Uniqueness of special Lagrangian submanifolds in hamiltonian isotopy orbits of Lagrangian submanifolds

"*Mirror Symmetry*", Hori, Katz, Klemm, Pandharipande, Thomas, Vafa, Vakil, Zaslow, Clay Mathematics Institute series. 2000 (2003??)

Moment maps, monodromy and mirror manifolds, math.DG/0104196

Special Lagrangians, stable bundles and mean curvature flow, with S.-T. Yau. math.DG/0104197

Stability conditions and the braid group, math.AG/0212214

Let $X = (X^n, \omega, \Omega)$ be an *almost Calabi-Yau manifold*:

- X is a complex manifold of dimension$_{\mathbb{C}}$ n,
- ω is a Kähler form on X,
- Ω is a nowhere-zero holomorphic n-form $\Omega \in H^0(K_X)$. "$\Omega = dz_1 \wedge \ldots \wedge dz_n$"

If Ω is compatible with the Kähler metric ω (i.e. $|\Omega| \equiv 1$; equivalently Ω is Levi-Civita parallel) then X is Calabi-Yau. By Yau's theorerm we can arrange this by altering ω if X is compact.

Let $L \subset X$ be a *Lagrangian* submanifold:

- L has real dimension n, and
- $\omega|_L \equiv 0$.

An easy local computation shows that $\Omega|_L = |\Omega| e^{i\theta}\,\mathrm{vol}_L$ for some phase θ. A *graded* Lagrangian (L, θ) is L together with a choice of a lift of $e^{i\theta}$ to a real-valued function

$$\theta :\ L \to \mathbb{R}.$$

All our Lagrangians will be (implicitly) graded; in particular we consider only Lagrangians whose Maslov class

$$H^1(L; 2\pi\mathbb{Z}) \ni [d\theta] = 0.$$

L is a *special Lagrangian* if $\theta \equiv$ constant. By rotating Ω we can assume without loss of generality that this *cohomological* constant (phase of $\int_L \Omega$) is zero; that is L satisfies

- $\mathrm{Im}\,\Omega|_L \equiv 0$; equivalently

- $\mathrm{Re}\,\Omega|_L \equiv |\Omega|\,\mathrm{vol}_L$.

The special Lagrangian condition is a first order elliptic PDE describing minimal submanifolds of X (in the *almost* CY case have to define volume with respect to $|\Omega|\,\mathrm{vol}_L$).

We will call a Lagrangian L *close to being special Lagrangian* if $\operatorname{Re}\Omega|_L$ is nondegenerate on L, i.e. defines a volume form. Equivalently, the phase function θ has bounded variation in $(-\pi/2, \pi/2)$. Such Lagrangians clearly have zero Maslov class and are gradeable.

Mirror symmetry

Mirror symmetry exchanges complex geometry (governed by Ω) on one Calabi-Yau with symplectic geometry (governed by ω) on a *mirror* Calabi-Yau.

Very roughly speaking, under mirror symmetry, Lagrangians correspond to holomorphic bundles on the mirror Calabi-Yau. (The former are defined using on the symplectic structure, the latter only the complex structure.) Special Lagrangians correspond to (perturbed or MMMS) Hermitian-Yang-Mills connections on the mirror bundle.

Theorem 1 Donaldson-Uhlenbeck-Yau
A holomorphic bundle E admits a Hermitian metric such that the induced connection is Hermitian-Yang-Mills iff E is (poly)stable. The metric is then unique.

(E is stable iff for all coherent subsheaves $F \hookrightarrow E$, we have $\mu(F) < \mu(E)$. Here

$$\mu(E) = \frac{\int_X c_1(E) \cup \omega^{n-1}}{\operatorname{rank} E}$$

is the only place that the Kähler form ω enters, just numerically through its cohomology class. Stable bundles are "generic"; they form a Zariski open subset.)

The theorem reflects the infinite dimensional Kähler geometry of the action of the gauge group on the space of connections on E. Moment maps, Yang-Mills flow, convexity...

To get similar structures on the space of Lagrangians, we need to complexify.

$$\mathcal{A} := \{\, (L, A) \ : \ L \subset X \text{ Lagrangian}, \\ A \text{ a flat } U(1) \text{ connection on } L \,\}$$

We restrict attention to Lagrangians close to being special Lagrangian, and connections on the trivial line bundle $L \times \mathbb{C}$.

The tangent space to $\mathcal{A}$ at (L, A)

$$Z^1(L, \mathbb{R}) \oplus i Z^1(L, \mathbb{R}) = \text{Defs of } L \oplus \text{Defs of } A$$

is naturally complex, and we can induce a non-degenerate 2-form on it by defining a metric on $Z^1_L \subset \Omega^1_L$ by

$$\begin{aligned} \langle a, b \rangle &= \int_L a \wedge (b \lrcorner \omega^{-1}) \lrcorner \operatorname{Im} \Omega \\ &= \int_L (a \wedge * b) \operatorname{Re} \Omega|_L. \end{aligned}$$

$C^\infty(X, U(1)) \ni \phi$ acts on $\mathcal{A}$ as the gauge transformation

$$e^{i\phi}|_L \ : \ A \mapsto A + id\phi.$$

This clearly complexifies via the above complex structure to give an action of $C^\infty(L, \mathbb{R})$ by exact deformations $d\phi \in \Omega^1_L$ of L – *hamiltonian deformations*.

Combining these two actions gives an action of $C^\infty(X, \mathbb{C}^\times) \ni f = he^{i\phi}$: as gauge transformations on A by $e^{i\phi}|_L$, and as hamiltonian isotopies of (L, A) induced from hamiltonian diffeomorphisms of X with hamiltonian h.

(I.e. by the time-one flow of its hamiltonian vector field $X_h = dh \lrcorner \omega^{-1}$.)

$C^\infty(X, \mathbb{C}^\times)$ preserves the complex structure on $\mathcal{A}$; $C^\infty(X, U(1))$ also preserves the symplectic structure and has a *moment map*

$$\mu : \mathcal{A} \to (C^\infty(X, \mathbb{R}))^*, \quad \mu(L, A) = \operatorname{Im}\Omega|_L.$$

(Here the current $\operatorname{Im}\Omega|_L$ is the linear functional on $C^\infty(X, \mathbb{R})$ taking f to $\int_L (f \operatorname{Im}\Omega)|_L$.)

The upshot of the general theory is that we expect the generic ("*stable*") Lagrangian (close to being special Lagrangian) to have in its hamiltonian isotopy class a zero of the moment map,

$$\operatorname{Im}\Omega|_L \equiv 0,$$

i.e. a *special Lagrangian*; this should then be *unique*.

We also expect to reach this special Lagrangian (zero of the moment map) via the gradient flow of $-||\mu||^2$; this can be shown to be *mean curvature flow*, which is indeed hamiltonian for Lagrangians of Maslov class zero.

Other considerations (angle criterion, an example of Harvey-Lawson and Joyce, Kontsevich's mirror conjecture) lead to the following Hitchin-Kobayashi-type conjecture.

Conjecture 1 *Take graded Lagrangians* (L_1, θ_1) *and* (L_2, θ_2)*, hamiltonian isotoped to intersect cleanly, and such that the graded (relative) Lagrangian connect sums* $(L_1 \# L_2, \theta_1 \# \theta_2)$ *exist. Then a Lagrangian* L *of Maslov class zero is said to be destabilised by the* L_i *if it is hamiltonian isotopic to such an* $L_1 \# L_2$*, and the phases (*real *numbers, induced by the gradings) satisfy*

$$\phi(L_1) \geq \phi(L_2).$$

If L *is close to being a special Lagrangian and is not destabilised by any such* L_i *then it should have a special Lagrangian in its hamiltonian deformation class, and this should be unique.*

If L*'s phase variation is sufficiently small then this special Lagrangian should be the infinite time limit of MCF of* L*.*

Infinitesimally, the linearised formal picture is clear; it becomes simple Hodge theory.

Deformations of Lagrangians are given by closed 1-forms $\ker d : \Omega^1(L; \mathbb{R}) \to \Omega^2(L; \mathbb{R})$, so that dividing by hamiltonian deformations we get

$$T_L(\{\text{Lagrangians/hamiltonian isotopies}\})$$

$$= \ker d / \operatorname{im} d = H^1(L; R).$$

If instead of dividing by hamiltonian deformations we impose the *special* condition $\operatorname{Im} \Omega|_L = 0$, we get a $\ker d^*$ slice

$$T_L(\{\text{Special Lagrangians}\}) =$$

$$\ker d \cap \ker d^* = H^1(L; R),$$

to the hamiltonian deformations.

The Kähler quotient and moment map picture is a vast nonlinear generalisation of this linear fact that the quotient of a vector space by a subspace is isomorphic to the orthogonal complement of the subspace.

Uniqueness

Even in infinite dimensional Kähler quotient problems, convexity properties of the moment map usually prove uniqueness of its zeros (modulo the action of the real group) in a complexified group orbit. The proof only relies on the existence of 1-parameter subgroups connecting any two points of the orbit.

Unfortunately this need *not* be the case here as most hamiltonian isotopies are *not* time independent. They also need not preserve the "close to special Lagrangian" condition.

But for those time-independent isotopies which *do* preserve this, we can demonstrate the general picture in our context as follows.

Theorem 2 *If two SLags L_0, L_1 are time-independent hamiltonian deformations of each other through Lagrangians L with $\mathrm{Re}\,\Omega|_L > 0$, then $L_0 = L_1$.*

Proof We are supposing that $L_0 = \Phi(L_1)$, where Φ is the time-one hamiltonian isotopy of a hamiltonian $h \in C^\infty(X, \mathbb{R})$; that is $\Phi = \Phi_1$, where

$$\Phi_0 = \mathrm{id}, \quad \frac{d}{dt}\Phi_t = X_h = dh \lrcorner \omega^{-1}.$$

(*Or*, we can allow h to be time independent as it follows L, in the sense that we have $h = h_t$ with $\Phi_t^* h_t|_L =$constant. Then there is no $X_h(h)$ term below.)

We compute, down the flow,

$$\begin{aligned}\frac{d}{dt}\int_L h\,\mathrm{Im}\,\Omega|_L &= \int_L \left\{X_h(h)\,\mathrm{Im}\,\Omega + h\,\mathcal{L}_{X_h}(\mathrm{Im}\,\Omega)\right\} \\ &= \int_L h\,d(X_h \lrcorner \mathrm{Im}\,\Omega) \\ &= \int_L (dh \wedge *dh)\,\mathrm{Re}\,\Omega|_L.\end{aligned}$$

So for phase θ lying in $(-\pi/2, \pi/2)$, i.e. $\mathrm{Re}\,\Omega|_L$ strictly positive,

$$\frac{d}{dt}\int_L h\,\mathrm{Im}\,\Omega|_L \geq 0,$$

with equality iff $h|_L = \text{constant}$.

But $\int_L h\,\mathrm{Im}\,\Omega$ is zero at $t = 0, 1$. Thus $h|_L \equiv$ constant and the two special Lagrangians coincide. $\square$

For the general case we mirror the algebro-geometric argument that a non-zero map between stable bundles of the same slope is an isomorphism. So we need to use the Floer homology of Fukaya-Oh-Ohta-Ono.

Call a connected graded Lagrangian L *unobstructed* if $w_2(L)$ is the restriction of a class $\in H^2(X;\mathbb{Z}/2)$ (e.g. if L is spin), and the [FOOO] obstructions to the existence of its Floer cohomology vanish. E.g. Lagrangian homology spheres are unobstructed in dimension 3 and above.

Graded Lagrangians have graded Floer homology groups. Suppose L_2 has phase 0 at an intersection point p with L_1, and let its tangent space be the x-axes:

$$L_2 = \{y_i = 0\}.$$

Then

$$L_1 = \{z_i = re^{i\alpha_i}\}, \qquad \alpha_i \in (0, \pi)\ \forall i.$$

Then $\sum \alpha_i = \theta_p(L_1)$ mod π and we define the Floer index at p to be

$$\text{ind}_p(L_2, L_1) := \frac{1}{\pi}\left(\sum \alpha_i - \theta_p(L_1)\right).$$

Notice that $\text{ind}_p(L_2, L_1) + \text{ind}_p(L_1, L_2) = n$.

It is not clear if the following theorem can be extended to obstructed Lagrangians. The role of Floer homology is analogous to that in the proofs of the Arnol'd conjecture.

Theorem 3 *There is at most one smooth special Lagrangian in the hamiltonian deformation class of an unobstructed Lagrangian L.*

In particular, special Lagrangian homology spheres are unique in their hamiltonian deformation class in dimension 3 and above.

Proof The key points are that Floer cohomology is independent of hamiltonian deformations [FOOO], and the zeroth order piece of $H^*(L)$ survives in $HF^*(L, L)$ for L with Maslov class zero [FOOO].

Therefore two graded special Lagrangians L_1, L_2 in the same hamiltonian deformation class satisfy

$$HF^0(L_1, L_2) = H^0(L_1; \mathbb{C}) = \mathbb{C},$$

and there must be at least one intersection point p of L_1 and L_2.

The (constant) phases of the L_i differ by $r\pi$ for some $r \in \mathbb{Z}$; we first show that $r = 0$.

Hamiltonian perturbation $\Longrightarrow$ transverse intersection point of L_is of Floer index 0, with the phases of the L_i at this point differing by $r\pi+\epsilon$.

Locally L_2 is graph of df in T^*L_1; f Morse index r, so $r \geq 0$.

Similarly there is a point of Floer index n (index 0 on swapping L_1, L_2); locally dg with g of Morse index $n + r$; so $r \leq 0$ also. Therefore $r = 0$ as required.

If the intersection points are isolated, one must have Floer index zero:

$$\sum \alpha_i = 0, \qquad \alpha_i \in (0, \pi) \ \forall i.$$

So the relative angles $\alpha_i = 0 \ \ \forall i$ and the L_i are tangent at p; there is no isolated transverse intersection point of Maslov class zero.

Near p write L_1 as the graph in T^*L_2 of df with $d^*df = 0$: f harmonic. By the maximum principle, f has no local maxima or minima. This, roughly, is the proof, but we must also deal with non-Morse f, i.e. non-transverse intersections.

We want to show that f is constant, so assume for a contradiction that the critical set of f is not all of L_2.

Perturb f inside a connected component of a small neighbourhood of its critical set such that its value is unchanged on the boundary, *where it attains its global maximum and minimum*, and is Morse in the interior.

By Morse theory we can then perturb f further to arrange its index 1 critical points to be lower (with respect to f) than all higher index points and then cancel any local minima with them. (There must be index 1 critical points if there are interior minima, by connectivity of our neighbourhood.)

So we get a hamiltonian perturbation of L_1 with no Floer index zero intersection points with L_2. Thus $HF^0(L_1, L_2) = 0$, a contradiction. $\square$