DUALITY AND TOPOLOGICAL STRINGS

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RELATIONS OF DIFFERENT AREAS OF MATHEMATICS

$$\begin{tikzcd}
\text{ENUMERATIVE GEOMETRY} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{KNOT THEORY} \arrow[d, Rightarrow] \\
\text{KODAIRA-SPENCER THEORY} \arrow[r, Rightarrow] & \text{MATRIX MODELS}
\end{tikzcd}$$

$\Sigma_g \to X$

(DEFORMATIONS OF COMPLEX STRUCTURES)

ALL THESE APEAR VERY DIFFERENT

AT FIRST SIGHT.

→ THESE THEORIES ARE

DIFFERENT CLASSICALY ($\hbar = 0$).

QUANTUM MECHANICALLY

THEY CAN BE THE SAME...

BECAUSE WHAT WE MEAN BY

$\hbar$

IS DIFFERENT IN EACH CASE.

THE POWER OF DUALITY
IS THAT IT GENERALLY
RELATES $\hbar$ OF ONE
THEORY TO A CLASSICAL
PARAMETER OF THE DUAL
AND VICE-VERSA.

A CANNONICAL EXAMPLE OF THIS IS MIRROR SYMMETRY, IN THE CONTEXT OF CLOSED  TOPOLOGICAL STRINGS ON ~~any~~ CALABI YAU.

A-MODEL :

COUNTING OF HOLOMORPHIC MAPS

$$\Sigma_g \longrightarrow X$$

GENUS g RIEMANN SURFACE

$$F_g = \sum_{\beta \in H_2(X,\mathbb{Z})} n_g^\beta \, e^{-t(\beta)}$$

FREE ENERGY

PLAYS ROLE OF $\hbar$

$$t(\beta) = \int_\beta k$$

B - MODEL :

VARIATIONS OF COMPLEX STRUCTURE

$\rightarrow$ KODAIRA - SPENCER THEORY

IT IS A POINT PARTICLE THEORY

$$F_0(t)$$

$\uparrow$

from classical geometry

MIRROR SYMMETRY RELATES

A-MODEL ON $\quad\quad$ B-MODEL ON

$$X \longleftrightarrow Y$$

$\hbar = e^{-t} \longleftrightarrow$ classical parameter

STRINGS ADMIT BOUNDARIES

ON D-BRANES

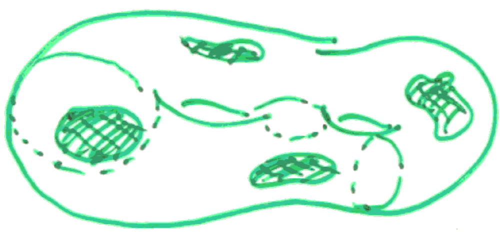
→ CONSIDER RIEMANN SURFACES WITH

BOUNDARIES ON L.

A- MODEL:

HOLOMORPHIC MAPS

$$\Sigma_{g,h} \longrightarrow X \swarrow M$$



$$K|_M = 0$$

$$\partial \Sigma_{g,h} \longrightarrow M$$

$$F_{g,h} = \sum_{\beta \in H_2(X/L)} n^{\beta}_{g,h}\, e^{-t(\beta)}$$

SIMPLE IN CERTAIN CASES:

E.G. WHEN $X = T^*M$,

~~THE~~ THERE ARE ONLY DEGENERATE MAPS,

WHERE RIEMANN SURFACES

DEGENERATE TO GRAPHS.

NOTE:

BY SUMMING OVER HOLES

$$\sum_h F_{g,h} \lambda^{2g-2} (\lambda N)^h$$

$\underline{t = \lambda N}$

 t +  $t^2 + \dots$

$$= F_g[t] \lambda^{2g-2}$$

"LARGE N DUALITY"

GET A CLOSED STRING AMPLITUDE.

t'HOOFT CONJECTURE '70s

BUT ... MAPS TO WHAT?

GOPAKUMAR & VAFA

'98

OPEN TOPOLOGICAL A-MODEL OF

N D-BRANES ON S^3

IN $X = T^*S^3$

= CLOSED A-MODEL STRING

THEORY ON

$$\hat{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$
$$\downarrow$$
$$\mathbb{P}^1$$

THE GRAPHS ARE RIBBON GRAPHS

OF CHERN-SIMONS THEORY

$$Z = \int \mathcal{D}A \; e^{k \int_M A dA + A^3}$$

$$= \exp F(N, \lambda) \qquad \lambda = \frac{2\pi}{k+N}$$

$$F = \sum_{g,h} F_{g,h} N^h \lambda^{2g-2+h}$$

WITTEN '92

CHERN - SIMONS THEORY IS SOLVABLE

AND COUNTING OF

MAPS TO $\mathbb{P}^1$ IS SIMPLE

→ ONE CAN EXPLICITLY CHECK

$$\int \mathcal{D}A \; \exp\left[k \int_{S^3} A\,dA + A^3\right] \qquad \lambda = \frac{2\pi}{k+N}$$

$$= \exp\left[\sum_n \frac{1 \times e^{-nt - N\lambda}}{n\left[2 \sin \frac{n\lambda}{2}\right]^2}\right]$$

X AND $\hat{X}$ ARE

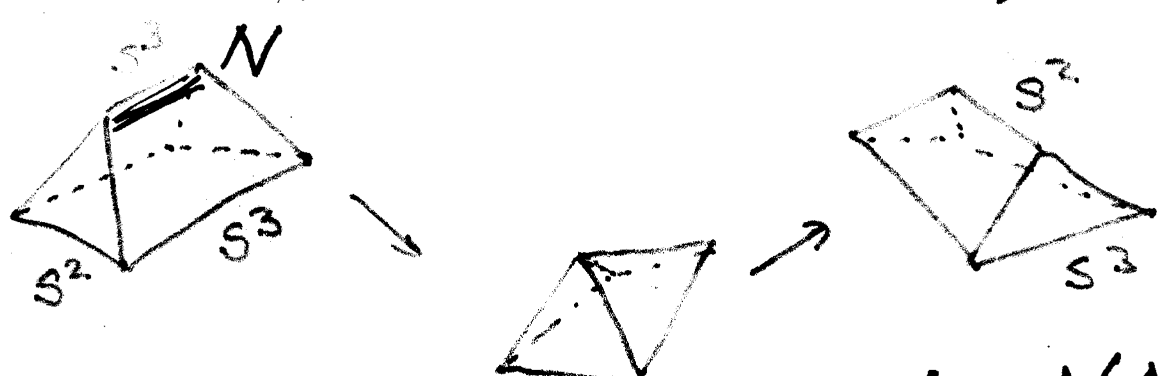
RELATED BY GEOMETRIC

TRANSITION

WHERE S^3 (AND D-BRANES
⟺ BOUNDARIES)

DISSAPEARS

AND IS REPLACED BY $\mathbb{P}^1$



QUADRIC
$x^2+y^2+z^2+w^2=0$
$\in \mathbb{C}^4$

$t = N\lambda$

SIZE OF $\mathbb{P}^1$

IN PRESENCE OF A NON-COMPACT

LAGRANGAN L $\quad L \cap S^3 = \gamma$

→ TOPOLOGICAL A-MODEL COMPUTES

AN INVARIANT OF S^3

WITH AN UNKNOT γ

OOGURI & VAFA '99

$$Z = \sum_R \langle \mathrm{Tr}_R V \rangle_{S^3} \, \mathrm{Tr}_R U \, e^{-\ell(R) r}$$

WITTEN '88

$$V = P e^{\oint_\gamma A}, \quad U = P e^{\oint A}$$

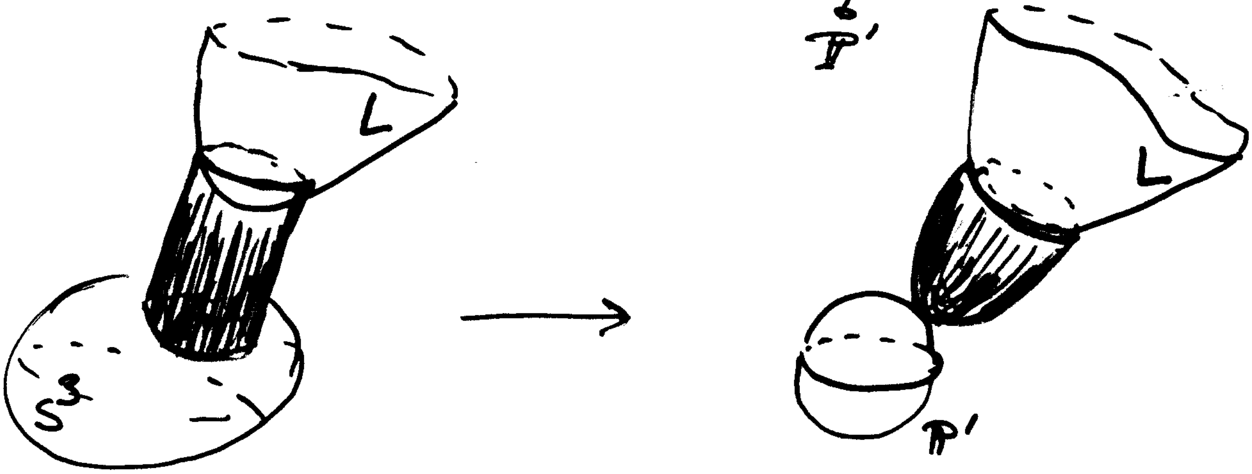
ON S3 $\qquad$ ON L

LARGE N DUAL THEORY:

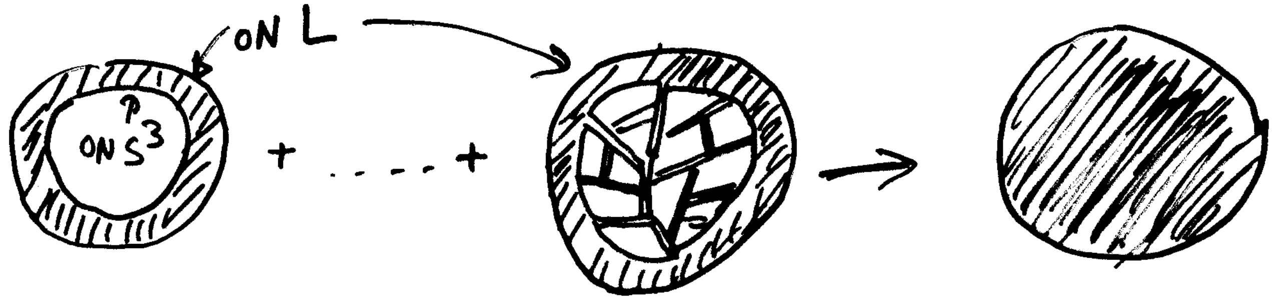
BY GEOMETRIC TRANSITION

→ GET A NON-COMPACT LAGRANGIAN

L ON $\hat{X} = \mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$



SUMMING OVER HOLES,



→ HOLES GET "FILLED IN".

HOWEVER:

THE CHERN-SIMONS CALCULATION

IS AMBIGUOUS!

→ DUE TO THE NEED TO DEFINE

THE SELF-LINKING # OF THE KNOT.

→ CORRESPONDING AMBIGUITIES

IN GROMOV-WITTEN THEORY

AND MIRROR B-MODEL.

M.A, A. KLEMM, & C. VAFA

C.C.M. LIU, S. KATZ

D-BRANES IN THE B-MODEL

↑

BOUNDARIES ON HOLOMORPHIC CYCLES

DIJKGRAAF & VAFA :

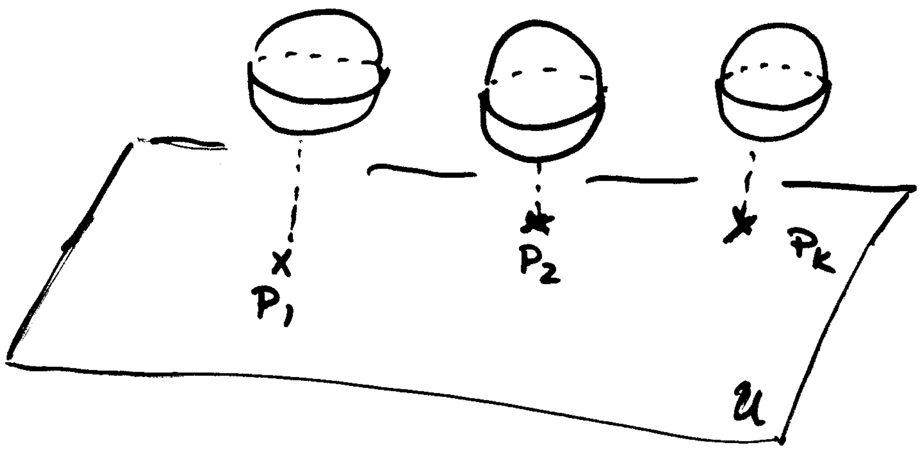
B-MODEL OF N D-BRANES ON

A $\mathbb{P}^1$ IS A MATRIX MODEL!

$Y =$ FIBRATION OF A_1 SINGULARITY OVER $\mathbb{C}$

$$xy = v^2 - \left(\frac{d}{du} W(u)\right)^2$$

$W(u) \to$ POLYNOMIAL IN u OF DEGREE $k+1$



$$\left.\frac{dW}{du}\right|_{P_i} = 0$$

GIVEN A COLLECTION OF

$$N = N_1 + \dots + N_k$$

D-BRANES, WHAT IS THE B-MODEL?

THE THEORY IS A MATRIX MODEL

$$Z = \frac{1}{\mathrm{vol}(U(N))} \int \mathcal{D}\phi \, \exp\left(-\mathrm{Tr}\, \frac{W(\phi)}{\lambda}\right)$$

$\Phi \rightarrow$ $N \times N$ HERMITIAN MATRIX

EIGENVALUES OF Φ:

WHERE THE D-BRANES ARE
IN THE u-PLANE

$\rightarrow$ CRITICAL POINTS ARE

$$W'(\phi) = 0$$

$\rightarrow$ HOLOMORPHIC CURVES.

RIBBON GRAPHS OF THE MATRIX MODEL ARE B-MODEL OPEN STRING DIAGRAMS

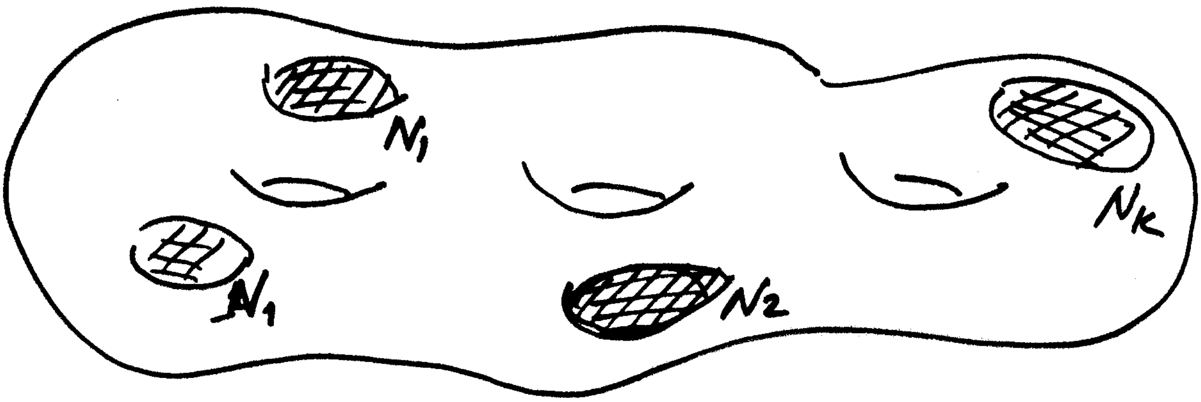
I.E.

$$\phi_{CRIT.} = \begin{pmatrix} \phi_1 \cdots \phi_1 \Big\} N_1 & & \\ & \ddots & \\ & & \phi_k \cdots \phi_k \Big\} N_k \end{pmatrix} \qquad W'(\phi_i) = 0$$

EXPANSION ABOUT ϕ_{CRIT} COMPUTES

$$Z = \exp\left(F(N_1, \ldots, N_k ; \Lambda) \right)$$

$$= \exp\Big(\sum_g \sum_{h_1 + \cdots + h_k = h} F_{g, h_1, \ldots, h_k} N^{h_1} \cdots N^{h_k} \Lambda^{2g-2+h} \Big)$$



$F_{g, h_1 \cdots h_k}$ - B MODEL FREE ENERGY WITH h_i HOLES ON i-th D-brane

SUMMING OVER HOLES:

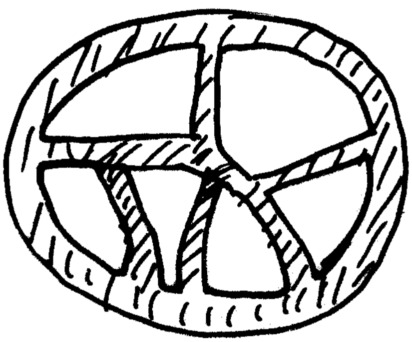
WHAT IS THE LARGE N DUAL GEOMETRY ?

(BASICALLY, QUESTION ABOUT GENUS Ø.)

$$\sum_{\{h_i\}} F_{g=0,h_1,\dots,h_k} (N_1\lambda)^{h_1}\cdots(N_k\lambda)^{h_k} \times \lambda^{-2} =$$

$$= F_{g=0}(t_1,\dots,t_k)\,\lambda^{-2}$$

→ THE SUM IS OVER PLANAR GRAPHS

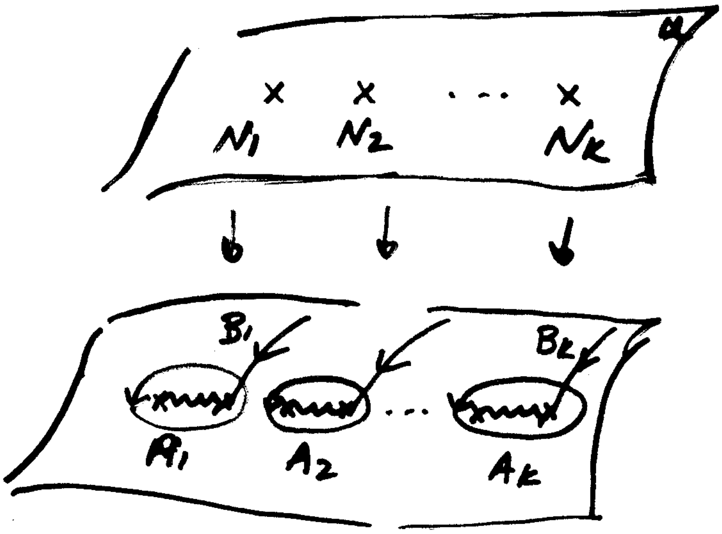


CLASSICAL $W'(\phi)=0$ SADDLE POINTS

OF

$$Z = \int \prod_i \mathcal{D}\phi_i \prod_{i<j} (\phi_i - \phi_j)^2 \, e^{-\sum_i \frac{W(\phi_i)}{\Lambda}}$$

SPREAD IN LARGE $N \to \infty$ LIMIT



$$xy = v^2 - (W'_{k+1}(u))^2$$

$$\downarrow$$

$$xy = v^2 - (W'_{k+1})^2 + f_k(u, t_i)$$

$$\Omega = \quad V(u)\,du$$

$\to$ GET A DISTRIBUTION OF EIGENVALUES

$$\rho(\phi)\, d\phi$$

$$\oint_{A_i} \rho(\phi)\, d\phi = N_i \Lambda = t_i$$

$$\oint_{B_i} \rho(\phi)\, d\phi = \partial_{t_i} F_0(t)$$

IN FACT:

TOPOLOGICAL A-MODEL OF

$$\hat{X} = \begin{array}{c} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

HAS A MATRIX MODEL DESCRIPTION:

CHERN SIMONS THEORY ON S^3

CAN BE WRITTEN AS A PARTICULAR

KIND OF MATRIX MODEL:

$$\underbrace{Z(\hat{X}) = \frac{1}{\mathrm{VOL}(U(N))} \int \mathcal{D}_H \phi \, e^{-\frac{\mathrm{Tr}\,\phi^2}{2\Lambda}}}$$

Φ – $N \times N$ HERMITIAN MATRIX

$\mathcal{D}_H \Phi \rightarrow$ MEASURE ON THE TANGENT SPACE OF $U(N)$-GROUP MANIFOLD

M.A., A. KLEMM,

M. MARINO & C. VAFA

THIS ALSO FOLLOWS FROM <u>MIRROR SYMMETRY</u>

$$\begin{tikzcd}
X = T^*S^3 \arrow[r] \arrow[d] & Y \arrow[d] \\
\hat{X} = \mathcal{O}(-1)\oplus\mathcal{O}(-1) \arrow[r] \arrow[d] & \hat{Y} \\
\mathbb{P}^1 &
\end{tikzcd}$$

$$\hat{Y}: \quad xy = (e^{v}-1)(e^{v+u}-1) + e^{-t} - 1$$

$$\uparrow$$

$$Y: \quad xy = (e^{v}-1)(e^{v+u}-1)$$ (i.e SMALL RESOLUTION OF THIS)

~~LOCAL VERSION:~~

N D-BRANES ON S^3 IN $X = T^*S^3$ $\xleftrightarrow[\text{SYMMETRY}]{\text{MIRROR}}$ N D-BRANES ON $\mathbb{P}^1$ IN Y

IN A LOCAL LIMIT

$\longrightarrow$ D-V MATRIX MODEL W/ $k=1$

$$Z = \frac{1}{\mathrm{Vol}(U(N))} \int \mathcal{D}\phi_H \, e^{-\frac{\mathrm{Tr}\,\phi^2}{2\lambda}}$$

$$\downarrow$$

$$\mathcal{D}\phi$$

$$xy = (e^{v}-1)(e^{u+v}-1) + e^{-t} - 1$$

$$\downarrow$$

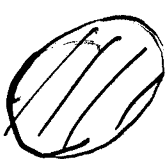
$$xy \approx v(u+v) + t$$

ADDING D-BRANES :

$$\hat{X} = \begin{array}{c} \mathcal{O}(-1)\oplus \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

WITH NON-COMPACT D-BRANE ON L

$L \sim \mathbb{R}^2\times S^1$

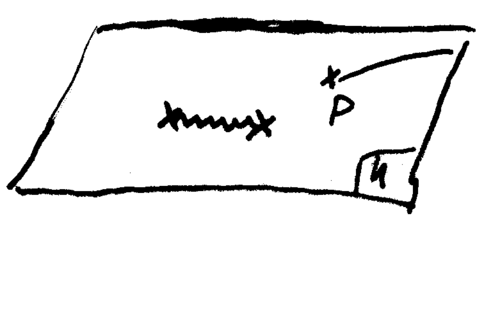


$$Z(\hat{X}, V) = \frac{1}{\mathrm{vol}(U(N))}\int \mathcal{D}_H\phi \; e^{-\frac{\mathrm{Tr}\,\phi^2}{2\lambda}} \sum_R \mathrm{Tr}_R\, e^{\Phi}\, \mathrm{Tr}_R V$$

$V =$

→ DISK AMPLITUDE IS ABEL-JACOBI MAP ON MIRROR RIEMANN SURFACE

M.A., C.VAFA



$$F_{0,1} = \int^{p} \rho(u)\, du$$

$$V = e^{-u_p}$$

LARGE N DUALITIES PROVIDE
A WAY TO CALCULATE
TOPOLOGICAL A-MODEL AMPLITUDES
TO ALL GENERA, EXACTLY

TO EXPLAIN THIS, ONE
NEEDS SOME RUDIMENTS
OF (TORIC) GEOMETRY.

M.A., M. MARINO, C. VAFA

CONSIDER $T^2 \times \mathbb{R}$ FIBRATION

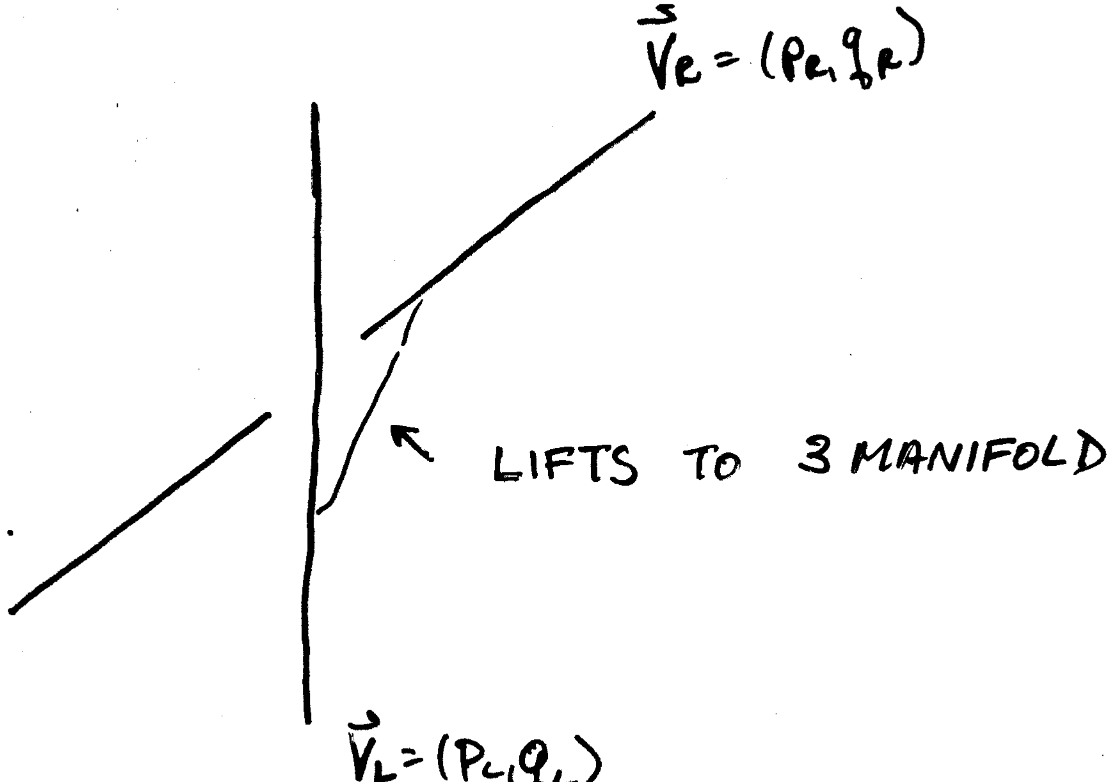
OVER $\mathbb{R}^3$

WHERE OVER LINES IN BASE

$\vec{V}_i = (p_i, q_i)$ 1-CYCLES

OF T^2 DEGENERATE.

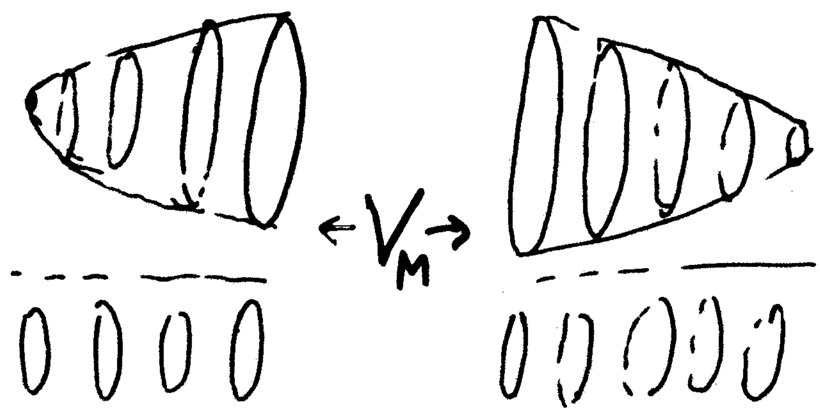
E.G.



THE THREE-MANIFOLD IN QUESTION

CAN BE OBTAINED BY

GLUING TOGETHER SOLID 2-TORI



BY AN $SL(2,\mathbb{Z})$ TRANSFORMATION

OF THEIR BOUNDARY

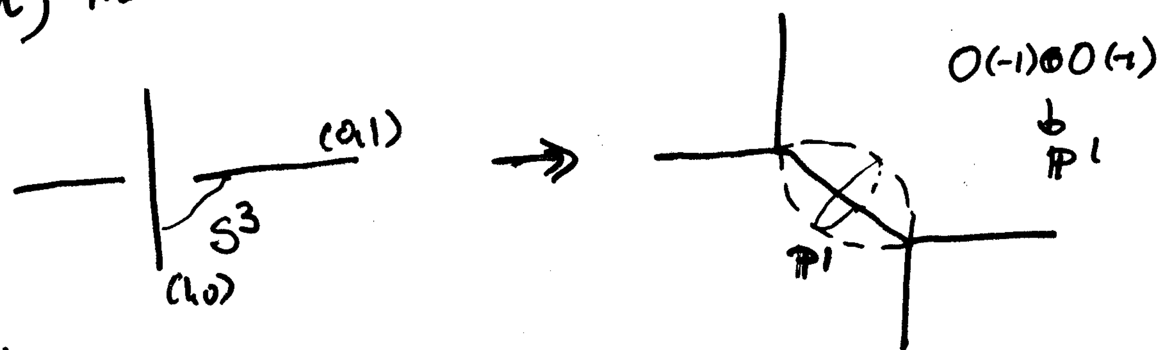
$$V_M: \begin{pmatrix} p_L \\ q_L \end{pmatrix} \to \begin{pmatrix} p_R \\ q_R \end{pmatrix}$$

IF $|\mathrm{DET}(V_M)| = m \Rightarrow S^3/\mathbb{Z}_m$

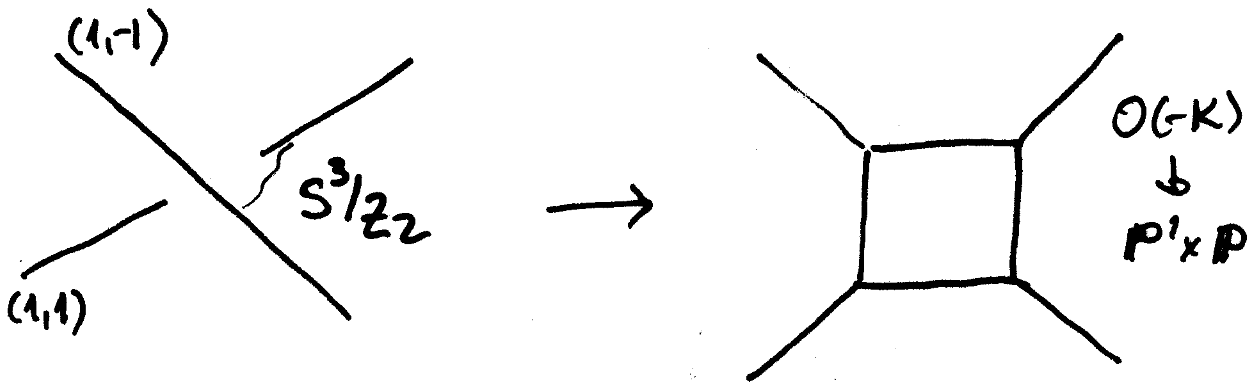
THIS HAS A GEOMETRIC TRANSITION WHEREBY

$$T^*(S^3/\mathbb{Z}_m) \to \begin{array}{c} A_{m-1} \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

E.g. i) $m=1$



ii) $m=2$



EXACT AMPLITUDE FOR $\begin{matrix} A_{m-1} \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$:

$T^*(S^3/\mathbb{Z}_m)$ HAS A

MATRIX MODEL DESCRIPTION

$$Z = \frac{1}{\mathrm{Vol}(U(N))} \int \mathcal{D}_H \phi \, \exp\left(-\frac{m \, \mathrm{Tr}\, \phi^2}{2\lambda}\right)$$

$\sin(\phi_i - \phi_j)^2$

→ HAS m VACUA

BECAUSE OF PERIODICITY OF MEASURE

→ HOW MANY D-BRANES IN EACH OF m VACUA

→ m CLASSES OF $\begin{matrix} A_{m-1} \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$

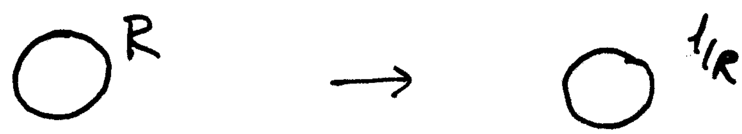
IN FACT,

VIEWED THIS WAY, IT IS

EASY TO MOTIVATE APEARANCE

OF MATRIX MODEL.

MIRROR SYMMETRY



U(N) BUNDLE ON R $\xrightarrow{\text{MODULI}}$ SPACE OF FLAT CONNECTIONS

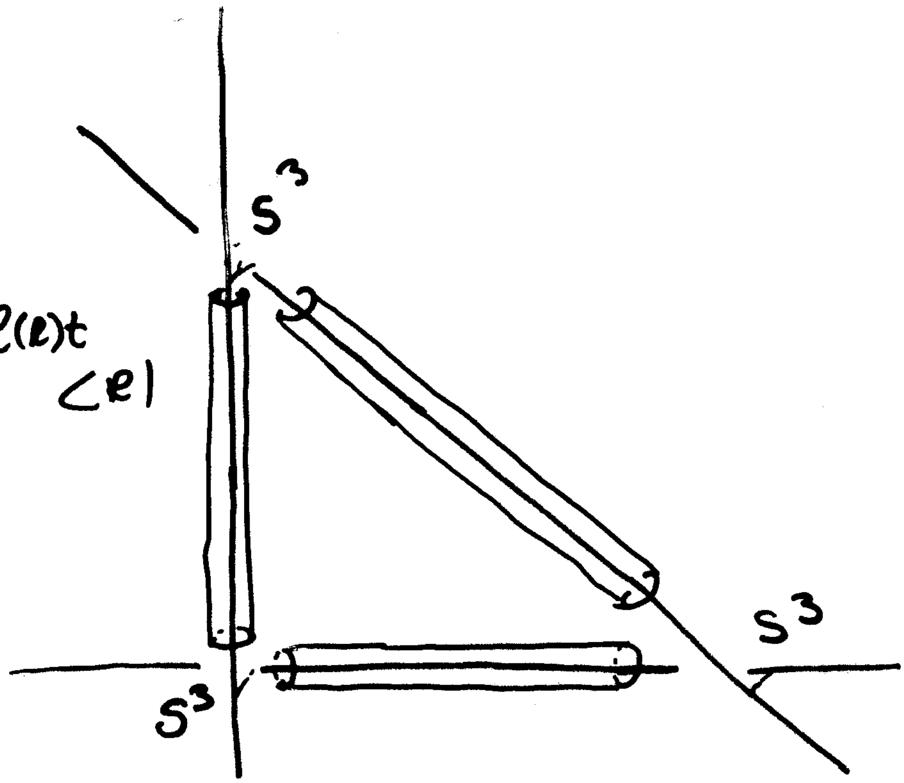
N POINTS ON DUAL CIRCLE

$$\Phi = \oint_{S^1} A$$

MORE EXAMPLES:

S^3 ON EACH NODE

$$\sum_R |R\rangle e^{-\ell(R)t} \langle R|$$

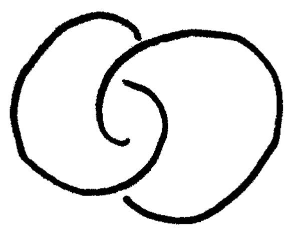


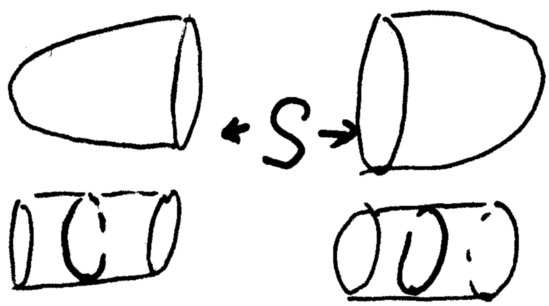
CONNECTED BY HOLOMORPHIC ANNULI.

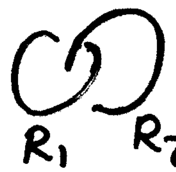
IN FACT, THE BOUNDARIES OF ANNULI ON EACH S^3

$\downarrow$

HOPF LINKS





$$Z = \sum_{R_1, R_2, R_3} Z\Big(S^3,$$


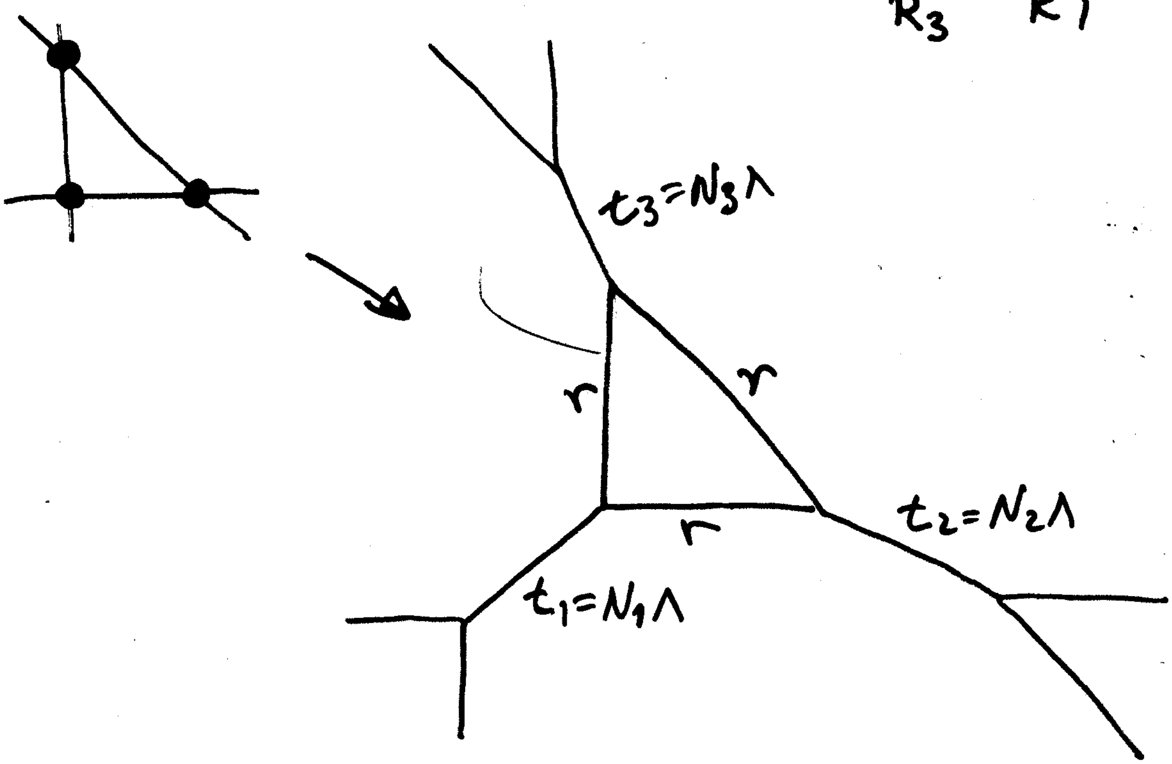
$$\Big)$$

$$\times Z\Big(S^3,$$


$$\Big) \times e^{-\sum_i \ell(R_i) r\xi}$$

$$\times Z\Big(S^3,$$


$$\Big)$$



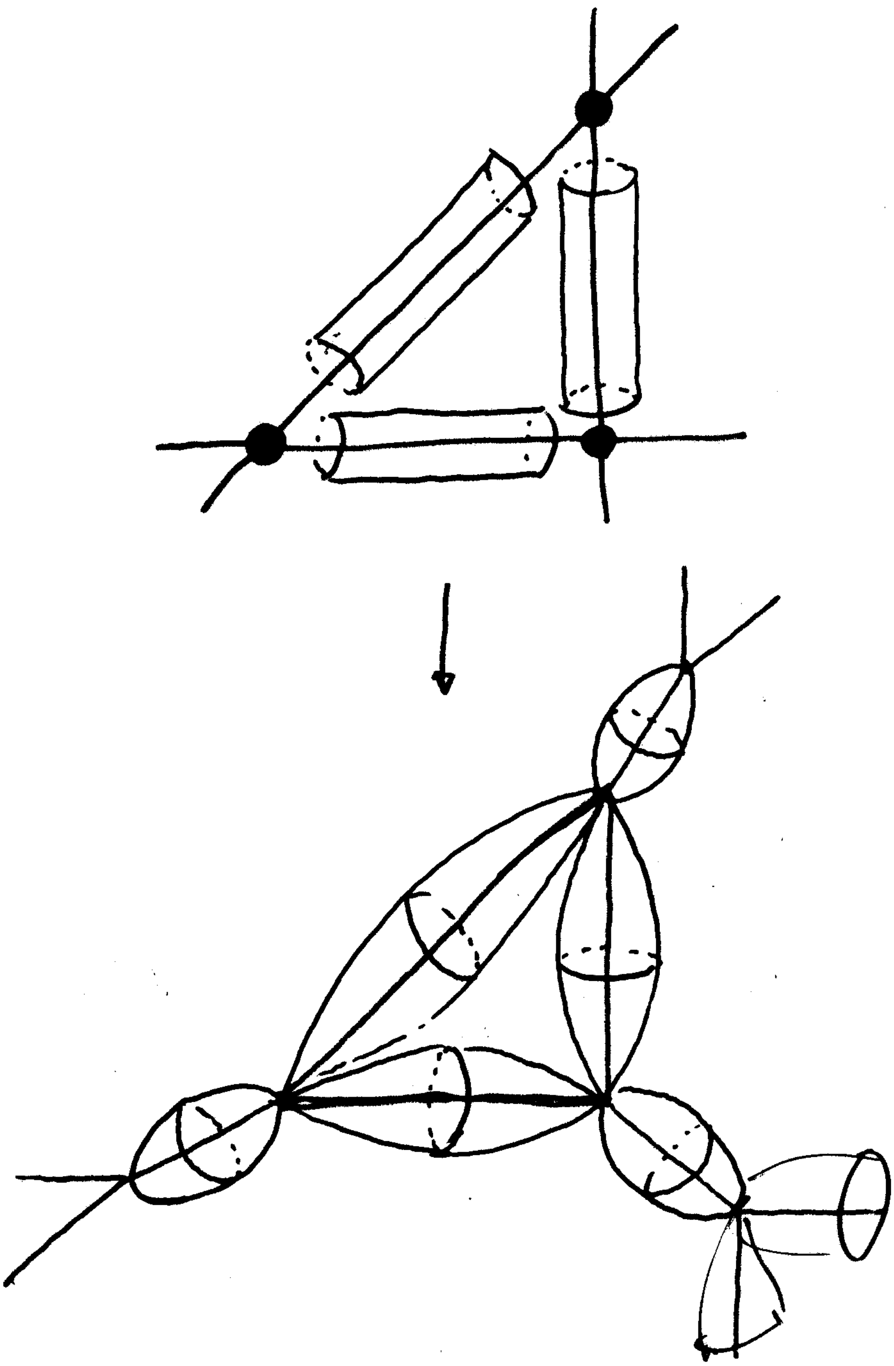
g	$d=1$	2	3	4	5	6	7	8	9
0	3	−6	27	−192	1695	−17064	188454	−2228160	27748899
1	0	0	−10	231	−4452	80948	−1438086	25301295	−443384578
2	0	0	0	−102	5430	−194022	5784837	−155322234	3894455457
3	0	0	0	15	−3672	290853	−15363990	649358826	−23769907110
4	0	0	0	0	1386	−290400	29056614	−2003386626	109496290149
5	0	0	0	0	−270	196857	−40492272	4741754985	−396521732268
6	0	0	0	0	21	−90390	42297741	−8802201084	1156156082181
7	0	0	0	0	0	27538	−33388020	12991744968	−2756768768616
8	0	0	0	0	0	−5310	19956294	−15382690248	5434042220973
9	0	0	0	0	0	585	−9001908	14696175789	−8925467876838
10	0	0	0	0	0	−28	3035271	−11368277886	12289618988434
11	0	0	0	0	0	0	−751218	7130565654	−14251504205448
12	0	0	0	0	0	0	132201	−3624105918	13968129299517
13	0	0	0	0	0	0	−15636	1487970738	−11600960414160
14	0	0	0	0	0	0	1113	−490564242	8178041540439
15	0	0	0	0	0	0	−36	128595720	−4896802729542
16	0	0	0	0	0	0	0	−26398788	2489687953666
17	0	0	0	0	0	0	0	4146627	−1073258752968
18	0	0	0	0	0	0	0	−480636	391168899747
19	0	0	0	0	0	0	0	38703	−120003463932
20	0	0	0	0	0	0	0	−1932	30788199027
21	0	0	0	0	0	0	0	45	−6546191256
22	0	0	0	0	0	0	0	0	1138978170
23	0	0	0	0	0	0	0	0	−159318126
24	0	0	0	0	0	0	0	0	17465232
25	0	0	0	0	0	0	0	0	−1444132
26	0	0	0	0	0	0	0	0	84636
27	0	0	0	0	0	0	0	0	−3132
28	0	0	0	0	0	0	0	0	55

Table 1: The integral invariants n^g_d for the local $\mathbb{P}^2$ case.

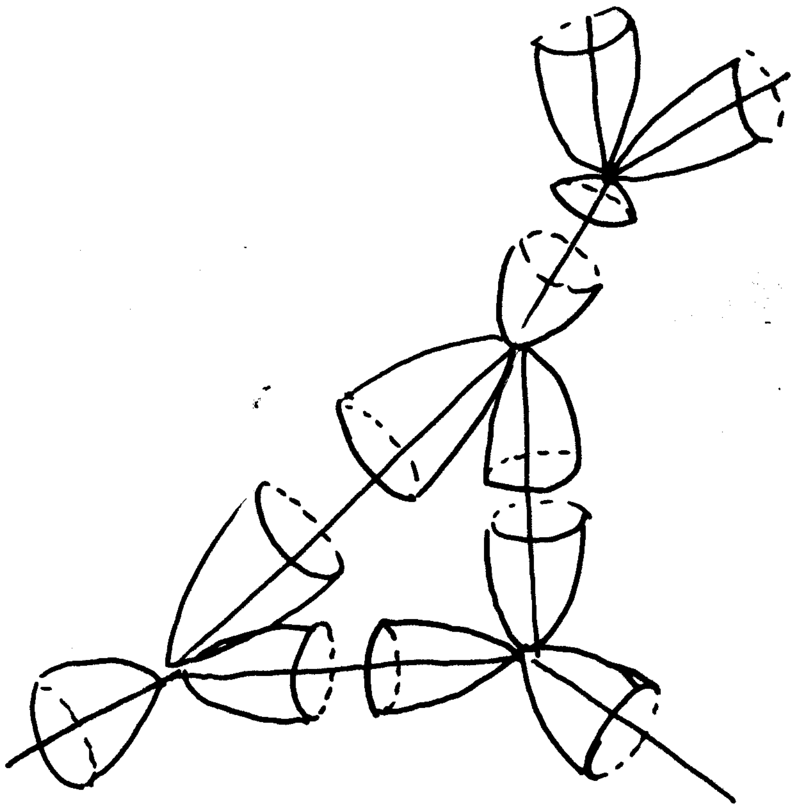
g	$d=10$	11	12
0	−360012150	4827935937	−66537713520
1	7760515332	−135854179422	2380305803719
2	−93050366010	2145146041119	−48109281322212
3	786400843911	−24130293606924	698473748830878
4	−5094944994204	210503102300868	−7935125096754762
5	26383404443193	−1485630816648252	73613315148586317
6	−111935744536416	8698748079113310	−572001241783007370
7	395499033672279	−42968546119317066	3786284014554551293
8	−1177301126712306	181202644392392127	−21609631514881755756
9	2978210177817558	−658244675887405242	107311593188998164015
10	−6445913624274390	2074294284130247058	−466990545582708577390
11	12001782164043306	−5702866358492557440	1791208287019324701495
12	−19310842755095748	13744538465609779287	−8085017394087513680618
13	26952467292328782	−29157942375100015002	183346122778010358924791
14	−32736035592797946	54641056077839878893	−495787827778769125835658
15	34693175820656421	−90735478019244786786	119723947998685791289164
16	−32151370513161966	133885726253316076984	−259634731498425150837576
17	26099440805196660	−175976406401479949154	506961721474582218552270
18	−18580932613650624	206477591201198965488	−893407075206205808615238
19	11609627766170547	−216671841840838260606	1424048002136300951108030
20	−6367395873587820	203674311322868998065	−2057099617415644933602618
21	3064262549419899	−171730940091766865658	2697839037217627321703085
22	−1292593922494452	130015073789764141299	−3217397468483821476968358
23	477101143946277	−88451172530198637924	3494176460021369389735746
24	−153692555590206	54098277648908454123	−3460084190968494003073062
25	43057471189239	−29751302949160261398	3127576636374963802648718
26	−10441089412308	14709694749741501501	−2582938330708242629937150
27	2177999212647	−6535189635435373326	1950461493734929553600580
28	−387688567518	2606677300588276035	−1347524558332336039964082
29	58269383541	−932238829973577348	852109374825775079556606
30	−7292193288	298408032566091294	−493309207337689509893062
31	745600245	−85297647759486510	261477149328500781917776
32	−60650490	21708810999461607	−126876156355185161374314
33	3773652	−4901354114590566	56339101711825399890960

Table 2: The integral invariants n_d^g for the local $\mathbb{P}^2$ case (continuation).

NOTE:



MORE NATURALLY:



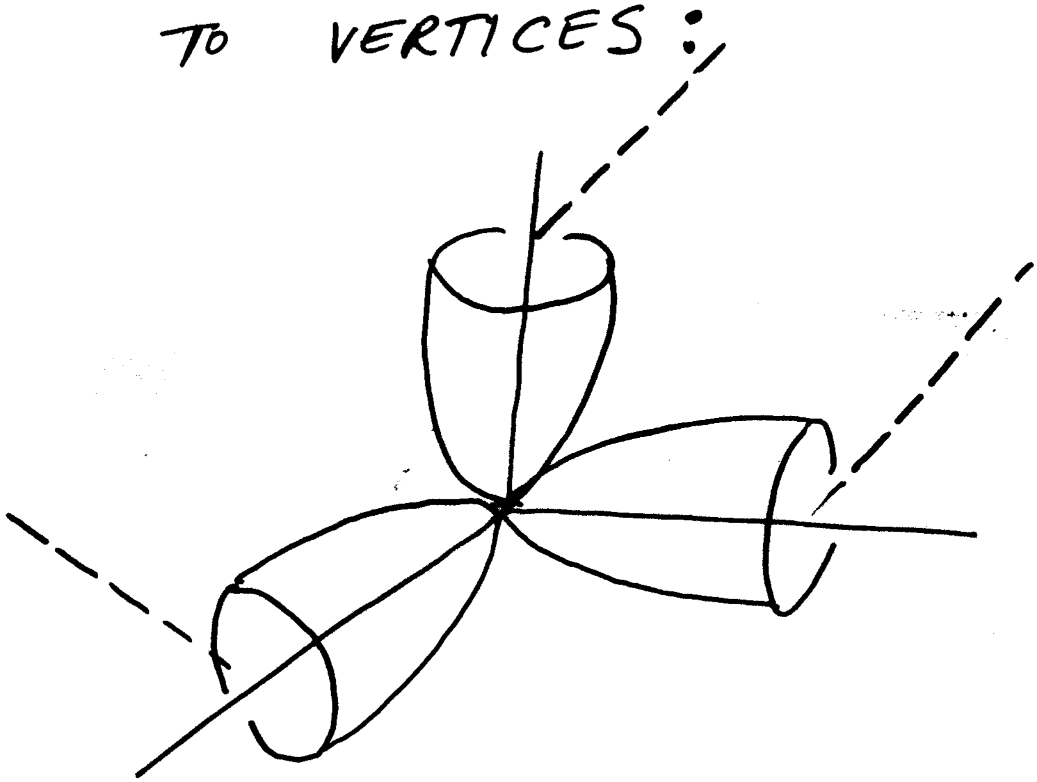
THERE ARE MORE ELEMENTARY

PIECES...

M. A., A. KLEMM, M. MARINO
& C. VAFA.

OBTAINED BY CUTTING THE AMPLITUDE

TO VERTICES:

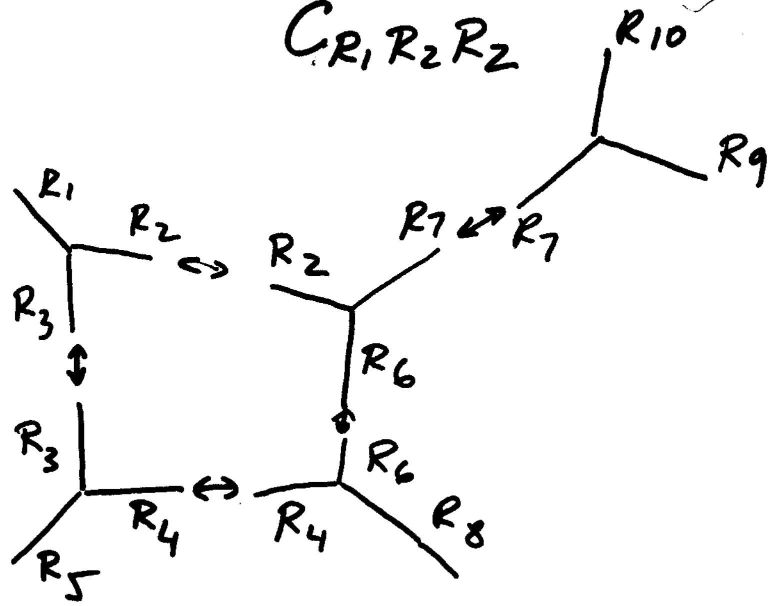


BY D-BRANES

$$= \sum_{R_1, R_2, R_3} C_{R_1 R_2 R_3}(g_s) \, \mathrm{Tr}_{R_1} V^{(1)} \, \mathrm{Tr}_{R_2} V^{(2)} \, \mathrm{Tr}_{R_3} V^{(}$$

ALL GENUS AMPLITUDES FOR ANY TORIC GEOMETRY ARE OBTAINED FROM

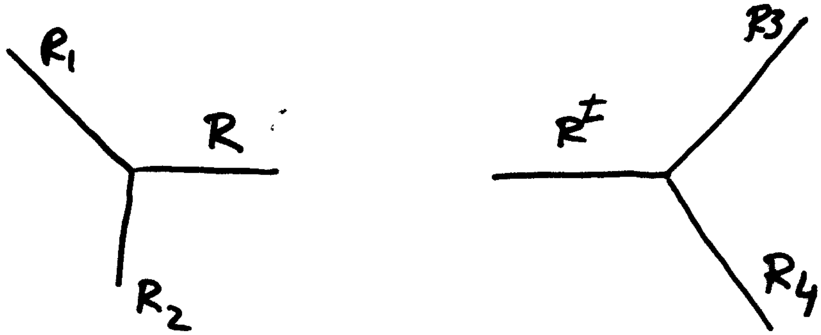
$C_{R_1 R_2 R_2}$



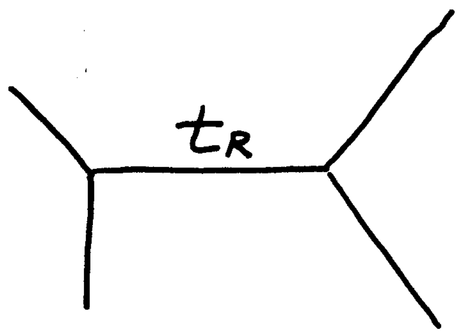
WHERE KAHLER MODULI

ARE LENGTHS OF

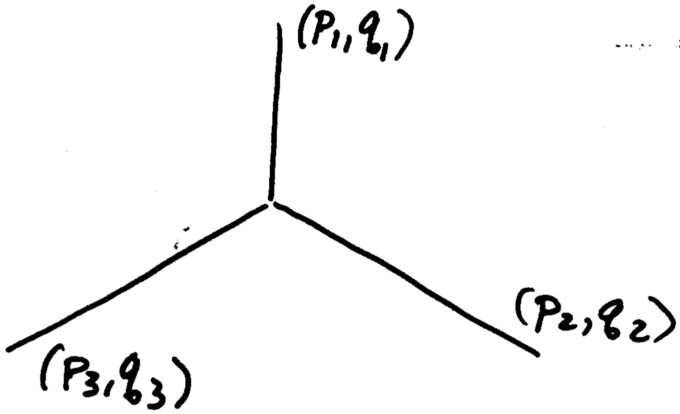
PROPAGATORS



$$\sum_R |R\rangle \, e^{-\ell(R)\, t_R} \langle R^t|$$



AND (p,q) LABELS OF VERTEX



ARE RELATED TO FRAMING

$$C^{(n_1,n_2,n_3)}_{R_1 R_2 R_3} = C^{(0,0,0)}_{R_1 R_2 R_3}\, q^{\sum_i \frac{k_{R_i}}{2} n_i}$$

$k_{R_i} \sim$ quadratic Casimir of R_i

$q = e^{\lambda}$