Instantons & Lagrangians in Manifolds w/ V.C.P. [Vector Cross Product].

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Joint work with JaeHyouk Lee.

- Vector Cross Product (V.C.P.)
  - Kähler / Symplectic Manifold

    $\partial \begin{pmatrix} \text{Holomorphic} \\ \text{Curve} \end{pmatrix} \subset$ Lagrangian Submanifold.

  - G_2 - manifold

    $\partial \begin{pmatrix} \text{Associative} \\ \text{submanifold} \end{pmatrix} \subset$ Coassociative submanifold.

- $\mathbb{C}$ - V. C. P.
  - Calabi - Yau Manifold.

    $\partial (SLag_\theta) \subset$ Complex Hypersurface and $SLag_{\theta + \pi/2}$.

  - Hyperkähler Manifold.

    $\partial \begin{pmatrix} \text{Holomorphic} \\ \text{Curve} \end{pmatrix} \subset$ Complex Lagrangian submanifold.

§ Symplectic Geometry [Review].

(M^{2n}, ω) $\qquad \omega \in \Omega^2(M)$.

$$\begin{cases} d\omega = 0 \\ \omega > 0. \end{cases}$$

Example. $\quad M = \mathbb{C}^n$

$$\omega = dx^1 \wedge dy^1 + \cdots\cdots + dx^n \wedge dy^n.$$

metric

$$g = (dx^1)^2 + (dy^1)^2 + \cdots\cdots + (dx^n)^2 + (dy^n)^2$$

complex str.

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j} \qquad \forall j.$$

complex structure.
(1-fold V.C.P.)

$\omega \in \Omega^2(M)$ $\qquad\qquad$ $J : T_M \to T_M, \; J^2 = -1$

via metric

$\underline{\omega(u, v) = g(Ju, v).}$

(M, ω, J, g) $\qquad$ Almost Kähler mfd.

$$(M^{2n}, g, \omega, J)$$

Definition: $A^2 \subset M$

<u>Holomorphic Curve</u> / <u>Instanton</u>

if A preserved by J.

$\Big[\underset{\text{Wirtinger}}{\Longleftrightarrow}$ A calibrated by ω, i.e. $\omega|_A = \nu_A \Big]$

$\Big[\Longrightarrow$ Absolute minimum area. $\Big]$

$$\mathrm{Area}(A) = \int_A [\omega]. \qquad \text{(topological)}$$

Remark: Boundary value problem

$$\partial A \subset C \subset M$$

$$\omega|_C = 0.$$

$$(M^{2n}, g, \omega, J)$$

Definition: $C^n \subset M^{2n}$

<u>Lagrangian</u> submanifold if

$$\omega|_C = 0$$

$$\dim C = \frac{1}{2} \dim M.$$

Remark:

$\#\{\text{instantons}\; A^2 \subset M\}$

(i) $\partial A = \phi$ $\rightsquigarrow$ Gromov-Witten Invariants

(ii) $\partial A \subset C$ $\rightsquigarrow$ Fukaya-Floer Category.

§. Vector Cross Product.

Example : $(\mathbb{R}^3, \times)$.
Example : (M^{2n}, J).

Definition: (M^m, g) Riemannian manifold.
[Gray].

$$\chi : \Lambda^r TM \longrightarrow TM.$$ r-fold. VCP

if. (i). $\chi(v_1, \ldots\ldots, v_r) \perp v_1$

(ii) $v_1, \ldots\ldots, v_r$: orthonormal
$\Rightarrow |\chi(v_1, \ldots\ldots, v_r)| = 1$

(i)$\Leftrightarrow$(i)' $\varphi(v_1, \ldots\ldots, v_r, v_{r+1}) \triangleq \langle \chi(v_1, \ldots, v_r), v_{r+1} \rangle$.

then $\varphi \in \Omega^{r+1}(M)$.

(iii) $d\varphi = 0$.

[$(\widetilde{\text{iii}})$ $\nabla \varphi = 0$ Integrable VCP]

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(M^m, g) w/ r-fold VCP.

$$\varphi \in \Omega^{r+1}(M) \longleftrightarrow \chi : \Lambda^r T_M \to T_M$$

$$\underline{\varphi(v_1, \ldots, v_r, v_{r+1}) = \langle \chi(v_1, \ldots, v_r),\ v_{r+1} \rangle.}$$

Def$^{\underline{n}}$. $A^{r+1} \subset M^m$ <u>instanton</u>

if A preserved by χ

$\left[\overset{\text{Lemma.}}{\Longleftrightarrow} \right.$ A calibrated by $\left. \varphi \right]$

Def$^{\underline{n}}$. $C^k \subset M^m$ <u>Lagrangian</u>

if $\begin{cases} \varphi|_C = 0 \\ \dim C = k = \dfrac{n+r-1}{2}. \end{cases}$

Remark : $\dim C > \frac{n+r-1}{2} \Rightarrow \varphi|_C \neq 0.$

§ (Unparametrized) Loop Space Interpretation.

$$(M\,,\,g) \qquad \varphi \in \Omega^{r+1}(M).$$

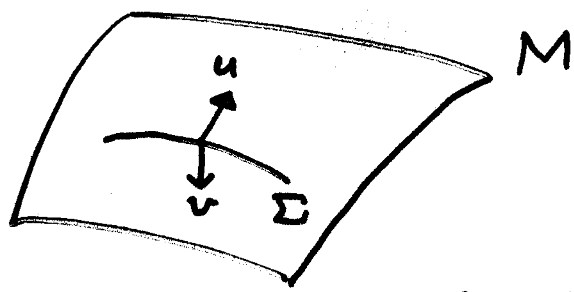
Fix ANY. Σ^{r-1}

$$\mathcal{L}_\Sigma M \triangleq \mathrm{Map}(\Sigma\,,M)_{\mathrm{embed}} \big/ _{\mathrm{Diff}^+(\Sigma)}.$$

Transgression $\longrightarrow$

$$\omega_{\mathcal{L}_\Sigma M} = \int_\Sigma \varphi \quad \in \Omega^2(\mathcal{L}_\Sigma M).$$

That is,



$u, v \in \Gamma(\mathcal{N}_{\Sigma/M}).$

$$\omega_{\mathcal{L}_\Sigma M}(u\,,v)$$

$$= \int_\Sigma \iota_{u\wedge v}\varphi$$

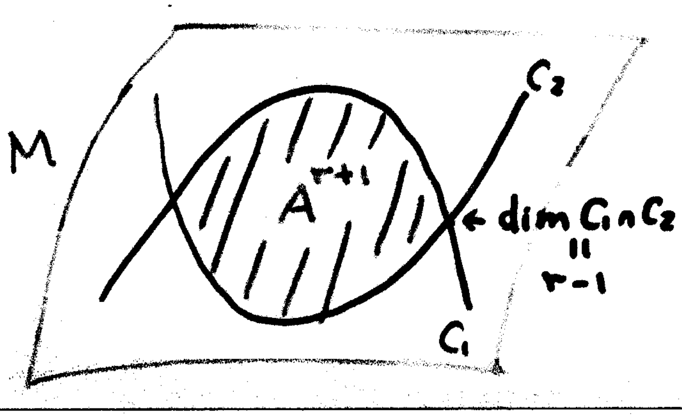
Theorem. (M^n, g) $\varphi \in \Omega^{r+1}(M)$

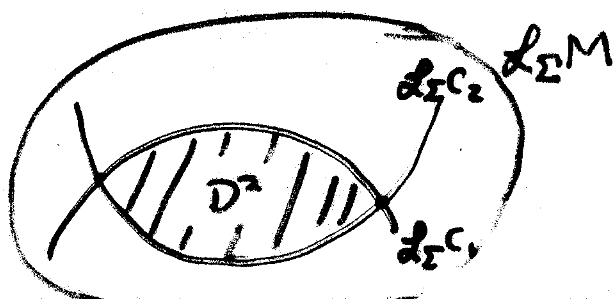
$\rightsquigarrow$ $\omega_{\mathcal{L}_\Sigma M} \in \Omega^2(\mathcal{L}_\Sigma M)$.

(1) φ : r-fold VCP on M $\iff$ $\omega_{\mathcal{L}_\Sigma M}$: 1-fold VCP on $\mathcal{L}_\Sigma M$. (i.e. Symplectic).

(2) $C \subset M$ φ-Lagrangian $(\dim C = (n+r-1)/2)$. $\iff$ $\mathcal{L}_\Sigma C \subset \mathcal{L}_\Sigma M$ Lagrangian

(3) $\underset{\underset{A^{r+1}}{\|}}{D^2 \times \Sigma} \subset M$ instanton $\iff$ $D^2 \subset \mathcal{L}_\Sigma M$ instanton (i.e. holomorphic curve).





§ Examples / Classification.

1° Kähler / Symplectic Manifolds.

$(r = 1)$.

$$(M^{2m}, \chi, \varphi) = (M, \underset{\substack{\text{cpx.}\\ \text{str.}}}{J}, \underset{\substack{\text{sympl.}\\ \text{form.}}}{\omega})$$

- Instanton
- Lagrangian

(usual def^{n}.).

2° Volume form $(r = n-1)$

(M^n, g) $\qquad \varphi = \nu_M \in \Omega^n(M)$

$$= \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

- Instantons $A^n \underset{\text{open}}{\subseteq} M^n$ (domain in M)

- Lagrangians $C^{n-1} \subset M^n$
  $\quad\|$
  Hypersurfaces

3° G_2-manifold $(r=2)$.

(M^7, g) $\qquad \varphi =: \Omega \in \Omega^3(M^7)$

e.g. $M^7 = X^6 \times S^1$

Calabi-Yau 3-fold.

$$\Omega_M = \mathrm{Re}\,\Omega_X + \omega_X \wedge d\theta$$

When $M^7 = \mathrm{Im}\,\mathbb{O}$

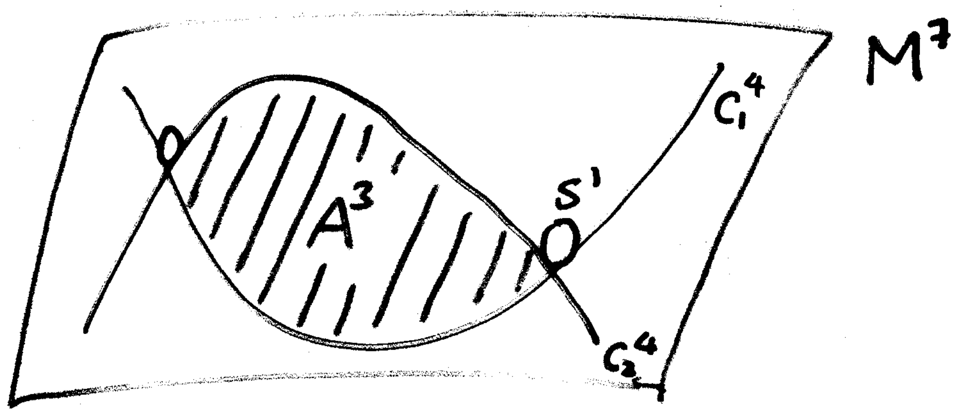
$$\chi(u, v) =: u \times v$$
$$= \mathrm{Im}(u \cdot v)$$

(M^7, g, Ω) : G_2-mfd.

$$\Omega(u,v,w) = g(u \times v, w).$$

- Instanton $\quad A^3 \subset M \quad$ preserved by $\times$
  $\|$ $\quad$ (Calib. by Ω).
  Associative

- Lagrangian $\quad C^4 \subset M \quad$ $\Omega|_C = 0$
  $\|$ $\quad$ (calib. by $*\Omega$).
  Coassociative.



Remark: Count A^3.
When $C_1 \sim C_2$. $\quad$ (L. - X.W. Wang).

$\# A^3$

$\approx \#$ holo. curves in C_1^4 $\quad$ (+ bubbles).

$\overset{\text{Taubes}}{\approx}$ Seiberg-Witten Inv. of C_1.

4°. Spin(7)-manifold M^8. $(r=3)$

[eg. $M^8 = \underset{G_2\text{-mfd.}}{Z^7} \times S^1$

$\varphi_M = \Omega_Z \wedge d\theta + *\Omega_Z.$

Example: $M = \mathbb{O}$

$$u \times v \times w = \chi(u, v, w) = \frac{1}{2}[u(\bar{v}w) - w(\bar{v}u)]$$

- Instanton $=$ Cayley $A^4 \subset M^8$ preserved by $\times$ (calib. by φ).

- Proposition: $\nexists$ φ-Lagrangian in any Spin(7)-manifold.

Remark: No other VCP — classification by Brown-Gray.

In fact, we can also allow

$\underline{r = 0}$. i.e.

(M^n, g) $\quad \varphi \in \Omega^1(M)$

$$\begin{cases} d\varphi = 0 \\ |\varphi| = 1. \end{cases}$$

Eg.

$f : M \longrightarrow S^1$ Riemannian submension.

$$\varphi = f^*(d\theta).$$

- Instanton $A^1 \subset M$
  $\|$
  Gradient Flow Line.

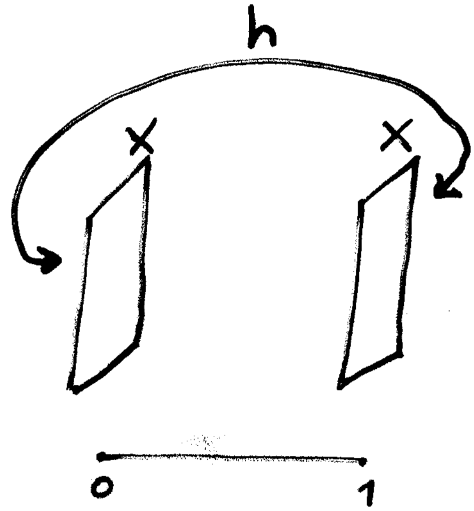
- Lagrangian ?

$$f : M^n \longrightarrow S^1$$

$$\Longleftrightarrow \quad M = X \times [0,1]/\sim$$

$$h : X \xrightarrow[\text{isometry}]{} X$$

(Mapping Cylinder).

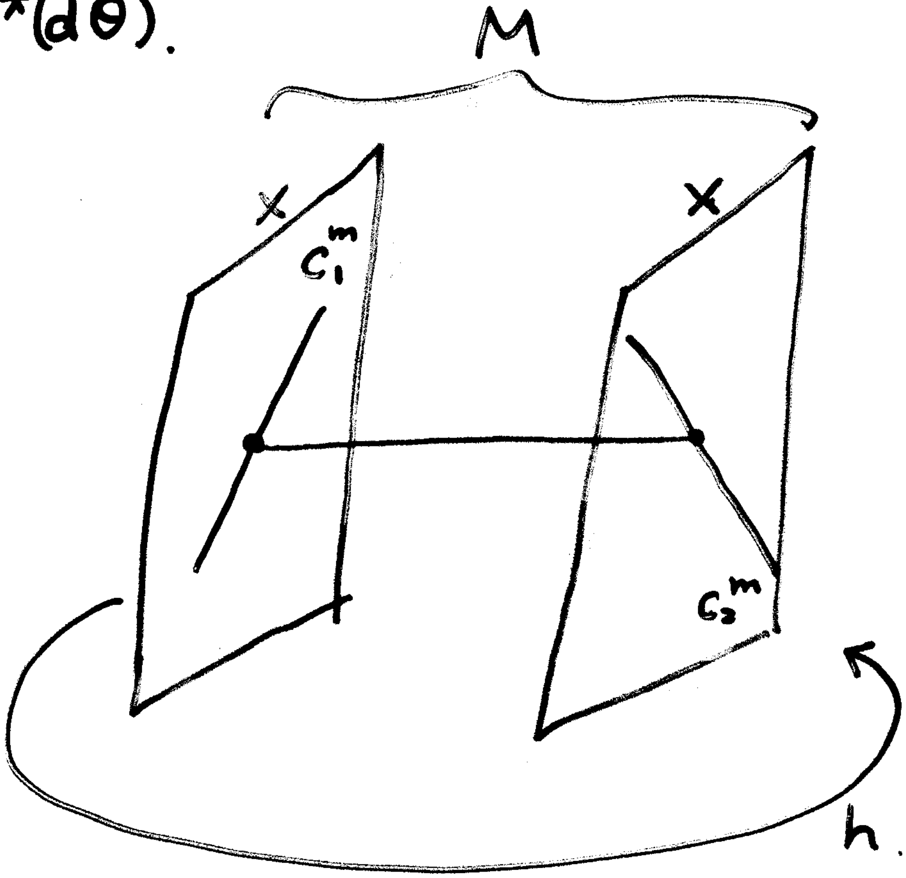


- Lagrangian.

$$C^m \subset X^{2m} \times \{t\} \quad \exists t.$$

[$r=0$ Continue ....].

$$\begin{array}{ccc} M^{2m+1} & & \varphi \in \Omega^1(M). \\ {\scriptstyle f}\big\downarrow & & \| \\ S^1 & & f^*(d\theta). \end{array}$$



Instantons bounding $c_1 \cup c_2$

$$= \sum_{k=-\infty}^{\infty} \left(\# c_1 \cap h^k(c_2) \right) t^k.$$

Remark: (M^n, g), $\varphi \in \Omega^{r+1}(M)$
VCP BUT $\boxed{d\varphi \neq 0}$

Assume

(*) $$| \iota_{u_1 \wedge \cdots \wedge u_{r+1}}(d\varphi)|^2 = 1 - |\varphi(u_1, \dots, u_{r+1})|^2$$

$\forall$ orthormal $u_1, \dots, u_{r+1}$.

- (*) $\Rightarrow$ $\mathbb{R}_+ \times M$ (cone).
  (r+1)-fold V.C.P.

  $\tilde{\varphi} = dt \wedge \varphi + t \wedge d\varphi$, $d\tilde{\varphi} = 0$.

  $\tilde{g} = t^2 g_M + dt^2$.

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r	M w/ (*)
0	Contact
1	Nearly C.Y. 3-fold.
2	Nearly G_2-mfd.
$n-1$	Volume form.

Remark (Continue)

(M^{2m+1}, g) $\quad \varphi \in \Omega^1(M)$ $\quad \boxed{r=0}$

$d\varphi$ satisfies $(*)$

$\Rightarrow M$: Contact.

- Instanton $\quad A^1 \subset M$

- Lagrangian $\quad C^m \subset M^{2m+1}$

  $\|$

  Legendrian

§ Complex V.C.P.

Definition: (M^{2n}, g, J). Kähler.

$$\varphi \in \Omega^{r+1,0}(M), \quad d\varphi = 0.$$

satisfying

$$|\, \iota_{u_1 \wedge \dots \wedge u_r} \varphi \,| = 1$$

for any orthonormal $u_1, \dots, u_r \in T^{1,0}M$.

is called a Complex Vector Cross Product.

Classification Theorem (Lee-L.)

(M^{2n}, g, J), $\varphi \in \Omega^{r+1,0}(M)$

r-fold $\otimes$. $\mathbb{C}$. V.C.P.

$\Rightarrow$ (1). Calabi-Yau. $(r = n-1)$.

φ = holomorphic Volume form.

(2) Hyperkähler $(r = 1)$.

φ = holomorphic Symplectic form.

Remark: $\nexists$ $\mathbb{C}$-analog. of G_2 or $Spin(7)$ type V.C.P.

Remark: $d\varphi = 0 \Leftrightarrow \nabla\varphi = 0$ for $\mathbb{C}$-VCP.

(M^{2n}, J, g), $\varphi \in \Omega^{r+1,0}(M)$.
$\mathbb{C}$VCP.

<u>Def$^{\underline{n}}$</u> (i) $A^{r+1} \subset M$ <u>Instanton</u>
if Calibrated by $\mathrm{Re}(e^{i\theta}\varphi)$
for some 'phase' θ.

(ii) $C^{2k} \subset M^{2n}$ <u>Lagrangian</u>
$k = (n+r-1)/2$ (of cx. type).
if $\varphi|_C = 0$.

Remark: $C \subset M$ Lagr. $\Rightarrow$ complex submfd.

Def. <u>Lagrangian</u> of <u>Real Type</u>

$$(M^{2n}, g, J)$$

$$\omega \in \Omega^2(M) \qquad \varphi \in \Omega^{r+1,0}(M)$$

$C^n \subset M^{2n}$.	$C^{2k} \subset M^{2n}$
(i) $\omega\|_C = 0$	$\varphi\|_C = 0$
(ii) $\mathrm{Re}(e^{i\theta}\varphi)\|_C = 0$	
real type	complex type

Remark: Both are good boundary value for Instantons
(Calib. by $\mathrm{Re}(e^{i\theta}\varphi)$).

Example $[r=1]$: <u>Hyperkähler</u>.

$$(M^{4m}, g, J, \omega = \omega_J)$$

$$\underset{\omega_I + i\,\omega_K.}{\underset{\parallel}{\varphi}} \in \Omega^{2,0}(M)$$

- Instanton. $A^2 \subset M$ calib. by $Re(e^{i\theta}\varphi)$.

  ||

  J_θ-holomorphic curve.

  $J_\theta = \cos\theta\, I + \sin\theta\, K$.

- Lagr. of Cx. Type. $C^{2n} \subset M^{4n}$. $\varphi|_C = 0$

  ||

  J- Complex Lagr.

- Lagr. of Real Type. $C^{2n} \subset M^{4n}$ $\omega|_C = 0$, $Re(e^{i\theta}\varphi)|_C = 0$.

  ||

  $J_{\theta+\pi/2}$-complex Lagr.

Example $[r = n-1]$ <u>Calabi-Yau</u>

$$(M^{2m}, g, J, \omega).$$

$\varphi \in \Omega^{n,0}(M)$ holo. vol. form.

- Instanton $A^m \subset M^{2m}$ Calib. by $\mathrm{Re}(e^{i\theta}\varphi)$.

  ||

  SLag w/ phase $= \theta$.

- Lagr. of $\mathbb{C}$-type $C^{2m-2} \subset M^{2m}$ $\varphi|_c = 0$.

  ||

  Complex Hypersurface.

- Lagr. of $\mathbb{R}$-type $C^m \subset M^{2m}$ $\omega|_c = 0$, $\mathrm{Re}(e^{i\theta}\varphi)|_c = 0$

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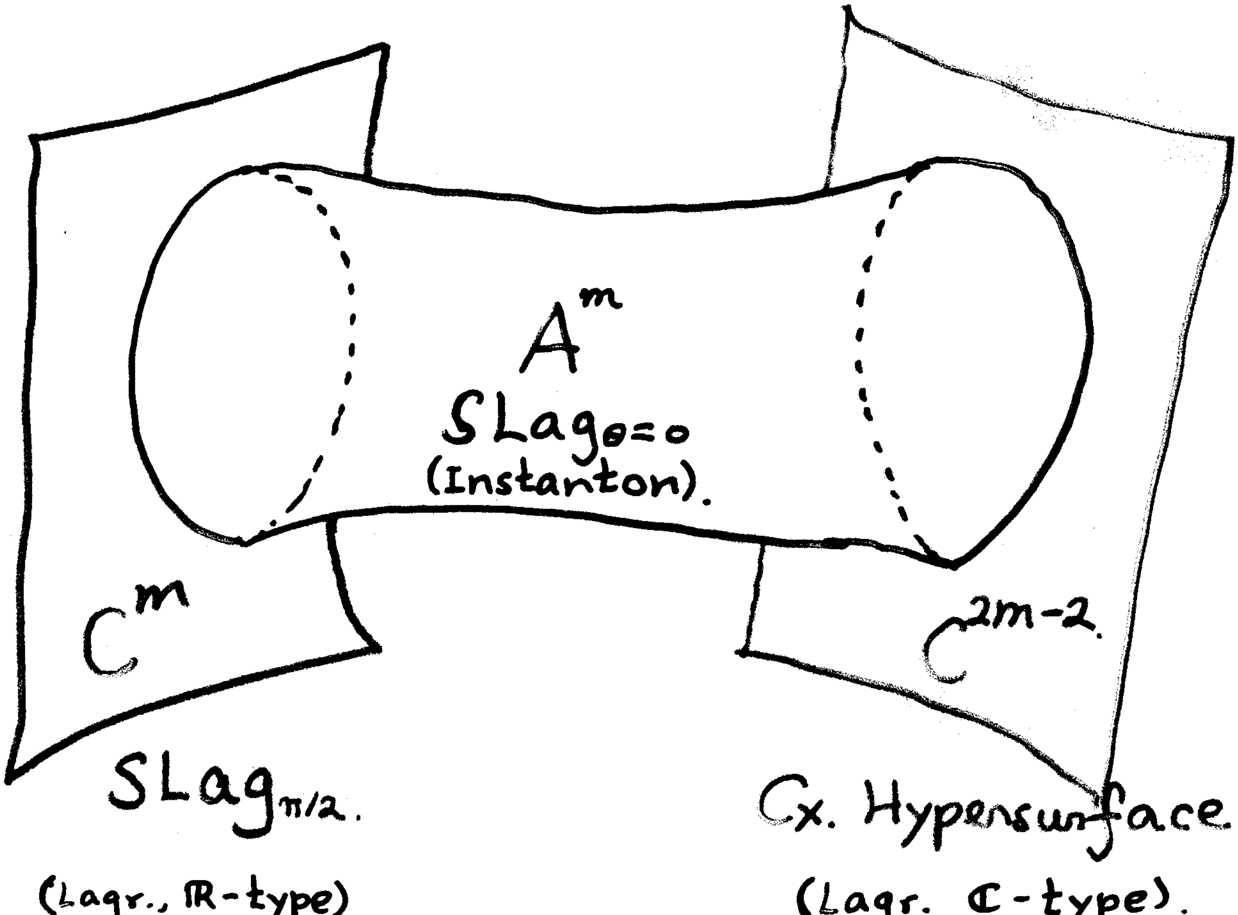
  SLag. w/ phase $= \theta + \frac{\pi}{2}$.

052 Remark: M^{2m} : C.Y.

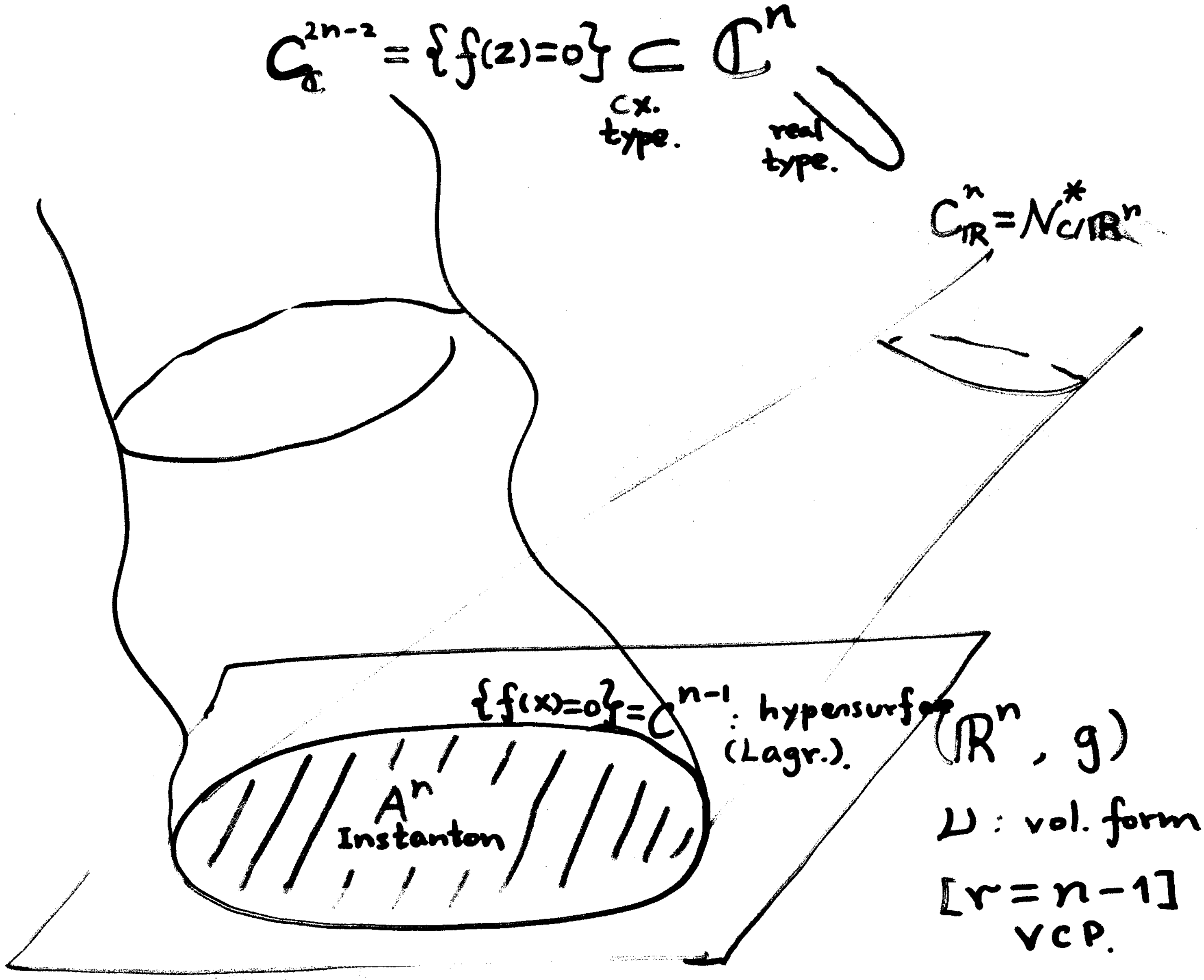


$SLag_0$ with boundary lying on $SLag_{\pi/2}$ (Dirichlet) or Cx. Hypersurface (Neumann)

— Schoen's school.

(A. Butscher, W.Y. Qiu).

Remark: Motivation of Lagr. of $\mathbb{R}$ vs $\mathbb{C}$ type. (Complexification).



§ Loop Space Interpretations of $\mathbb{C}$. VCP.

Recall: $[r = n-1]$

$$(M^n, g), \quad \nu \in \Omega^n(M). \quad (n-1)\text{-fold.}$$

$$\rightsquigarrow$$

$$\mathcal{L}_{\Sigma^{n-2}} M = \frac{\mathrm{Map}(\Sigma^{n-2}, M^n)}{\mathrm{Diff}(\Sigma)}, \quad \omega_{\mathcal{L}_\Sigma M} \in \Omega^2(\mathcal{L}_\Sigma M)$$

1-fold. (symplectic).

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M^{2n} : Calabi-Yau. $(n-1)$-fold $\mathbb{C}$VCP.
(M, g, J, ω), $\varphi \in \Omega^{n,0}(M)$.

$$\Big\downarrow$$

$$\frac{\mathrm{Map}(\Sigma^{n-2}, M)}{\mathrm{Diff}(\Sigma)} \quad : \quad \text{Hyperkähler}$$

Symplectic Quotient ??

- For ω / M induces a symplectic str. ω_{Map} on $\mathrm{Map}(\Sigma, M)$. Need FIX. $\nu_\Sigma \in \Omega^{n-2}(\Sigma)$. i.e.

$$\omega_{\mathrm{Map}}(u, v) = \int_\Sigma \omega(u, v)\, \nu_\Sigma.$$

- Problem: ω_{Map} is preserved only by $\mathrm{Diff}(\Sigma, \nu_\Sigma)$, not $\mathrm{Diff}(\Sigma)$.

Moment Map (Donaldson, Hitchin).

$$\begin{tikzcd} \mathrm{Diff}(\Sigma, \nu_\Sigma) \arrow[d, bend left] \\ \mathrm{Map}(\Sigma, M) \arrow[r, "\mu"] & \Omega^1(\Sigma)/d\Omega^0(\Sigma). \end{tikzcd}$$

$$\mu(\Sigma \overset{f}{\hookrightarrow} M) = \alpha$$

where $d\alpha = f^* \omega$.

$$\frac{\mu^{-1}(0)}{\mathrm{Diff}(\Sigma, \nu)} \quad : \quad \text{Symplectic (NOT Hyperkähler)}$$

$$\| $$

$$\frac{\mathrm{Map}(\Sigma, M)}{\mathrm{Diff}(\Sigma, \nu)}.$$

<u>Want</u> $\cdot // \mathrm{Diff}(\Sigma)$

Note $\quad (\Sigma \overset{f}{\hookrightarrow} M) \in \mu^{-1}(0)$

$\iff \omega|_\Sigma = 0 \qquad$ (i.e. isotropic).

$\Rightarrow$

$$0 \to S \longrightarrow N_{\Sigma/M} \longrightarrow T^*_\Sigma \longrightarrow 0.$$

Proposition (Lee-L.).

~~Then~~ $\Gamma(\Sigma, S)$ determines an (integrable) distribution on $\mu^{-1}(0)$, which is $\mathrm{Diff}(\Sigma)$-invariant.

Moreover,

$$\mathrm{Leaf}/\mathrm{Diff}(\Sigma)$$

is <u>Hyperkähler</u>.

i.e. $\quad$ ~~Ma~~ "$\mathrm{Map}(\Sigma, M)_{\mathrm{emb}} /\!\!/ \mathrm{Diff}(\Sigma)$".

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M^{2n} Calabi-Yau. $\rightsquigarrow$ $\mathcal{L}_\Sigma M$ Hyperkähler
$\parallel$
"$\mathrm{Map}(\Sigma^{n-2}, M) // \mathrm{Diff}(\Sigma)$".

$\omega \in \Omega^2(M) \rightsquigarrow \tilde{\omega} \in \Omega^2(\mathcal{L}_\Sigma M)$

$\varphi \in \Omega^{n,0}(M) \rightsquigarrow \tilde{\varphi} \in \Omega^{2,0}(\mathcal{L}_\Sigma M)$

holo. volume form — holo. sympl. form.

$A^n \subset M \longleftrightarrow D^2 \subset \mathcal{L}_\Sigma M$

$A^n = D^2 \times \Sigma$

instanton (i.e. $\mathrm{SLag}_{\theta+\pi/2}$) — instanton (i.e. holo. curve).

$C^{2n-2} \subset M \longleftrightarrow \mathcal{L}_\Sigma C \subset \mathcal{L}_\Sigma M$

Lagr. $\mathbb{C}$-type. (i.e. Cx. hypersurface). — J-complex Lagr.

$C^n \subset M^{2n} \longleftrightarrow$ "$\mathcal{L}_\Sigma C$" $\subseteq \mathcal{L}_\Sigma M$

$$\text{"}\mathcal{L}_\Sigma C\text{"} = \frac{\mathrm{Map}(\Sigma, C) \cap \overbrace{\mu^{-1}(0)}^{\text{automatic}}}{\mathrm{Diff}(\Sigma)}$$

Lagr. $\mathbb{R}$-type (i.e. SLag_θ). — J_θ-complex Lagr.

<u>Remark</u>: MCF (Mean Curv. Flow) on manifolds w/ $\mathbb{C}$-VCP (C.Y. & H.K.).

<u>Calabi-Yau case</u>: $C^n \subset M^{2n}$: CY.

$\omega|_C = 0$ (i.e. $\mathbb{R}$-Lagr.) $\iff T_xC \subset T_xM$ calibrated by $\mathrm{Re}(e^{i\theta(x)}\varphi(x))$.

for some $\theta(x): C \longrightarrow S^1$.

$$\Rightarrow \quad \iota_{\vec{H}}\,\omega = d\theta$$

$$\Rightarrow \begin{cases} \text{MCF preserve Lagr.} \\ \text{Gr. pt.: } \theta \equiv \text{const (sLag.)}. \end{cases}$$

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$(M^{4n}, \omega_J, \Omega_J = \omega_I + i\omega_K)$
Hyperkähler.

If $C^{2n} \subset M^{4n}$ s.t. $\forall x \in C$
$T_x C \subset T_x M$ Calib. by $\Omega_{J_{\theta(x)}}$
$\exists\ \theta(x) \in S^2_{\text{twistor}}$,

then C is called a __$\mathbb{C}$-Lagrangian__.

__Theorem__ (L. and Tom Wan).

- $\underline{\iota_{\vec{H}} \Omega_J = \partial_J \theta}$

$J = J_{\theta(x)}$.

- MCF preserves $\mathbb{C}$-Lagr.
- Min. $\mathbb{C}$-Lagr. $\Rightarrow$ $\theta : C \longrightarrow S^2$ anti-holomorphic.
- $\theta(x) \equiv$ const. $\Leftrightarrow$ Complex Lagrangian (calibrated
(i.e. Min. $\mathbb{C}$-Lag. + $[\theta(x)] = 0 \in [C, S^2]$.)

Comparisons:

(1) Vector Cross Product

$r=$	$n-1$	1	2	3
VCP	Oriented	Kähler	G_2-mfd	Spin7-mfd
$\mathbb{C}$-VCP.	Calabi-Yau	Hyperkähler.		

(2) Geometry / Normed Algebras $\mathbb{A}$.

$\mathbb{A}=$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{A}$-mfd	Manifold	Kähler	Quaternionic Kähler	Spin7-mfd
Special $\mathbb{A}$-mfd	Oriented	Calabi-Yau	Hyperkähler	G_2-mfd.