Analytic Methods for SLAGs

I. Deformation & Gluing Thms

II. MCF & Variational Methods

(M^{2n}, g, J) Kähler (Almost Kähler)

$$\omega(X,Y) = g(JX,Y)$$

$d\omega = 0$

- Kähler-Einstein $Ric(g) = cg$
- $\Sigma^n \subseteq M^{2n}$ Lagrangian $\omega|_\Sigma = 0$

$$\Leftrightarrow J(T\Sigma) = (T\Sigma)^\perp$$

- How to construct Σ minimal, Lagrangian?

<u>I.</u> Def + Gluing Thms

Assume M is CY: $Ric(g) = 0$

$+ \nabla\alpha = 0$

α parallel $(n,0)$ form

<u>Def:</u> Σ is SLAG if Σ Lag + $Re(\alpha)|_\Sigma = dv_\Sigma$

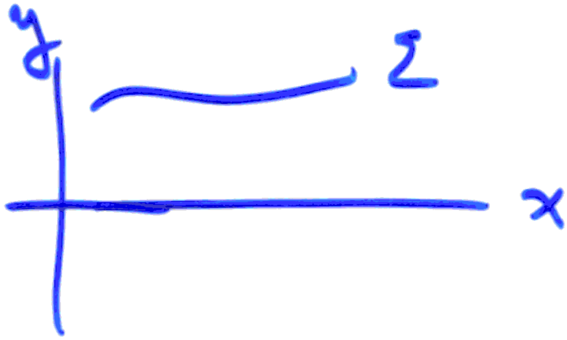
- SLAG $\Longrightarrow$ Σ is homologically volume min (In part., $H \equiv 0$)

<u>Ex:</u> $\mathbb{C}^n \approx \mathbb{R}^{2n}$, $\cdot$, J

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}, \quad J\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}$$

$$\omega = \sum_j dx^j \wedge dy^j$$

$$\Sigma : \ y^j = \frac{\partial u}{\partial x^j}$$



$$\Delta u = \det(\mathrm{Hess}\, u) \qquad (n=3)$$

Fully nonlinear 2nd order scalar PDE.

- Simplest thing to do is "perturbation theory". Look for small sol'ns or small deformations of given sol'n.

i) Moduli Space of sol'ns in given CY

ii) How moduli space varies under def of CY structure.

McLean's Thm: $\Sigma^n \subseteq M^{2n}$

Smooth SLAG $\Rightarrow$ Space of SLAG's in a nbhd of Σ is a smooth mfld of dim $= b_1(\Sigma)$.

$$T_\Sigma \mathcal{M} \cong \{ \underset{\substack{\cap \\ \text{1-form}}}{\eta} : d\eta = 0, \ d^*\eta = 0 \}$$

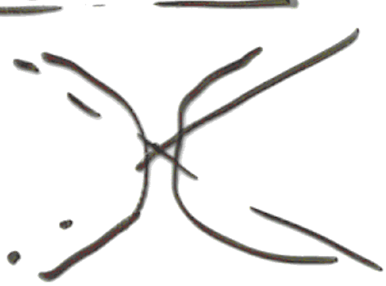
- Similar for def of ambient CY structure

Singular SLAGs may arise as limits of smooth SLAGs. (Boundary Strata)

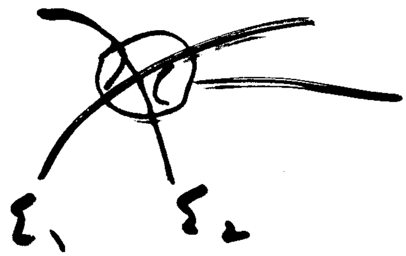
Isolated Multiplicity one sing:

- Transverse Double Point

A. Butscher: Local model: Lawlor neck



Σ_0 asympt. to pair of SLAG Planes in $\mathbb{C}^n$.



Patch in suitable scaled Version of Lawlor neck

& Use def theory to find exact SLAG

- Argument requires careful analysis of asympt. of Lawlor neck.

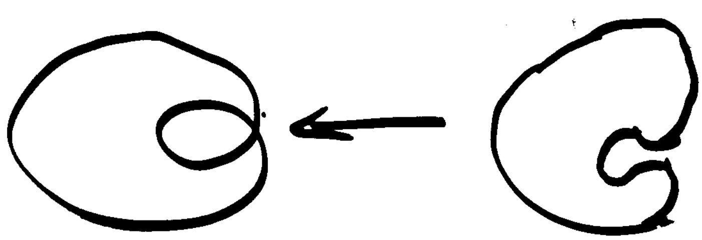
- Yng-Ing Lee

- For which pairs of SLAG planes are there Lawlor necks:

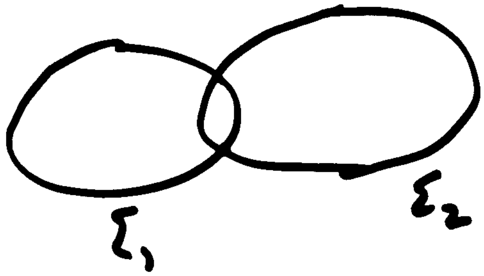
  Always for $n=2,3$

  Not always for $n \geq 4$

- $\Sigma^n \subseteq M^{2n}$ Connected Immersed SLAG with transverse double pts $\Longrightarrow$ Σ can be approx by smooth embedded SLAG



Gluing for Disconnected SLAGs



Transverse Double pts

Note: Dimension Count

i) Σ connected Trans. Dble Pt.

$b_1(\tilde{\Sigma}) = b_1(\Sigma) + 1$ $\quad \tilde{\Sigma}$ smoothed Lag.

Gluing const adds a dim to $\mathcal{M}$

ii) $\Sigma = \Sigma_1 \cup \Sigma_2$

$b_1(\tilde{\Sigma}) = b_1(\Sigma_1) + b_1(\Sigma_2)$

$\therefore$ Don't expect desing. to work in $\mathcal{M}$.

Dan Lee: Found explicit def of ambient CY structure to do desing in nearby CY mflds.

Flat Tori: T^{2n} $(n=3)$

$\exists\ \Sigma_1, \Sigma_2$ linear Lag. subsp.

intersecting trans. at a pt.

$\exists$ Nearby Flat torus for which

$\Sigma_1 \# \Sigma_2$ exists as SLAG (Cor. of D. Lee)

(Non Flat SLAGs)

(Reminiscent of holo subm.)

D. Joyce: General framework in which

Butscher's analysis works.

$\tilde{\Sigma}$ regularization

Compactifying $\mathcal{M}$: Issues

i) $\overline{\mathcal{M}}$ exists in weak (Flat norm) topology (G.M.T.) Each $\Sigma \in \overline{\mathcal{M}}$ is an SLAG variety

ii) What is possible singular structure of Σ? Experience for Vol min varieties suggests that <u>multiplicity</u> is the major complication.

Almgren's Thm (2000 pages)

$\exists\, S^{\text{closed}}$ $\quad \dim(S) \leq n-2$

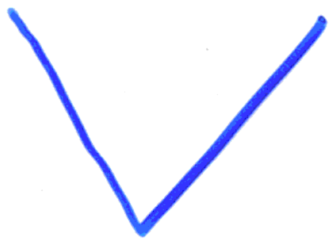
$\Sigma \sim S$ is smooth embedded SLAG

Can one describe a nbhd of Σ in $\overline{\mathcal{M}}$?

Is there an "easy" proof of Almgren for SLAG? (Yes for holo subv.)

Problem: $n=3$ Better understanding of:

SLAG cones



$\Sigma \subseteq \mathbb{R}^6$

$\Sigma \cap S^5 = \Sigma_0^2 \subseteq S^5$

Σ_0 is a min. Leg. surface in S^5

T^2 examples: Clifford Torus (Harvey Lawson)

$\cap|$

singular SLAG foliat.

∞ many other T^2's constructed by

M. Haskins.

- Find higher genus Σ_0 (should be possible)
- Classify? Under assumption Σ lies in (partial) SLAG foliation?

Part II : Global Constructions

M^{2n} $\sigma \in H_n(M, \mathbb{Z})$

- Can we find Lagrangian rep?
- Σ may be singular
- $n=2$ Yes $\Leftrightarrow$ $\int_\sigma \omega = 0$ (S-Wolfson)

Prop (Wolfson) $\exists$ singular Lag rep of σ

$\Leftrightarrow$ $\sigma \cap [\omega] = 0$

- Proof gives polyhedral rep. Further restrictions to find $\Sigma \in \sigma$ which is a smooth Lag immersion
- $n=2$ Need $\int_\sigma c_1 = 0$ to make Σ smooth embedding
- $n=3$ $\pi_1(M) = 0$ all of $H_3(M, \mathbb{Z})$ canbe rep by Lag immersions.

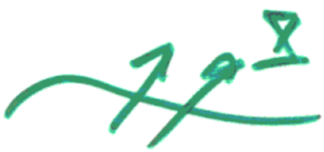
Recall that if M is CY & Σ is SLAG then Σ min volume in its homology class (Calibrated)

- Suggests that SLAG could be produced by volume min <u>among</u> <u>Lagr.</u>

Why should this work?

- Mean curv flow / Direct minimization (Variational Method)

<u>Key Pt</u>: Σ Lag $\Sigma \subseteq M$ K-E



$$\delta V(X) = -\int_\Sigma \langle X, H\rangle \, d\mu$$

H = mean curv

- $H = J\nabla\beta$ β multi-valued hamiltonian on Σ

- CY case α parallel $(n,0)$ form

$$\alpha|_\Sigma = e^{i\beta} \, d\mu$$

Heuristic Argument:

Any Lag $\Sigma \in \sigma$ is either minimal $(H=0)$ or it can be replaced by Σ_1 deformation of Σ. Σ_1 Lag + has strictly smaller volume than Σ.

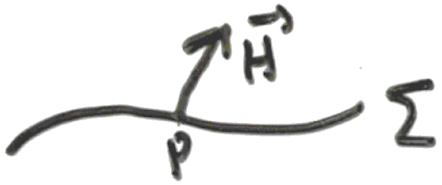
$\therefore$ Smallest such Σ must be minimal (SLAG)

How to produce a smallest Lag Σ?

i) Mean Curv Flow

ii) Variational Method

MCF:



$$\frac{dP}{dt} = \vec{H}(p)$$

$$p \in \Sigma_0$$

- Σ_0 Lag
  $\Rightarrow \Sigma_*$ Lag
- $|\Sigma_*| \downarrow$

Problem: Σ_t is smooth for small $t>0$ but may become singular at a finite time.

Ex: $n=1$ MCF converges for embedded

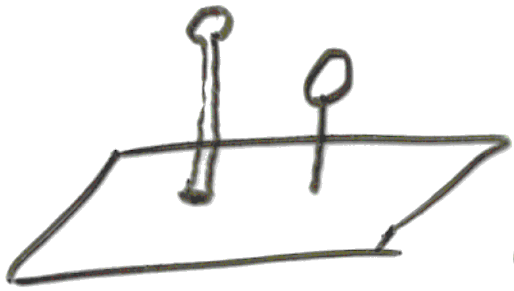
Σ_0

MCF shrinks Σ_0 to pt + Σ_t becomes round

MCF moves Σ_0 to a geodesic.

Sing will develop for immersed curves

Ex:



Plane with small area filigree

MCF will become singular

· Weak solution needed in general

$n \geq 2$ LMCF

R. Thomas / Yau

Formulate conjecture / Proof with certain symmetry (reducing problem to evolution for curves)

M. Wang – Long-time existence + convergense under special initial cond.

E. Goldstein, J. Pacini MCF with symmetry

S, Wolfson: $\exists$ "small" def of SLAG Σ_0

for which MCF becomes sing.

(Not quite counter-example to Thomas/Yau-

Idea: Add high curvature filigree ~~to~~ smooth SLAG. Conj: This can be done w. hamiltonian def.

Problem: 1. Does Σ_0 embedded

$\Longrightarrow \Sigma_t$ embedded ?

Yes for $n=1$. Probably not for $n \geq 2$

2. Can one define weak sol'n for MCF which preserves Lag for Σ_t?

(Very difficult in general.)

<u>Variational Approach:</u> w. J. Wolfson

Want to min volume among all

Lag $\Sigma \in \sigma$, $\sigma \in H_n(M, \mathbb{Z})$

$|\Sigma_i| \longrightarrow \inf$

$\Sigma_i \to \Sigma$ $\quad |\Sigma| \leq \inf$

Want $\Sigma \in$ Class under consideration

- Allow very gen. sing
- Need Σ Lag (Yes if orientable)

Σ Least vol. Lag. Integral Current

Need some reg of Σ:

(Heuristic: Σ smooth $\Rightarrow$ Σ is minimal)

Prove Σ is smooth (can allow some sing.; e.g. $D\beta \in L^2(\Sigma)$.)

Larger Class of Σ: Hamiltonian Station.

$$\delta V(J\nabla h) = 0$$

$\uparrow$ Ham. v. f.

E.L. Eqn: $H = J\nabla\beta$ **F. Helein**

$$\Delta\beta = 0$$

(Third order Eqn)

Thm: $\Sigma^2 \subseteq M^4$ Σ minimizing among Lag. rep of homology class

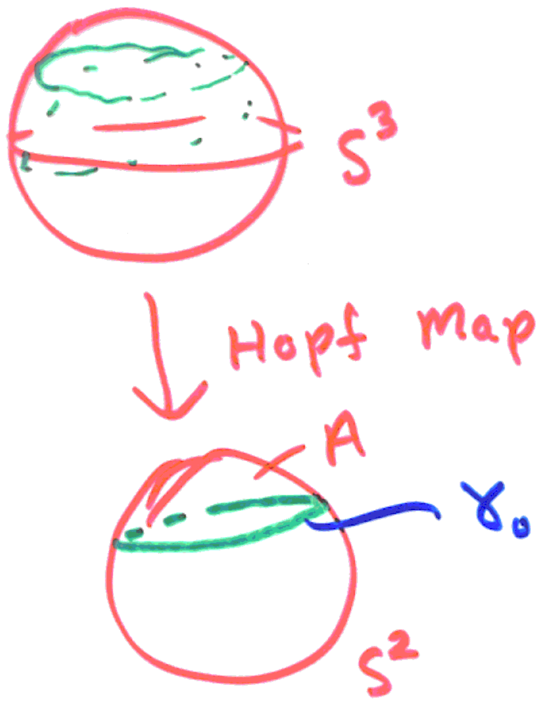
$\Rightarrow$ Σ is smooth ham. stationary except at $P_1, .., P_k, Q_1, .., Q_l$

i) P_j branch pts.

ii) Q_j Nonflat sing. pts.

Q_j

Hamiltonian Stationary Cones:



γ_0 has a closed horizontal lift $\iff$ A is a rational mult. of π.

Lift γ is Legendrian curve + $C(\gamma)$ = cone over γ is Ham. St.

γ $(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+$

(p,q) rel. prime

Index$(C) = p-q \in \mathbb{Z}$.

Prop: At least one of $|p-q|=1$ cones is area min among Lag. comp.

Prop: $\displaystyle\sum_{j=1}^{\ell} \mathrm{Ind}(Q_j) = \frac{1}{2\pi}\int_\sigma c_1(M)$

($=0$ in KE case)

Cor: Σ is minimal $\Longleftrightarrow$ $\ell = 0$ (No Q_j)

Note: $2 \cdot C$ is **not** minimizing

$\pi_2(M)$ $\quad \sigma \in \pi_2(M)$, $\|\sigma\| = \inf\{|\Sigma| : \Sigma \in \sigma\}$

Def: σ is 2-stable if $\|2\sigma\| = 2\|\sigma\|$

Thm: If σ is (approximately) 2-stable then least area $\Sigma \in \sigma$ is minimal

Cor: If σ is 2-stable, then it is approx 2-stable for deformations of KE structure on M, + hence has M.L. rep.

Remark: Any connected min Lag in CY is 2-stable.

Legendrian Least vol. Prob

$\mathbb{R}^{2n+1}$ (x, y, φ)

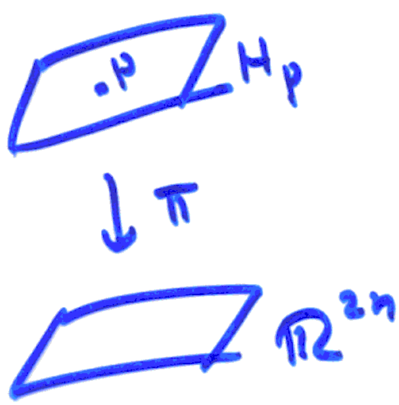
$\downarrow$

$\mathbb{R}^{2n}$, $\cdot$, J, ω $\qquad \omega = d\eta$

$$\eta = \tfrac{1}{2}(x\,dy - y\,dx)$$

Contact form: $\theta = d\varphi - \eta$

Horizontal Distr. $\quad \theta = 0$



$\Sigma^n \subseteq \mathbb{R}^{2n+1}$

Leg. if $\theta|_\Sigma = 0$

$T_p\Sigma \subseteq H_p$

$\Sigma_0 = \pi(\Sigma)$ is Lag.

Conversely: "Exact" Lag can be lifted

$\mathbb{R}^{2n+1}$ – Heisenberg Group

Horizontal Plateau Prob.

Group of contact transf. of $\mathbb{R}^{2n+1}$ is <u>much</u> larger than group of Symplectic transf. of $\mathbb{R}^{2n}$.

$\Rightarrow$ Larger def. space for Σ Leg.

(Crucial for regularity)

For $k \leq n$, can study k-dim'l Plateau problem (horizontal) in $\mathbb{R}^{2n+1}$

$n=2$, $k=2$ S-W

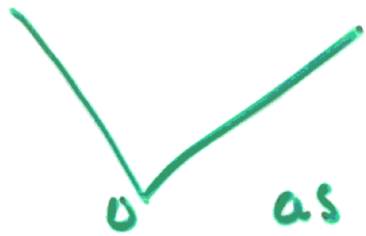
$n \geq 3$, $k=2$ W. Qiu

($n=1$, Isop. Prob in Plane)

$$g_\varepsilon = dx^2 + dy^2 + \frac{\theta^2}{\varepsilon^2}$$ Riem metric on $\mathbb{R}^{2n+1}$

Geom as $\varepsilon \to 0$ converges to Carnot/horiz. geometry.

<u>Cones:</u> $C(\gamma)$.



Cannot be smoothed as a lag surface by any def localized near 0

$\nexists$ Lag sol'n of MCF with $C = \Sigma_0$.

$S^2 \subseteq M^4$, CY

By making a cut in Σ from Q_1 to Q_2 + adding a Lag strip with small area can kill Maslov index

Lag MCF should exist only for very short time + converge back to Σ (If Σ is minimizing)