymbol{\mu}|^2 \right) |\mathbf{p}^{n+1/3}| dx \right] \tag{27}$$

where $\boldsymbol{\lambda}^n$ and $\mathbf{p}^{n+1/3}$ are known.

- From the Euler-Lagrangian equation of (27), we get that the solution $\boldsymbol{\lambda}^{n+1/3}$ is the solution of following linear equation:

$$\gamma \frac{\boldsymbol{\lambda}^{n+1/3} - \boldsymbol{\lambda}^n}{\tau} - \nabla(2b|\mathbf{p}^{n+1/3}|\nabla \cdot \boldsymbol{\lambda}^{n+1/3}) = 0 \text{ in } \Omega. \tag{28}$$

Use periodic boundary condition for the above equation, (28) can be efficiently and easily solved by the FFT.

Computing $\left(\mathbf{p}^{n+2/3}, \boldsymbol{\lambda}^{n+2/3}\right)$ from (23)

- One can view system (23) as the Euler-Lagrange equation of the following minimization problem:

$$\min_{(\mathbf{q},\boldsymbol{\mu})\in S} \left[\int_\Omega |\mathbf{q} - \mathbf{p}^{n+1/3}|^2 dx + \gamma \int_\Omega |\boldsymbol{\mu} - \boldsymbol{\lambda}^{n+1/3}|^2 dx\right]. \tag{29}$$

- Problem (29) can be solved point-wise, reducing, a.e. on Ω, to the following finite dimensional constrained minimization problem:

$$(\mathbf{p}^{n+2/3}(x), \boldsymbol{\lambda}^{n+2/3}(x)) = \operatorname{argmin}_{(\mathbf{q},\boldsymbol{\mu})\in\sigma} j_{n+1/3}(\mathbf{q}, \boldsymbol{\mu}; x) \tag{30}$$

where $\sigma = \{(\mathbf{q}, \boldsymbol{\mu}) \in \mathbf{R}^2 \times \mathbf{R}^2,\ \mathbf{q}\cdot\boldsymbol{\mu} = |\mathbf{q}|,\ |\boldsymbol{\mu}| \leq 1\}$.

Computing $\left(\mathbf{p}^{n+2/3}, \boldsymbol{\lambda}^{n+2/3}\right)$ from (23)

- Note that $\forall(\mathbf{q}, \boldsymbol{\mu}) \in \mathbf{R}^2 \times \mathbf{R}^2$, it has

$$j_{n+1/3}(\mathbf{q}, \boldsymbol{\mu}; x) = \left|\mathbf{q} - \mathbf{p}^{n+1/3}(x)\right|^2 + \gamma \left|\boldsymbol{\mu} - \boldsymbol{\lambda}^{n+1/3}(x)\right|^2$$

- Due to $\sigma = \{(\mathbf{q}, \boldsymbol{\mu}) \in \mathbf{R}^2 \times \mathbf{R}^2,\ \mathbf{q} \cdot \boldsymbol{\mu} = |\mathbf{q}|,\ |\boldsymbol{\mu}| \leq 1\}$, we may decompose it to σ_0 and σ_1 by

$$\sigma_0 = \{(\mathbf{q}, \boldsymbol{\mu}) \in \mathbf{R}^2 \times \mathbf{R}^2, \mathbf{q} = \mathbf{0}, |\boldsymbol{\mu}| \leq 1\},$$

$$\sigma_1 = \{(\mathbf{q}, \boldsymbol{\mu}) \in \mathbf{R}^2 \times \mathbf{R}^2, \mathbf{q} \neq \mathbf{0}, \mathbf{q} \cdot \boldsymbol{\mu} = |\mathbf{q}|,\ |\boldsymbol{\mu}| = 1\}.$$

Clearly, it has $\sigma = \sigma_0 \cup \sigma_1$.

Minimizing the functional in (30) over σ_0

- **Over** σ_0, the minimization problem (31) reduces to

$$\min_{\boldsymbol{\mu}\in\mathbf{R}^2, |\boldsymbol{\mu}|\leq 1} \left|\boldsymbol{\mu} - \boldsymbol{\lambda}^{n+1/3}(x)\right|. \tag{34}$$

- Clearly, the solution of problem (34) is given by

$$\boldsymbol{\lambda}_0^{n+1/3}(x) = \frac{\boldsymbol{\lambda}^{n+1/3}(x)}{\max(1, |\boldsymbol{\lambda}^{n+1/3}(x)|)}. \tag{35}$$

- Concerning $\mathbf{p}_0^{n+1/3}(x)$ by the definition of σ_0, we have

$$\mathbf{p}_0^{n+1/3}(x) = \mathbf{0}. \tag{36}$$

Minimizing the functional in (30) over σ_1

- **Over** σ_1, the minimization problem (32) reduces to

$$\inf_{(\mathbf{q},\boldsymbol{\mu})\in\mathbf{R}^2\times\mathbf{R}^2,\mathbf{q}\neq\mathbf{0},\mathbf{q}\cdot\boldsymbol{\mu}=|\mathbf{q}|,|\boldsymbol{\mu}|=1}\left[|\mathbf{q}-\mathbf{p}^{n+1/3}(x)|^2+\gamma|\boldsymbol{\mu}-\boldsymbol{\lambda}^{n+1/3}(x)|^2\right]. \tag{37}$$

- Denote $\mathbf{x}=\mathbf{p}^{n+1/3}(x)$ and $\mathbf{y}=\boldsymbol{\lambda}^{n+1/3}(x)$. Due to $|\boldsymbol{\mu}|=1$, problem (37) has

$$\inf_{(\mathbf{q},\boldsymbol{\mu})\in\mathbf{R}^2\times\mathbf{R}^2,\mathbf{q}\neq\mathbf{0},\mathbf{q}\cdot\boldsymbol{\mu}=|\mathbf{q}|,|\boldsymbol{\mu}|=1}\left[\frac{1}{2}|\mathbf{q}|^2-\mathbf{q}\cdot\mathbf{x}-\gamma\boldsymbol{\mu}\cdot\mathbf{y}\right] \tag{38}$$

- Set $|\mathbf{q}|=\theta$; since $\boldsymbol{\mu}=\mathbf{q}/|\mathbf{q}|$, thus it implies that

$$\mathbf{q}=\theta\boldsymbol{\mu},\quad \theta>0. \tag{39}$$

Minimizing the functional in (30) over σ_1

- Relation of $\mathbf{q} = \theta\boldsymbol{\mu}, \theta > 0$ allows us to substitute problem (38) by the following constrained minimization problem in $\mathbf{R}^3$

$$\inf_{(\theta,\boldsymbol{\mu})\in\mathbf{R}\times\mathbf{R}^2,\theta>0,|\boldsymbol{\mu}|=1} \left[\frac{1}{2}\theta^2 - \theta\boldsymbol{\mu}\cdot\mathbf{x} - \gamma\boldsymbol{\mu}\cdot\mathbf{y}\right]. \tag{40}$$

- To solve (40), we observe that the above problem is equivalent to

$$\inf_{\theta>0} \min_{\boldsymbol{\mu}\in\mathbf{R}^2,|\boldsymbol{\mu}|=1} \left[\frac{1}{2}\theta^2 - \theta\boldsymbol{\mu}\cdot\mathbf{x} - \gamma\boldsymbol{\mu}\cdot\mathbf{y}\right]. \tag{41}$$

- To minimize on a closed set of $\mathbf{R}^3$, the problem that we finally consider is the following variant of problem (41)

$$\min_{\theta\geq 0} \min_{\boldsymbol{\mu}\in\mathbf{R}^2,|\boldsymbol{\mu}|=1} \left[\frac{1}{2}\theta^2 - \theta\boldsymbol{\mu}\cdot\mathbf{x} - \gamma\boldsymbol{\mu}\cdot\mathbf{y}\right]. \tag{42}$$

Minimizing the functional in (30) over σ_1

- To solve: $\min_{\theta \geq 0} \min_{\boldsymbol{\mu} \in \mathbf{R}^2, |\boldsymbol{\mu}|=1} \left[\frac{1}{2}\theta^2 - \theta\boldsymbol{\mu} \cdot \mathbf{x} - \gamma\boldsymbol{\mu} \cdot \mathbf{y}\right]$, we may solve it by the following two steps:

- **i)** θ being fixed, the solution $\boldsymbol{\mu}^*(\theta)$ of problem (42) is given by

$$\boldsymbol{\mu}^*(\theta) = \frac{\theta\mathbf{x} + \gamma\mathbf{y}}{|\theta\mathbf{x} + \gamma\mathbf{y}|} \tag{43}$$

ii) When $\boldsymbol{\mu}^*(\theta)$ obtained, implying that problem (42) reduces to

$$\min_{\theta \geq 0} \left[\frac{1}{2}\theta^2 - |\theta\mathbf{x} + \gamma\mathbf{y}|\right] \text{(Solved by fixed point)} \tag{44}$$

Minimizing the functional in (30) over σ_1

- Once θ^* is computed, $\boldsymbol{\lambda}_1^{n+2/3}(x)$ and $\mathbf{p}_1^{n+2/3}(x)$ are obtained by:

$$\boldsymbol{\lambda}_1^{n+2/3}(x) = \frac{\theta^* \mathbf{p}^{n+1/3}(x) + \gamma \boldsymbol{\lambda}^{n+1/3}(x)}{|\theta^* \mathbf{p}^{n+1/3}(x) + \gamma \boldsymbol{\lambda}^{n+1/3}(x)|}, \tag{45}$$

and

$$\mathbf{p}_1^{n+2/3}(x) = \theta^* \boldsymbol{\lambda}_1^{n+2/3}(x), \tag{46}$$

- Take two sets of $\left(\mathbf{p}_1^{n+2/3}(x), \boldsymbol{\lambda}_1^{n+2/3}(x)\right)$ respectively computed on σ_0 and σ_1 to compute the energy of (30), then select the set with the smaller energy, see (33).

Selection of the parameter γ

- We intend to select the parameter γ so that the two terms in (33) are balanced. We note that

$$\boldsymbol{\lambda}(t) = \frac{\mathbf{p}(t)}{|\mathbf{p}(t)|}.$$

Thus

$$\frac{\partial \boldsymbol{\lambda}}{\partial t} = \lim_{\tau \to 0} \frac{1}{\tau}\left(\frac{\mathbf{p}(t+\tau)}{|\mathbf{p}(t+\tau)|} - \frac{\mathbf{p}(t)}{|\mathbf{p}(t)|} \right). \tag{47}$$

- Due to the following relation:

$$\left| \frac{\mathbf{p}}{|\mathbf{p}|} - \frac{\mathbf{q}}{|\mathbf{q}|} \right|^2 = \frac{|\mathbf{p}|^2}{|\mathbf{p}|^2} + \frac{|\mathbf{q}|^2}{|\mathbf{q}|^2} - \frac{2\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = 2\left(1 - \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\right) \le 2\left(1 - \frac{2\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}|^2 + |\mathbf{q}|^2}\right) = 2\frac{|\mathbf{p}-\mathbf{q}|^2}{|\mathbf{p}|^2 + |\mathbf{q}|^2}, \tag{48}$$

Selection of the parameter γ

- One has

$$\left|\frac{\mathbf{p}(t+\tau)}{|\mathbf{p}(t+\tau)|}-\frac{\mathbf{p}(t)}{|\mathbf{p}(t)|}\right| \leq \frac{\sqrt{2}|\mathbf{p}(t+\tau)-\mathbf{p}(t)|}{\sqrt{|\mathbf{p}(t+\tau)|^2+|\mathbf{p}(t)|^2}}.$$

- Let $\tau \to 0$, we get from (47) that

$$\left|\frac{\partial \boldsymbol{\lambda}}{\partial t}\right| \leq \frac{1}{|\mathbf{p}|}\left|\frac{\partial \mathbf{p}}{\partial t}\right|.$$

- For small τ, the minimizer of (29) verifies

$$\frac{|\mathbf{p}^{n+2/3}-\mathbf{p}^{n+1/3}|^2}{2\tau}+\gamma\frac{|\boldsymbol{\lambda}^{n+2/3}-\boldsymbol{\lambda}^{n+1/3}|^2}{2\tau} \approx \frac{\tau}{2}\left(\left|\frac{\partial \mathbf{p}}{\partial t}(t^{n+1/3})\right|^2+\gamma\left|\frac{\partial \boldsymbol{\lambda}}{\partial t}(t^{n+1/3})\right|^2\right). \quad (49)$$

Selection of the parameter γ

- According to the above estimate, to balance these two terms, we just need to choose

$$\gamma = |\mathbf{p}^{n+1/3}|^2.$$

- In order to avoid the case that $|\mathbf{p}^{n+1/3}| \approx 0$, we choose in practice

$$\gamma = \max(|\mathbf{p}^{n+1/3}|^2, \hat{\alpha}), \tag{50}$$

where $\hat{\alpha}$ is a given small number. In this work, we empirically choose $\hat{\alpha} = \sqrt{\tau}$.

Computing $\mathbf{p}^{n+1}$ and λ^{n+1} from (24)

- Clearly we have

$$\boldsymbol{\lambda}^{n+1} = \boldsymbol{\lambda}^{n+2/3}. \tag{51}$$

- For $\mathbf{p}^{n+1}$ subproblem, the multi-valued equation in (24) is the Euler-Lagrange equation of the following minimization problem

$$\mathbf{p}^{n+1} = \arg\min_{\mathbf{q}\in\Sigma_f} \left[\frac{1}{2}\int_\Omega \left|\mathbf{q} - \mathbf{p}^{n+2/3}\right|^2 dx + \frac{\tau}{2}\int_\Omega \left|v_{\mathbf{q}} - f\right|^2 dx \right], \tag{52}$$

the function $v_{\mathbf{q}}$ is defined by (9).

Computing $\mathbf{p}^{n+1}$ and λ^{n+1} from (24)

- Suppose that Ω is the rectangle $(0, L) \times (0, H)$. It was mentioned that we shall use periodic boundary condition for the subproblem. Define $\mathcal{H}_p^1(\Omega)$, a space of doubly periodic functions, by

$$\begin{aligned}\mathcal{H}_p^1(\Omega) = &\{v \in \mathcal{H}^1(\Omega); v(0, x_2) = v(L, x_2), \\ &\text{a.e. on } (0, H); v(x_1, 0) = v(x_1, H) \text{ a.e. on } (0, L)\}\end{aligned} \tag{53}$$

- From the definition of Σ_f, problem (52) is equivalent to

$$\mathbf{p}^{n+1} = \nabla u^{n+1}$$

with

$$u^{n+1} = \arg \min_{v \in \mathcal{H}_p^1(\Omega)} \left[\frac{1}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{\tau}{2} \int_\Omega |v - f|^2 \, dx - \int_\Omega \mathbf{p}^{n+2/3} \cdot \nabla v dx \right]$$

Computing $\mathbf{p}^{n+1}$ and λ^{n+1} from (24)

- Function u^{n+1} is the unique solution of the following well-posed linear variational problem in $\mathcal{H}^1_p(\Omega)$:

$$\begin{cases} u^{n+1} \in \mathcal{H}^1_p(\Omega), \\ \int_\Omega \nabla u^{n+1} \cdot \nabla v dx + \tau \int_\Omega u^{n+1} v dx = \int_\Omega \mathbf{p}^{n+2/3} \cdot \nabla v dx + \tau \int_\Omega f v dx, \end{cases} \tag{54}$$

where $\forall v \in \mathcal{H}^1(\Omega)$.

Computing $\mathbf{p}^{n+1}$ and λ^{n+1} from (24)

- The problem (54) still has a unique solution which is the weak solution of the following problem:

$$\begin{cases} -\nabla^2 u^{n+1} + \tau u^{n+1} = -\nabla \cdot \mathbf{p}^{n+2/3} + \tau f, \quad \text{in } \Omega \\ u^{n+1}(0, x_2) = u^{n+1}(0, x_2) \text{ a.e. on } (0, H); \\ u^{n+1}(x_1, 0) = u^{n+1}(x_1, H) \text{ a.e. on } (0, L), \\ \dfrac{\partial u^{n+1}}{\partial x_1}(0, x_2) = \dfrac{\partial u^{n+1}}{\partial x_1}(L, x_2) \text{ a.e. on } (0, H) \\ \dfrac{\partial u^{n+1}}{\partial x_2}(x_1, 0) = \dfrac{\partial u^{n+1}}{\partial x_2}(x_1, H) \text{ a.e. on } (0, L). \end{cases} \tag{55}$$

which is also solved by FFT method.

Summerized Algorithm

Algorithm 2: A schematic description of the algorithm for problem (3)

Input: The inputted image f, the parameters a, b, γ and τ.
Output: The computed image u^*.

Initialization: $n = 0$, $u^0 = f$, $\mathbf{p}^0 = \nabla f$, $\boldsymbol{\lambda}^0(x) = \begin{cases} \mathbf{p}^0(x)/|\mathbf{p}^0(x)|, & \text{if } \mathbf{p}^0(x) \neq 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$ $x \in \Omega$.

While: $\|u^{n+1} - u^n\|/\|u^{n+1}\| > tol$ and $n < \mathrm{M}_{iter}$

1. Solve system (22) to obtain $(\mathbf{p}^{n+1/3}, \boldsymbol{\lambda}^{n+1/3})$
2. Solve system (23) to obtain $(\mathbf{p}^{n+2/3}, \boldsymbol{\lambda}^{n+2/3})$
3. Solve system (24) to obtain $(u^{n+1}, \mathbf{p}^{n+1}, \boldsymbol{\lambda}^{n+1})$
4. Check convergence and go to the next iteration or stop.

End While.

If iterations stop, take $u^* = u^{n+1}$.

tol: the stopping criterion tolerance; M_{iter}: the maximum of iterations; $||\cdot||$ represents L_2 norm.

Advantages:

- The time-discretization step is, essentially, the only parameter one has to choose.

- The results produced by the new method are less sensitive to parameter choice.

- For the same stopping criterion tolerance, the new method needs less iterations than its counterpart - ALM method.

Moreover, the new method has a lower cost per iteration than ALM.

Numerical Results

- Numerical Discretization: Finite difference for discretization on Ω.
- Implemented in MATLAB(R2016a) on a laptop of 8Gb RAM and Intel(R) Core(TM) i7-7500 CPU: @2.70 GHz 2.90GHz.
- See relative error (ReErr) of the solution is smaller than the predefined tolerance *tol*, i.e.,

$$\text{ReErr} = \frac{\|u^{n+1} - u^n\|_2}{\|u^{n+1}\|_2} < tol, \tag{56}$$

where *tol* is a pre-defined positive value.

Image Smoothing

- $a = 0.1$, $b = 0$ (the ROF model), $tol = 1 \times 10^{-5}$.

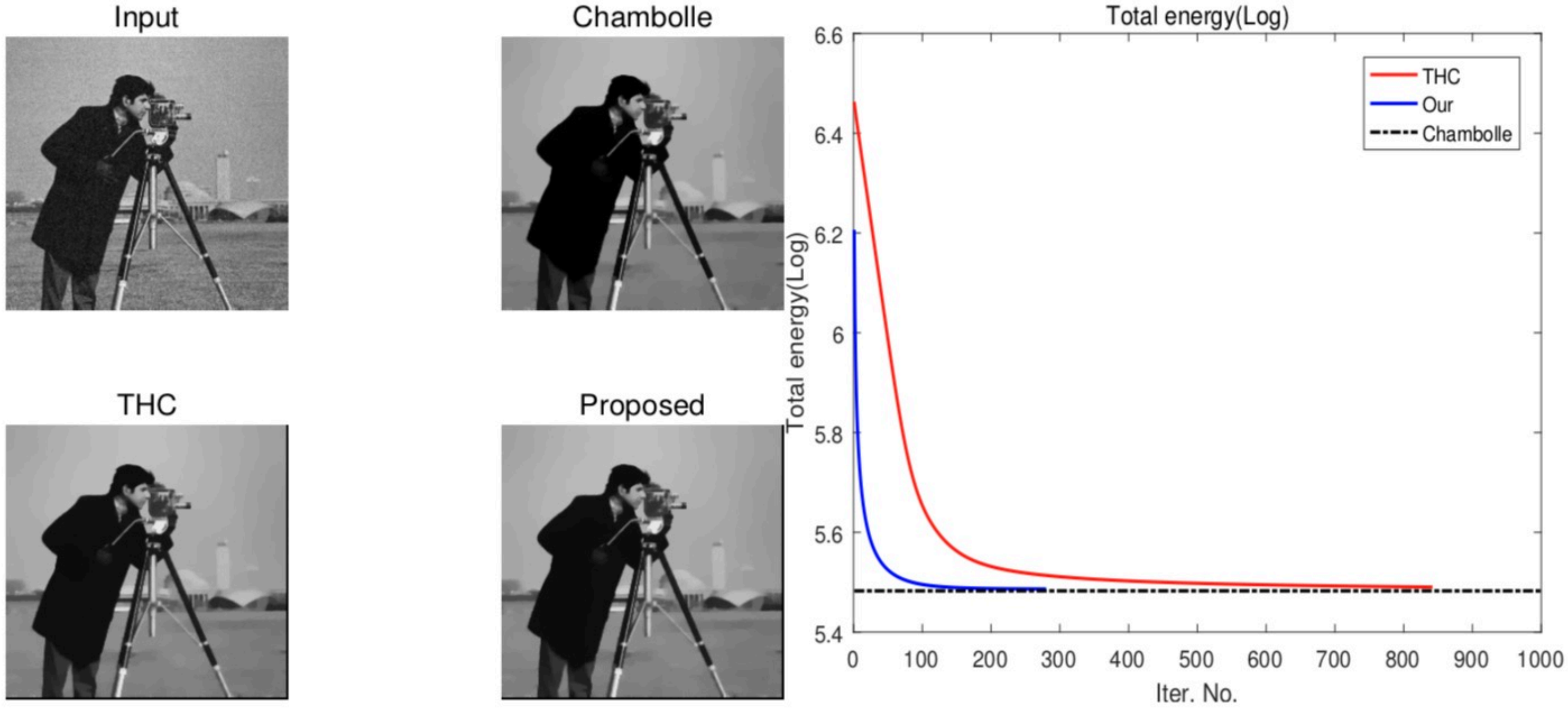
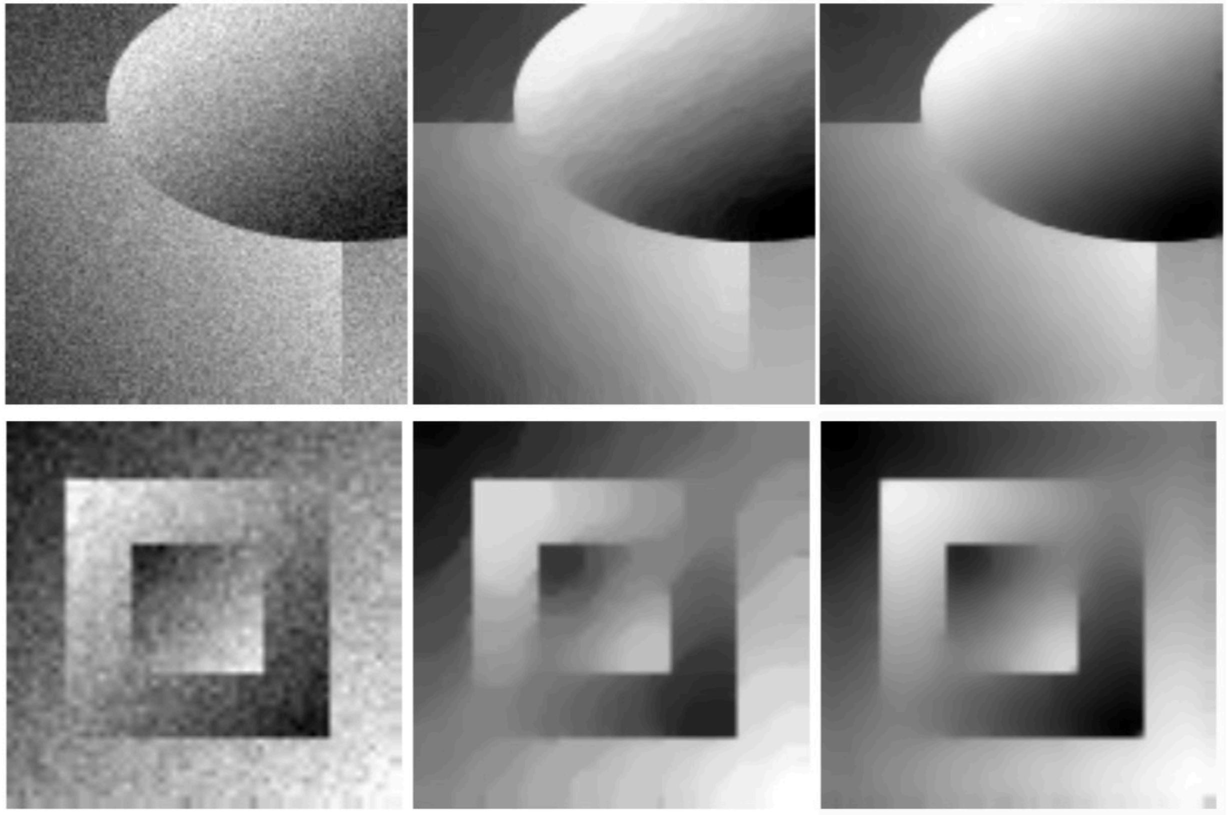
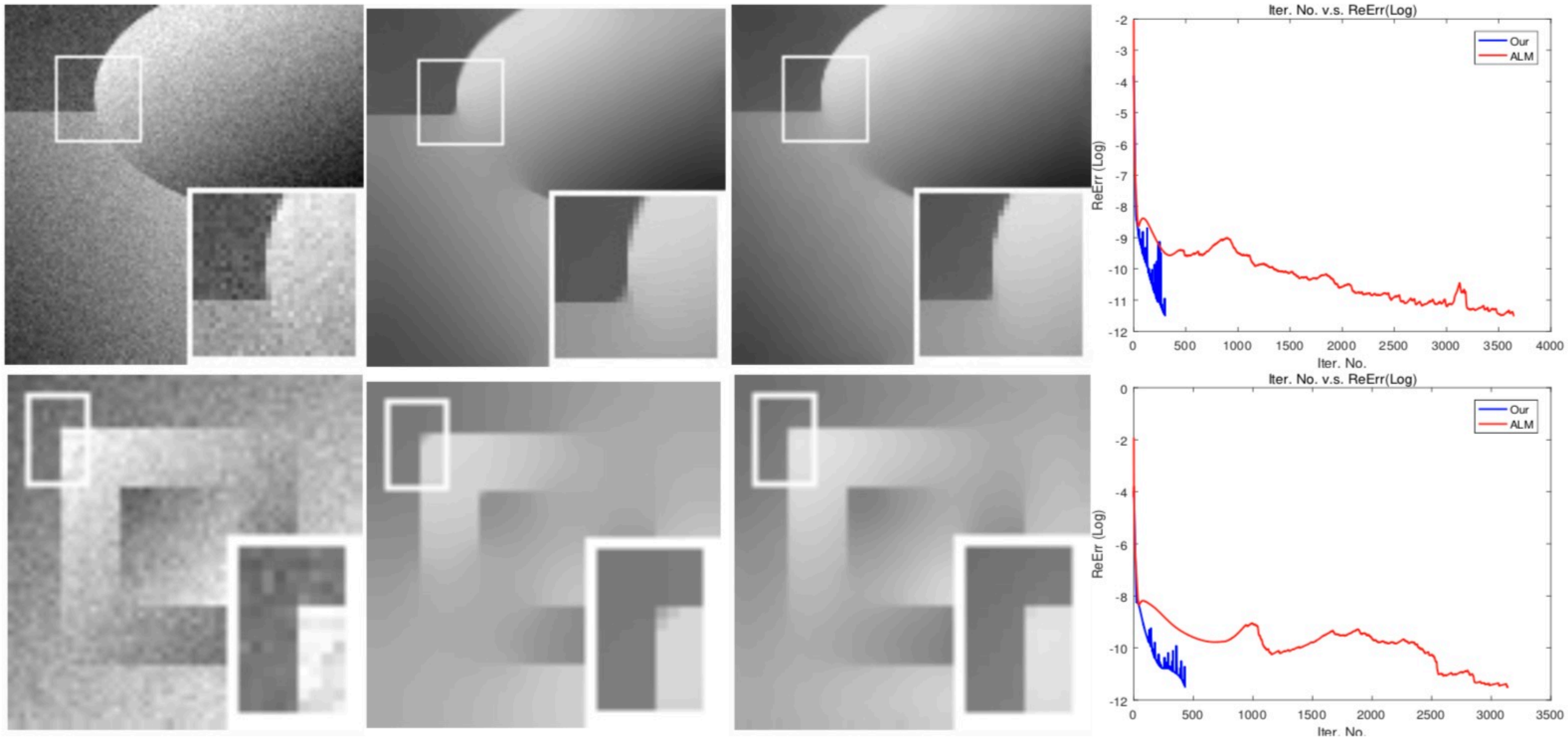


FIGURE: The results (the first two columns), and energy plots (the right column) by the three compared methods are shown. Note that the black dashed lines in the energy plots represent the final energy of the ROF model solved by the Chambolle's method. From this figure, we know that the results of three compared approaches for ROF model finally converge to the same energy value, which demonstrates the correctness of the proposed method.



Noisy Image TV Elastica

FIGURE: Image smoothing using the ROF and Euler elastica models. Left: Noisy images; Middle: ROF model treated by Chambolle's method; Right: the Euler elastica model treated by the proposed algorithm. The parameters of our method are all set as $a = b = 0.1$, $\tau = 0.1$ and $\gamma^n = \max\left(|\mathbf{p}^{n+1/3}|, \sqrt{\tau}\right)$.



Noisy THC Proposed ReErr

FIGURE: First row: the THC method and the proposed method both obtain good result with $r_1 = 0.01$, $r_2 = 10$, $r_3 = 100$ for the THC method and $\tau = 0.1$ for the proposed method. Second row: the THC method performs not so good with same parameters while the proposed method performs still good. Default parameters for both methods: $a = b = 0.1$ and $tol = 1 \times 10^{-5}$.

- If we increase the number of iteration (Iter) with the same $r_1 = 0.01$, $r_2 = 10$, $r_3 = 100$ the THC method could get the correct result (the left image is the result of Fig. 3 for the THC method).

Iter = 3314 Iter = 10000 Iter = 30000

- Or, if we tune the parameters slightly as $r_1 = 0.05$, $r_2 = 10$ and $r_3 = 100$, the THC may get good results.

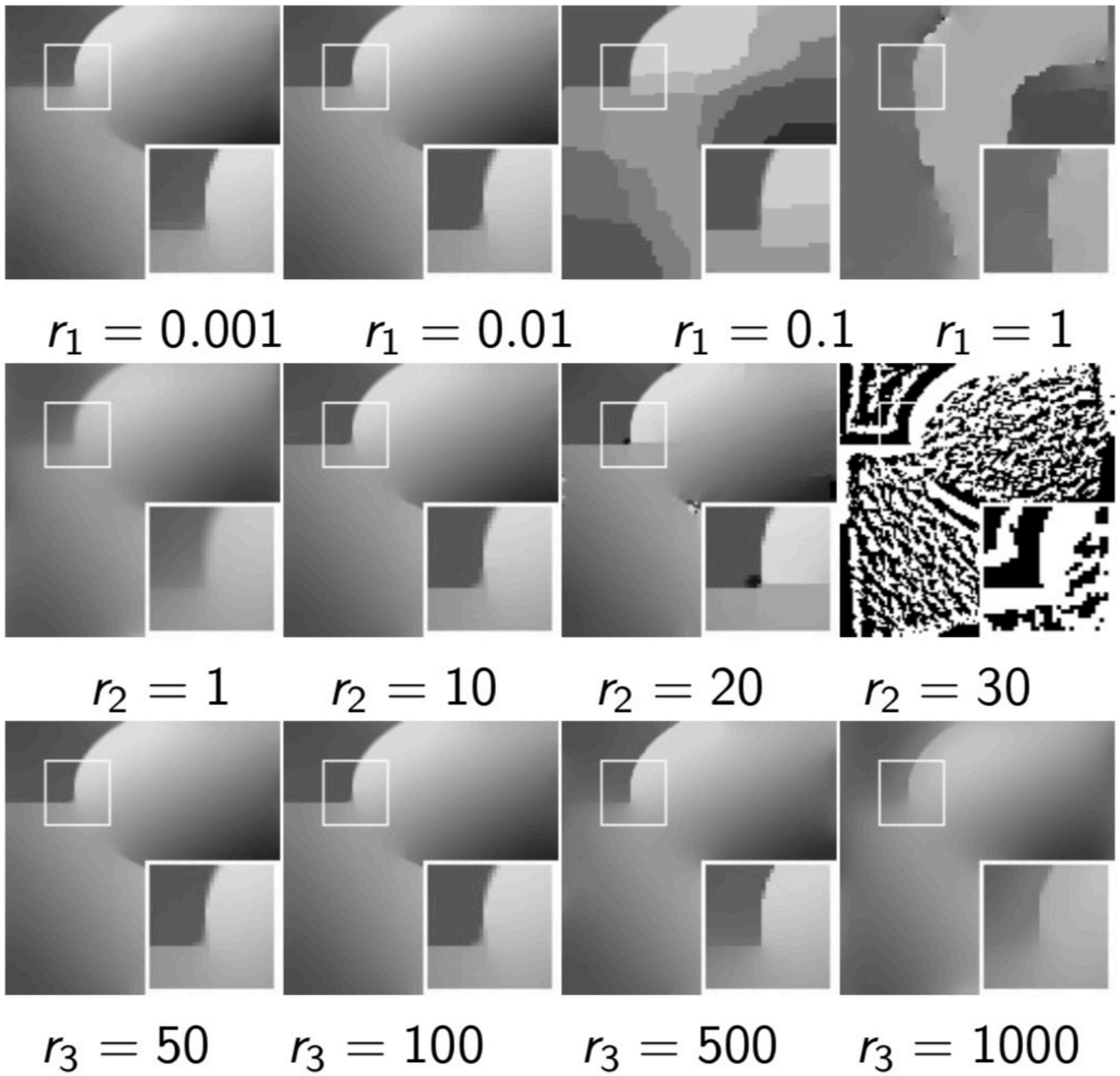
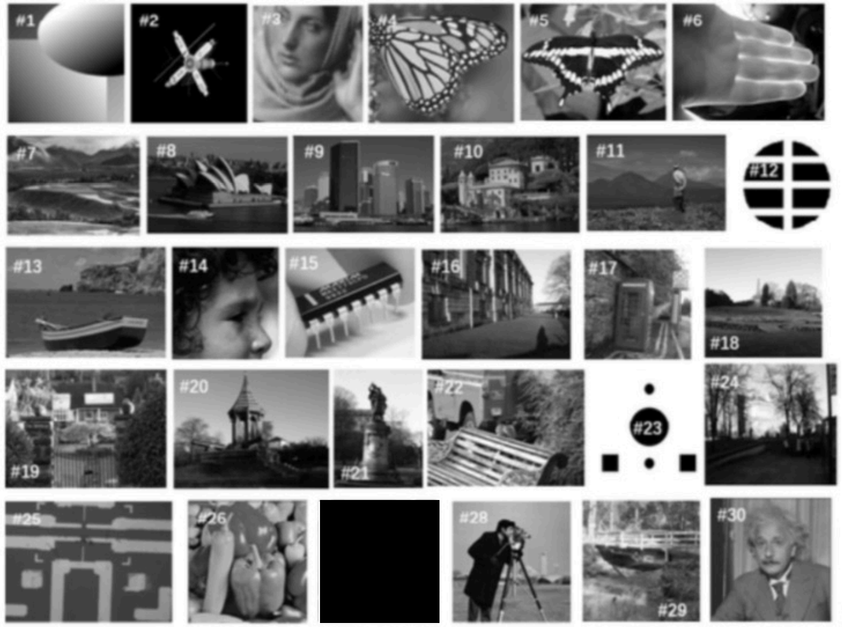


FIGURE: Sensibility of the THC method to parameters. Varying one parameter and keep others two unchanged (default: $r_1 = 0.01$, $r_2 = 10$ and $r_3 = 100$). With $a = b = 0.1$ and $tol = 1 \times 10^{-5}$.

TABLE: The iterations, total computational time and average computational time (per iteration) for Fig. 3 when reaching the tolerance (Unit: second).

Image	Method	Iterations	Time (s)	Average Time (s)/per iteration
ball (128 × 128)	**Proposed**	306	2.57	0.008
	THC	3648	37.19	0.010
square (60 × 60)	**Proposed**	434	1.16	0.002
	THC	3339	10.27	0.003
star(100 × 100)	**Proposed**	562	3.70	0.006
	THC	2234	17.58	0.007
Lena (256 × 256)	**Proposed**	462	15.21	0.033
	THC	808	31.18	0.039

- 

(30 test images)

TABLE: The average computational time (per image) of the proposed method and the THC method on 30 test images with default parameters (Unit: second).

-

tol	Method	$std = 0.1$	$std = 0.05$	$std = 0.02$
$tol = 1 \times 10^{-5}$	**Proposed**	7.2	7.2	6.1
	THC	74.9	70.3	70.4
$tol = 5 \times 10^{-5}$	**Proposed**	4.5	3.7	3.1
	THC	34.4	29.4	29.6
$tol = 1 \times 10^{-4}$	**Proposed**	1.9	1.9	1.6
	THC	8.1	8.2	8.0

Image Segmentation

- For a given image f, the Chan-Vese segmentation model with the Euler elastica energy of the interface is shown as follows:

$$\min_{\phi}\left[\int_{\Omega}\left(a+b\left|\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|}\right|^{2}\right)|\nabla H(\phi)|dx + \int_{\Omega}\eta\left(f-c_1\right)^2 H(\phi)+\left(f-c_2\right)^2\left(1-H(\phi)\right)dx\right] \quad (57)$$

ϕ: a level set function, $H(\cdot)$: Heaviside function, c_1, c_2: two scalars, and a, b, η: positive parameters.

- Introduce a new function $v = H(\phi)$, due to $\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|} = \nabla\cdot\frac{\nabla H(\phi)}{|\nabla H(\phi)|}$, then the model (57) can be rewritten as follows:

$$\min_{v\in[0,1]}\left[\int_{\Omega}\left(a+b\left|\nabla\cdot\frac{\nabla v}{|\nabla v|}\right|^{2}\right)|\nabla v|dx+\int_{\Omega}\eta\left(f-c_1\right)^2 v+\left(f-c_2\right)^2(1-v)dx\right], \quad (58)$$

Image Segmentation

- To solve (58) under the framework of the proposed method, we only need to **keep the same formulas of $\mathbf{p}^{n+1/3}$, $\boldsymbol{\lambda}^{n+1/3}$ and $\mathbf{p}^{n+2/3}$, $\boldsymbol{\lambda}^{n+2/3}$ problems**, and **rewrite the $\mathbf{p}^{n+1}$ problem** (52) as follows,

$$\min_{v\in[0,1]} \left[\int_\Omega \frac{|\nabla v - \mathbf{p}^{n+2/3}|^2}{2\tau} dx + \int_\Omega \eta \left(f - c_1\right)^2 v + \left(f - c_2\right)^2 (1 - v) dx\right], \tag{59}$$

where $\nabla v = \mathbf{q}$.

Image Segmentation

- Let u^{n+1} be the minimizer of (59) without the constraint $v \in [0, 1]$. It is easy to derive that the solution is a weak solution of:

$$-\nabla \cdot \left(\frac{\nabla u^{n+1} - \mathbf{p}^{n+2/3}}{\tau} \right) = g, \quad \text{in } \Omega, \tag{60}$$

where $g = -\eta (f - c_1)^2 + (f - c_2)^2$.

- The solution of (60) is not unique. However, we may modify (60) by adding a time stabilization term $\frac{u^{n+1} - u^n}{\tau}$ to get an approximate solution for (60):

$$-\nabla \cdot \left(\frac{\nabla u^{n+1} - \mathbf{p}^{n+2/3}}{\tau} \right) + \frac{u^{n+1} - u^n}{\tau} = g, \quad \text{in } \Omega \tag{61}$$

Solved as previous method of image smoothing!

- After computing u^{n+1}, it is easy to compute:

$$\mathbf{p}^{n+1} = \nabla u^{n+1}.$$

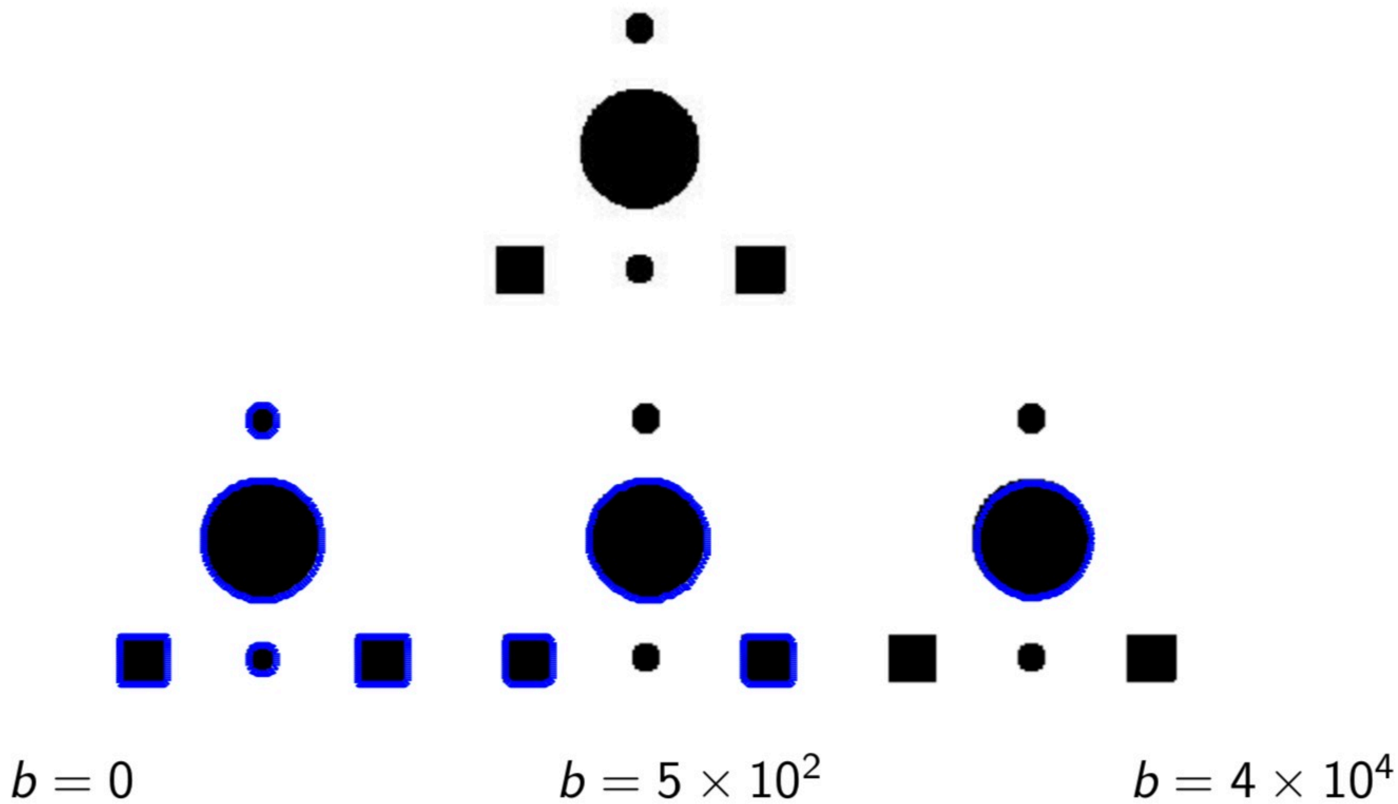
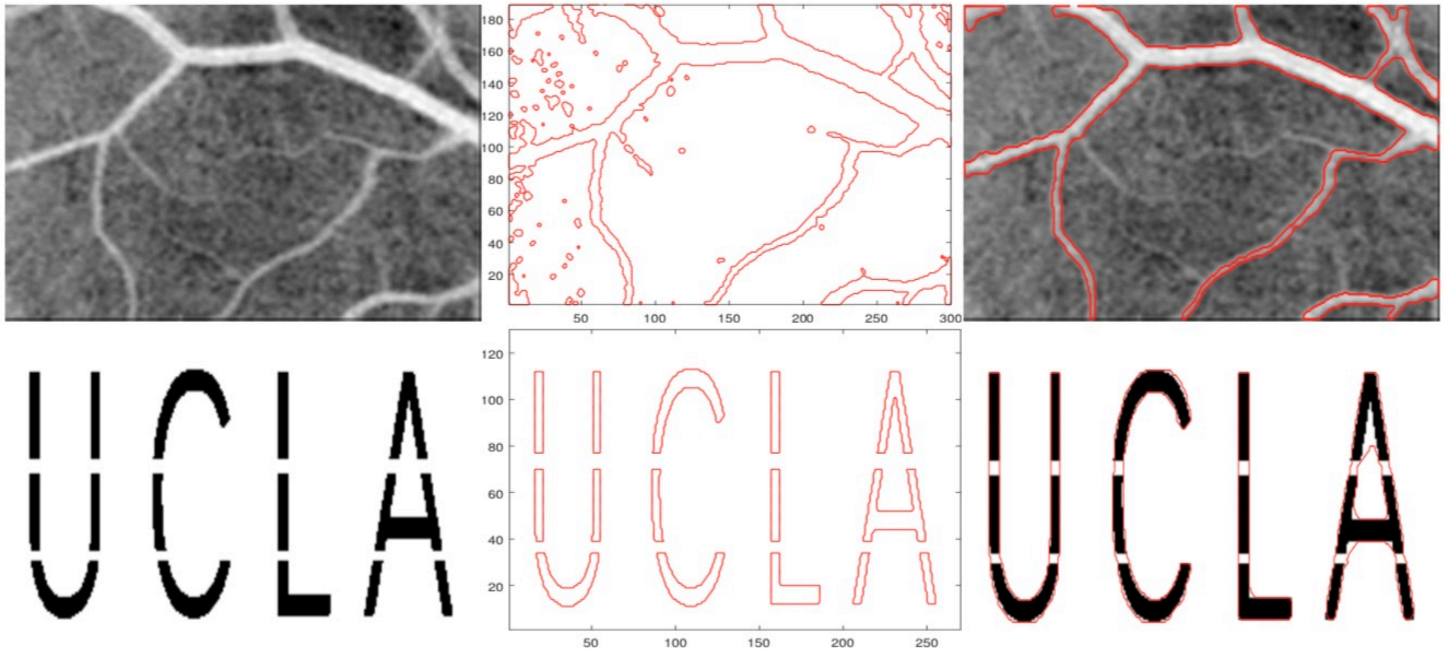


FIGURE: A synthetic image (the first row) and its three different segmentation results (the second row, blue lines) by applying the proposed algorithm to the Euler elastica based segmentation model. Different curvature parameter $b = 0, 5 \times 10^2, 4 \times 10^4$ are used while fixing other parameters as $a = 1$, $\eta = 1$, $\tau = 0.005$ and $\gamma = 0.1$.

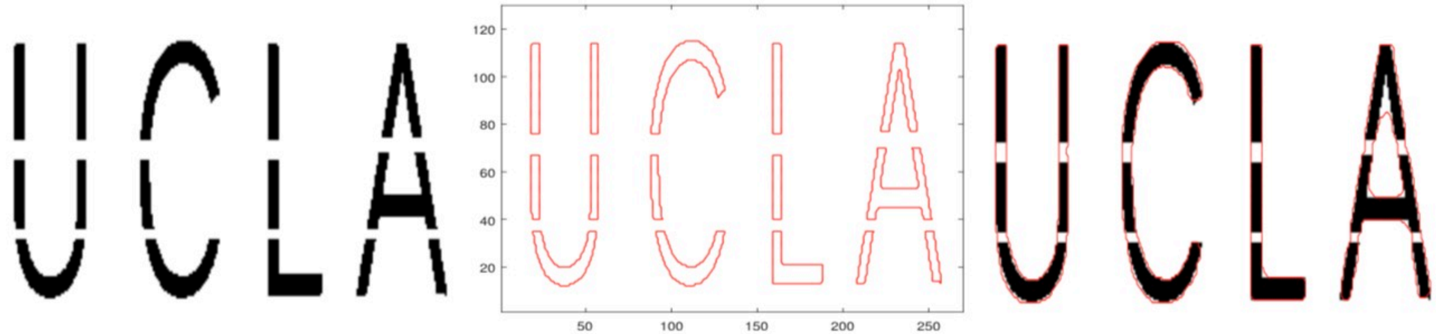
Segmentation Results



Input Initial contour Proposed

FIGURE: The results of image segmentation by the proposed method on the examples "vessel" and "UCLA". The parameters for "vessel": $a = 1 \times 10^{-3}$, $b = 1$, $\eta = 1$, $\tau = 0.03$, and $\gamma = 0.1$, while parameters for "UCLA" examples: $a = 0.6$, $b = 100$, $\eta = 0.5$, $\tau = 0.005$, and $\gamma = 0.1$.

Segmentation Results



Input   Initial contour   Proposed

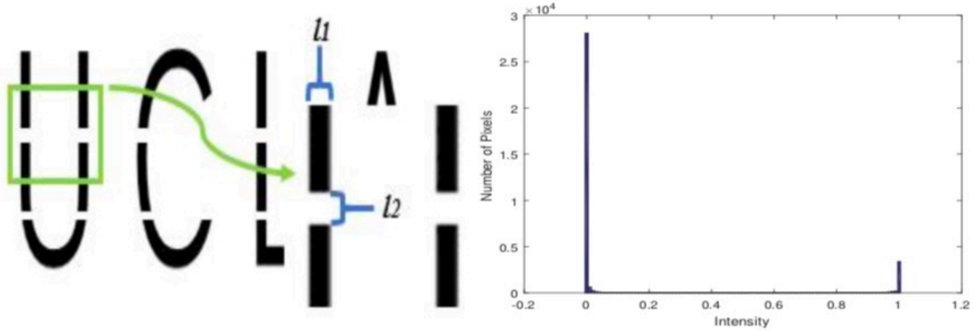


FIGURE: This example of "UCLA" is with bigger broken gaps, i.e., the "UCLA" with the gap relation $l_2 > l_1$. The parameters for "vessel" are: $a = 1 \times 10^{-3}$, $b = 1$, $\eta = 1$, $\tau = 0.03$, and $\gamma = 0.1$, while parameters for "UCLA" examples are all set as: $a = 0.6$, $b = 100$, $\eta = 0.5$, $\tau = 0.005$, and $\gamma = 0.1$.

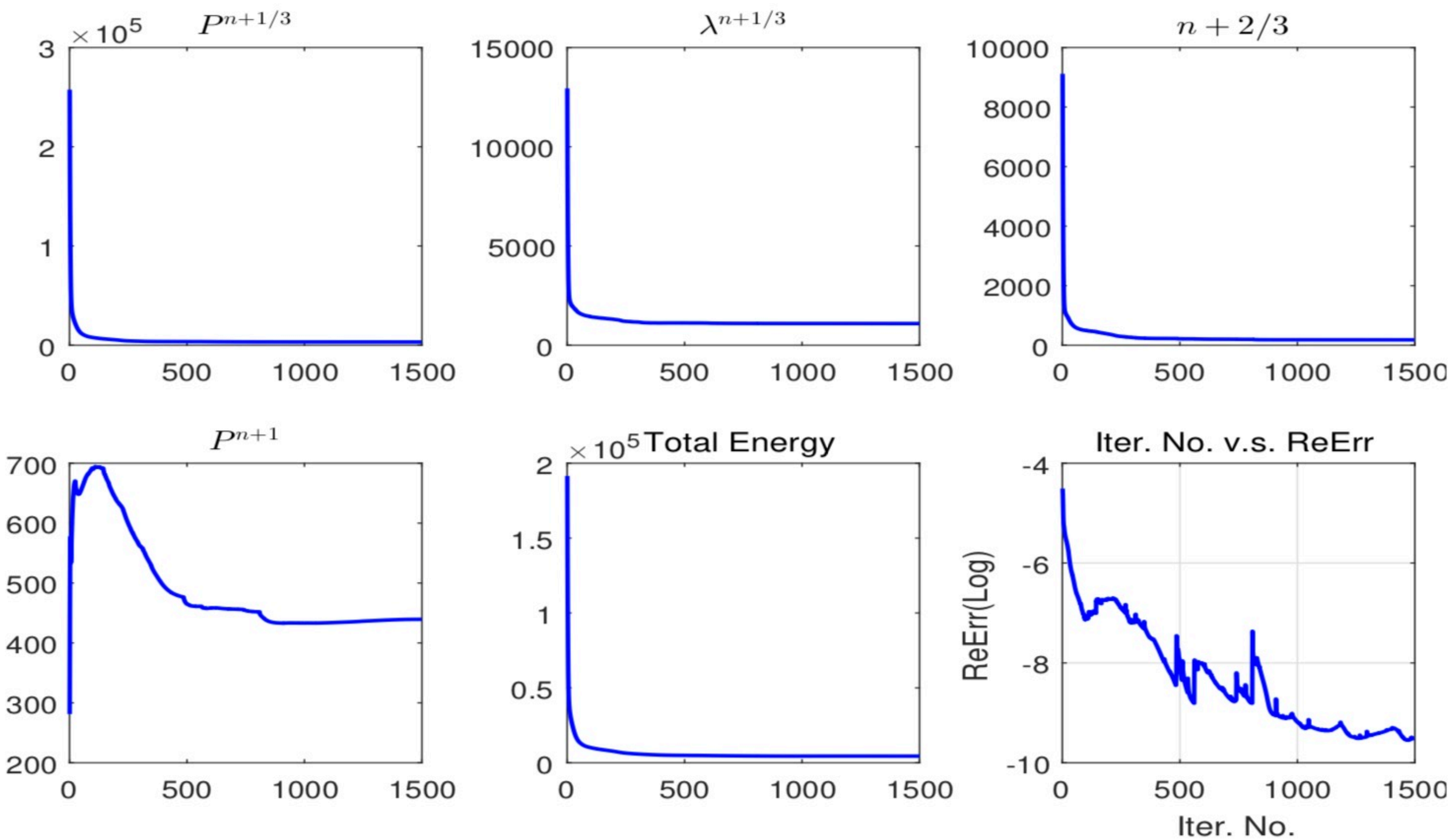
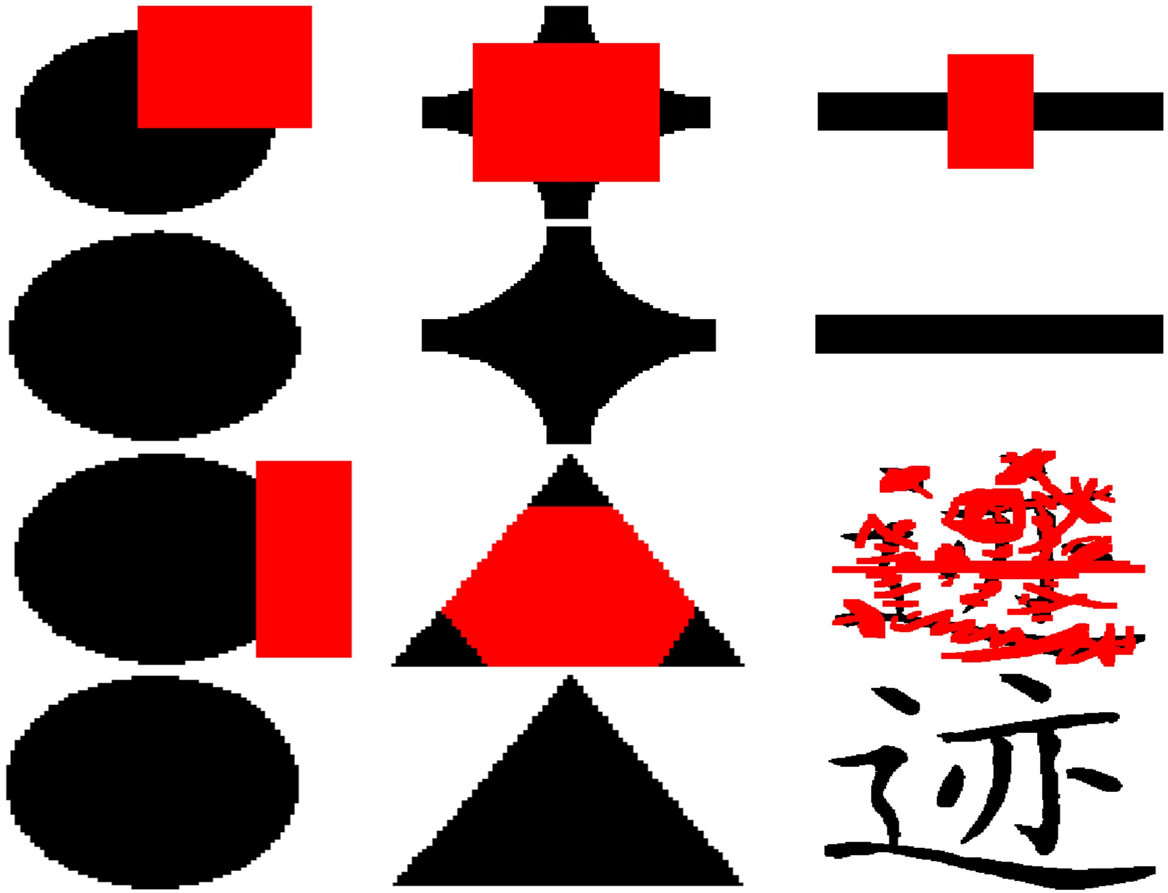
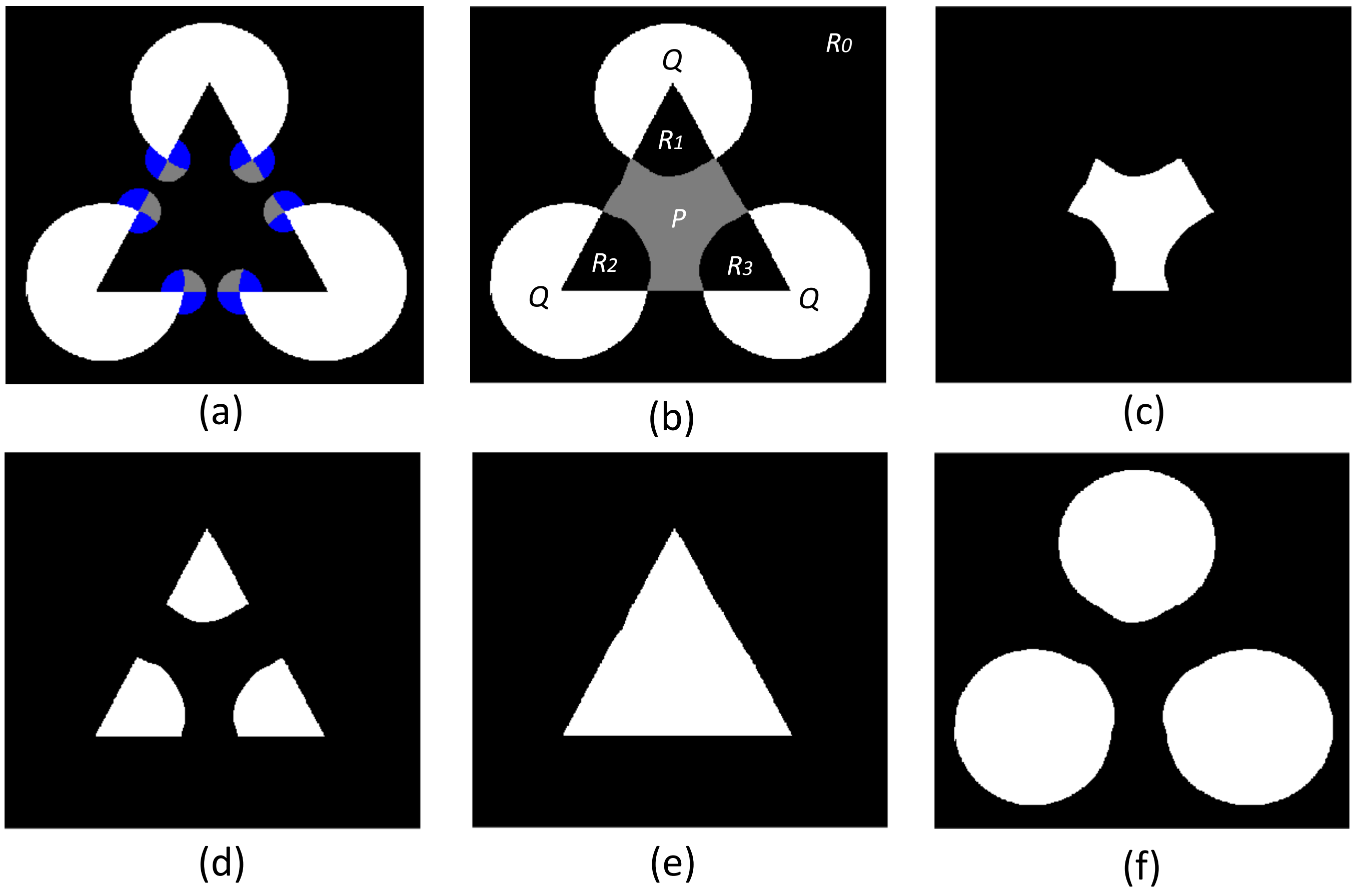


FIGURE: The energy plot of $\mathbf{p}^{n+1/3}$ subproblem (25), $\boldsymbol{\lambda}^{n+1/3}$ subproblem (27), $(n+2/3)$ subproblem (29), $\mathbf{p}^{n+1}$ subproblem (59) and the total energy (58).

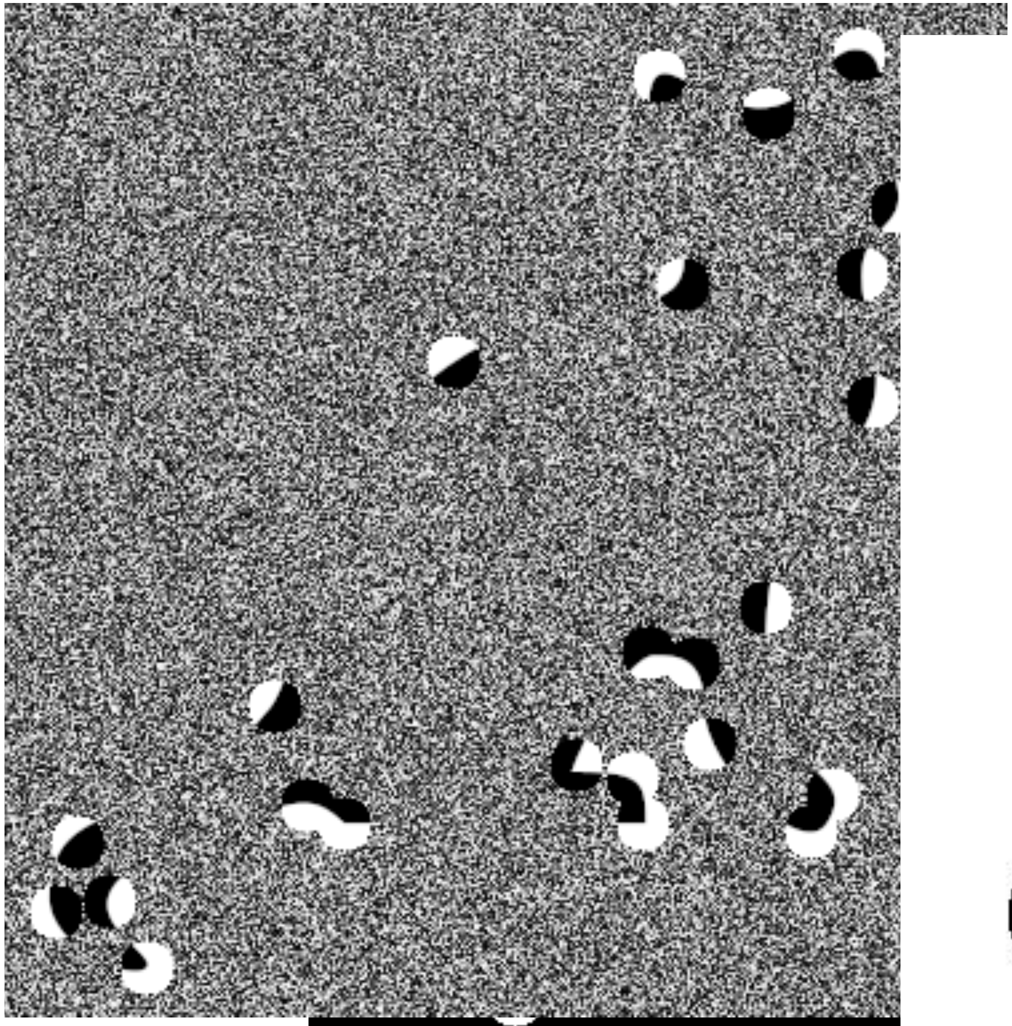
Numerical Tests:Binary image Inpainting:

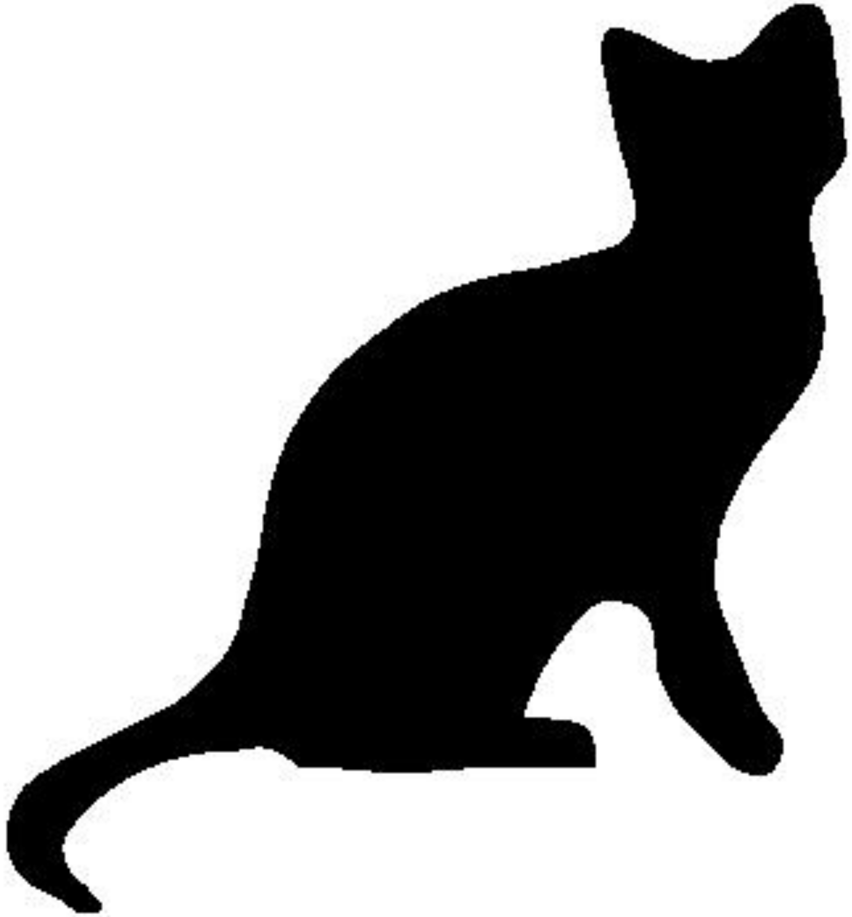


Numerical Tests:Illusaory contours:



Numerical Tests:Binary image Inpainting:





Thank you!