Gromov-Witten invariants on CY 3-folds and their Physical Interpretation

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What are Gromov - Witten invariants ?

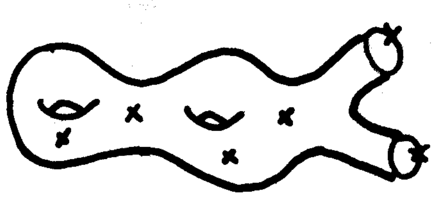
String theory: consists of maps

$$\phi : \Sigma_{g\,h\,\underset{\sim}{n\,m}} \longrightarrow M$$

genus   boundary components   marked points

$\Sigma_{g\,h\,nm}$: **Worldsheet:**

oriented Riemann surface, i.g. with boundaries



$\Sigma_{2\,2\,4\,2}$

M : **Target ~~space~~:**

some 10d manifold

$$M_{10} = \mathbb{R}_{3,1} \times K_6$$

4d Minkowski space   "compact" Calabi-Yau physical content is encoded in

Variational Integrals over those maps

E.g. partition function:

$$Z(G,B) = \int \mathcal{D}\phi \, \mathcal{D}g \, e^{-S(\phi, g, G, B)}$$

G, B: metric & 2-form field on M

not varied over ~ Background fields

limit approach: 1. quantized

g: - metric on $\Sigma_{g,n,m,n}$

classically one can chose

conformal gauge $g_{\alpha\beta} = \delta_{\alpha\beta}$

- in quantum theory this is

only true for supersymmetric WS

spectrum if $\dim(M) = 10$

$$\Rightarrow \quad Z(G,B) = \int \mathcal{D}\phi \; e^{-S(\phi, G, B)}$$

Action:

$$S(\phi, G, B) = \int_\Sigma d^2\sigma \left(G_{\mu\nu}\, \partial_\alpha \phi^\mu\, \partial^\alpha \phi^\nu + i\, B_{\mu\nu}\, \varepsilon_{\alpha\beta}\, \partial^\alpha \phi^\mu\, \partial^\beta \phi^\nu \right)$$

$+$ fermionic terms

Assume M symplectic with

J almost complex structure

ω symplectic form $(1,1)$

$G_{\mu\nu} = J^\rho{}_\mu\, \omega_{\rho\nu}$ positive

with $z = \frac{1}{\sqrt{2}}(\sigma_1 + i\sigma_2)$, $\bar{z} = \frac{1}{\sqrt{2}}(\sigma_1 - i\sigma_2)$

$\Rightarrow$ Action:

$$S(\phi, G, B) = 2i \int_\Sigma dz \wedge d\bar{z}\; G_{\bar{i}j}\, \partial_z \phi^{\bar{i}}\, \partial_{\bar{z}} \phi^j - \int_\Sigma dz \wedge d\bar{z}\, (iG_{\bar{i}j} + B_{\bar{i}j}) (\partial_z \phi^{\bar{i}}\, \partial_{\bar{z}} \phi^j - \partial_z \phi^j\, \partial_{\bar{z}} \phi^{\bar{i}})$$

$+$ fermionic terms

Action:

$$S(\phi, G, B) = 2i \int_\Sigma dz \wedge d\bar{z} \; g_{\bar{i}j} \partial_z \phi^{\bar{i}} \partial_{\bar{z}} \phi^i$$

$$+ \int_\Sigma \phi^* (\omega + B) \quad + \text{ fermionic}$$

Saddle points ?

Equation of motion : $\bar{\partial} \phi^i = \partial \phi^{\bar{i}} = 0$

Such solution are called Instantons

J holomorphic curves

$\phi : (\Sigma, j) \to (M, J) \qquad J \circ d\phi = d\phi \circ j$

Instanton Action:

$$S(\phi^I, G, B) = \int_\Sigma \phi^{I*} (\omega + B)$$

If J, ω exists, one can write fermionic terms with $\mathcal{N}=2$ world sheet supersymmetry

(L. Alvarez-Gaumes & D. Freedman 88)

$$|\Psi_\pm\rangle = G_\pm |\phi\rangle$$

fermion state

boson state

<u>Chiral Fields :</u>

		h	Q
U(1) current	$J(z)$	1	0
Energy momentum tensor	$T(z)$	2	0
Super currents	$G_\pm(z)$	$\frac{3}{2}$	± 1

two topological twists

$$T(z) \to T(z) + \frac{1}{2}\partial_z J(z) \qquad\qquad T(z) \to T(z) - \frac{1}{2}\partial_z J(z)$$

$\hat{G}^+ = H^+ \quad h=2$

$G^\pm$

$\hat{G}^- = Q^- \quad h=1$

$\tilde{G}^- = Q^+ \quad h=1$

$G^\pm$

$\tilde{G}^- = H^- \quad h=2$

Q^- $\qquad\qquad$ Q^+

Currents odd statistic

From these BRST operators can be defined:

$$\hat{Q}^- = \oint Q^- \, dz \qquad \hat{Q}^+ = \oint Q^+ \, dz$$

two chiral halfs: $T(z), J(z), G_\pm(z)$
$\bar{T}(\bar{z}), \bar{J}(\bar{z}), \bar{G}_\pm(\bar{z})$

$$Q^A = \hat{Q}^+ + \hat{\bar{Q}}^- \qquad (Q^{A/B})^2 = 0$$
$$Q^B = \hat{Q}^- + \hat{\bar{Q}}^-$$

Cohomological states of these in the Hilbertspace of the $\mathcal{N}=2$ superconformal theory

define the

topological A-model

&

topological B-model

<u>States</u>: For string theory on M

$$V \in A^{\bullet}(M)$$

$$\{Q^A, O_V\} = O_{dV}$$

i.e. the local operators of the <u>A-model</u> are identified with $\underbrace{}_{\text{elements of}} H^k(M)$

<u>localisation:</u> (by supersymmetry)

All contributions to the A-model correlation functions are from the saddle points of the action discussed

E.g.

$$Z = \int \mathcal{D}\Phi \; e^{-S(\Phi, G, B)}$$

$$\Big\downarrow \text{ localizes}$$

$$Z = \sum_{\substack{\text{top. type} \\ \text{of } \Phi_{cl}}} \int_{\mathcal{M}(\Phi_{cl})} e^{\text{top}}(\Phi_{cl})$$

Φ_{cl} holomorphic curves

For full string theory we need conformal invariance on World-sheet.

- cancellation of conformal anomaly

  $\dim(M) = 10$

- living in 4d

  $M = \mathbb{R}_{3,1} \otimes K_6$

- maintaining conformal symmetry at all σ-model loop order $\beta_N = 0$

  $\Leftrightarrow \quad R_{ij} = 0 \quad$ Ricci-flat

- super symmetry

  J, ω exists

- Yau $\quad c_1(TM) = 0 \quad \Leftrightarrow \quad R_{ij} = 0$

$\Rightarrow K_6$ Calabi-Yau complex 3-fold

However as far as the topological A-model is concerned we do not need conformal invariance

any almost complex symplectic manifold will do.

Closed "String" A-model:

$$\log(Z) =: F = \sum_{g,0,\beta} \underbrace{\int_{\overline{\mathcal{M}_{g,0}(M,\beta)}} e^{top}(g,0,\beta)}_{\text{Gromov-Witten-Invariant}} \cdot \underbrace{e^{-\int_\beta \phi^*(\omega+iB)}}_{\text{Instanton action}}$$

Kontsevich

$\overline{\mathcal{M}_{g,n}(M,\beta)}$ <u>stable</u> compactification of moduli space of maps from genus g Riemann surface with n-marked points such that image curve:

$$\phi_*[\Sigma_{g,n}] = \beta \in H_2(M,\mathbb{Z})$$

Riemann-Roch theorem gives for expected dimension of the moduli space

$$\text{vdim}\, \overline{\mathcal{M}_{g,n}(M,\beta)} = \int_\beta c_1(TM) + (\dim M - 3)(1-g) + n$$

Calabi-Yau special because

$$\text{Vdim}\, \overline{\mathcal{M}_{g0}(M,\beta)} = 0$$

important fact : $\exists$ virtual fundamental

class $[\mathcal{M}_{g,n}(M,\beta)]^{vir} \in H_{2 vdim}(M,\mathbb{Q})$

in particular for CY 3 folds:

$$F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$$

$$= \sum_{g=0}^{\infty} \lambda^{2g-2} \sum_{\beta} \deg [\overline{\mathcal{M}}_{g,0}(M,\beta)]^{vir}$$

For toric Calabi-Yau 3-folds:

1) Example : non-compact CY-3 fold

B Fano variety (toric)

3-fold $\mathcal{O}(K) \to B$

example $\mathcal{O}(-3) \to \mathbb{P}^2$

$\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ etc.

higher codim

$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$

2) Example: compact CY-3 fold

A_Δ toric variety associated

Δ reflexive polyhedron in $4_{\mathbb{R}}$-dim

$\mathcal{O}(-K) \in A_\Delta$ Calabi-Yau 3fold (Batyrev)

Example: Quintic in $\mathbb{P}^4$

$\mathcal{O}(5) \in \mathbb{P}^4$

- $\mathbb{C}^*$ action on toric variety can be pulled back to $M = [M_{g,n}(M, \beta)]$
- Atiyah-Bott localisation can be used to evaluate integrals of type

$$\int_M \phi = \sum_F \int_F \left(\frac{i_F^*(\phi)}{e(N^{vir}_{F/M})} \right)$$

equivariant cohomology class

- $i_F : F \hookrightarrow M$ Fixpoint locus under induced $\mathbb{C}^*$ action
- $e(N^{vir}_{F/M})$ Euler class of the normal bundle of F in M

F is given by the graphs Γ

the equivariant maps

$$\deg [\mathcal{M}]^{vir} = \sum_{\Gamma}' \frac{1}{|A_\Gamma|} \int_{\mathcal{M}_\Gamma} \frac{i^*_\Gamma(\phi)}{e(N_\Gamma^{vir})}$$

Kontsevich 95

Generalisation to open topological

A-model Katz, Liu, ...

$$\phi : (\Sigma, \partial\Sigma) \longrightarrow (\mathcal{M}, L)$$

$\mathrel{\rotatebox{90}{$\hookleftarrow$}}$

<u>Lagrangian</u>

Submanifold

$$L^d \hookrightarrow \mathcal{M}^{2d}$$

$$\omega|_L = 0$$

again the physical context preserving

$N=1$ supersymmetry requires L to

be a <u>special Lagrangian</u> in CY-3 fold

$\Omega|_L = e^{i\varphi} \operatorname{vol}(L) \qquad \omega|_L = 0$

In the open case:

- $\beta \in H_2(M, L; \mathbb{Z})$ relative homology class
- $\gamma_1, \ldots \gamma_n \in H_1(L, \mathbb{Z})$

- Choice

$$\mu(\phi^* TM, (\phi|_{\partial\Sigma})^* TL) = \mu$$

~~related to framing in dual~~

Chern-Simons Witten hep-th 9207094
Aganagic, Vafa, AK hep-th 1010?5045

- $M_{(g,h)(n,m)}(M, L \mid \beta; \gamma_1 \ldots \gamma_n; \mu)$

Moduli space of open J-holomorphic curves

theory essentially goes through Liu

- $\exists$ stable compactification

$\overline{M_{(g,h)(n,m)}}$

- $\exists$ virtual fundamental class

$[\overline{M_{(g,h)(n,m)}}]^{vir}$

$\mathrm{vdim}_{\mathbb{R}} = \mu + (\dim M - 3)(2 - 2g - h) + 2n + m$

15 - $F_{g,\beta,\gamma_1 \ldots \gamma_n, \mu} = \deg [\overline{M_{(g,h)(n,m)}}]^{vir} = \sum_{\Gamma}' \frac{1}{|A_\Gamma|} \int_{M_\Gamma} \frac{i_\Gamma^*(\Phi)}{e(N_\Gamma^{vir})}$

i.e. there are localisation formulas, e.g.

for $\mathcal{O}(K) \to B$ $\quad B$ toric, Brane Harvey & Lawson type

<u>Example:</u> $B = \mathbb{P}^2$ Graber & Zaslow

Mirror calculation exist

so far only for disks Aganagic Klemm, Vafa physikal interpretation of Gromov-Witten sums "Instanton sum"

- closed string Gromov-Witten-Invariants on Calabi-Yau 3-fold.

$\mathbb{R}_{3,1} \otimes CY$

valid type II string compactification

$\exists$ two covariant constant spinors:

$$\eta, \bar{\eta} \Rightarrow \Omega_{abc} = \Gamma^{\alpha\beta\gamma}_{abc} \eta_\alpha \eta_\beta \eta_\gamma$$

$\exists$ Ω holomorphic (3,0) form

$$\Updownarrow$$

$$R_{i\bar{j}} = 0$$

$$\Updownarrow$$

$$c_1(TM) = 0$$

From $\eta, \bar{\eta}$ one gets $N=2$ supergravity in 4 d, with the string scale l_s as ultraviolett cut-off. $\Rightarrow$ Consistent theory of gravity in 4 d.

How are string amplitudes related to field theory amplitudes

String correspondence principle

$$\text{String theory} \overset{l_s \to 0}{\Longrightarrow} \text{general relativity \& gauge theory}$$

Take an amplitude calculated in String theory

$$\lambda = \sum \Big\{$$

$$+ \dots \Big\}$$

and take $l_s \to 0$ limit ↷ Field theory amplitude

This way one can reconstruct the low energy effektive action for the massless fields:

$$S_{eff} = \int d^{10}x \underbrace{\sqrt{g}\, R}_{\text{Einstein term}} + (\underset{\text{Gauge kinetic term}}{F^2} + \dots)$$

topological string amplitudes are special: they come only from <u>classical solutions</u> and do <u>not depend on α' corrections</u>

- they immediatly exact contributions to the low energy effective action

- this is reflected by non-renormalisation statements in $\mathcal{N}=2$ $\mathcal{N}=1$ super symmetric theories Seiberg (93-95)

<u>$\mathcal{N}=2$ 4d theories</u>

- $g=0$ contribution $\mathcal{F} = F_0(t_1, \ldots, t_n)$ calculates the Kähler potential

$$K = \log\left((t^a - \bar{t}^a)(\mathcal{F}_a - \bar{\mathcal{F}}_a) - 2(\mathcal{F} + \bar{\mathcal{F}}) \right)$$

$$\mathcal{F}_a = \frac{\partial}{\partial t^a}\mathcal{F}$$

which gives rise to the metric of the gauge kinetic terms

$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ gauge kinetic terms

- $g > 0$ contribution $\mathcal{F}_g(t_1 \ldots t_n)$ gives exact couplings of the terms

$$R_+^2 F_+^{2g-2} \mathcal{F}_g(t_1 \ldots t_n)$$

R_+ self dual part of curvature 2-form

F_+ " " " Graviphoton Field strength

- one can decouple gravity to obtain the gauge kinetic terms of Yang-Mills theory.

- this gives an exact relation between invarianst of Yang-Mills gauge Instantons and Gromov-Witten Invarianst.

Gopakumar & Vafa interpreted

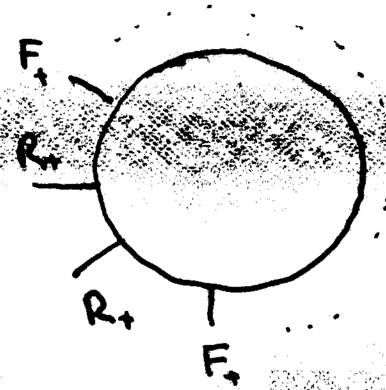
$$F(\lambda) = \sum' \lambda^{2g-2} \mathcal{F}_g(t)$$

as five dimensional amplitude

from 11 d supergravity (M-theory)

compactified on CY

$F_+^{2g-2} R_+^2$



is generated by a loop contribution

in 5d field theory, with massive

BPS - States in the loop

- this BPS- States come from M- Membrane wrapping holomorphic curves in M

- From loop (Schwinger (48)) calculating given mass $\sim \beta$, spin $\sim g$

One obtaines

Integer contribution
BPS states with (mass, spin)

$$F(\lambda) = \sum'_{\beta, r \geq 0, k > 0} \alpha_r^\beta \frac{1}{k} \left(2 \sin \frac{k\lambda}{2}\right)^{2r-2} e^{-k\int_\beta \phi(\omega)}$$

$\mathcal{N}=1$ theories in 4d can be obtained by various breaking mechanism:

Consider a <u>special Lagrangian 3-cycle L</u> in a Calabi-Yau geometry

- $\exists$ $\mathcal{N}=1$ effective theory on the $\mathbb{R}^{3,1}$ part of 6-brane wrapping L
- If $H_1(L,\mathbb{Z})=0$, one has $\dim(H_1(L,\mathbb{Z}))$ moduli in the $\mathcal{N}=1$ theory $+$ $H_2(M,L,\mathbb{Z})$ bulk moduli

The superpotential is given by

$$W = \sum_{\beta,\gamma,\mu}{}' \deg \left[\mathcal{M}_{01}(M,L,\beta,\gamma,\mu) \right]^{\mathrm{vir}} \cdot \underbrace{e^{-\int_\beta \Phi(\omega+J)}}_{\text{bulk moduli}} \; \underbrace{e^{-\int_\gamma (\partial\phi)^*(A+i^*J)}}_{\text{boundary moduli}}$$

- since last year there exists a matrix model formulation (Dijkgraaf & Vafa) for the calculation of superpotentials in many $\mathcal{N}=1$ theories.

=> suggest a matrix model calculation for open Gromov Witten invariants.

- planar diagram $\mathcal{N}^0$ => Disks
- nonplanar diagrams $\mathcal{N}^{-1}$ => crosscaps
- nonplanar diagrams at $\mathcal{N}^{-2n}$ => higher genus GWI

- annulus amplitude is physically most interesting because if $\mathcal{N}=1$ gauge kinetic terms.