Brian White's JAMS papers

Let $\mathcal{K} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ be a mean convex flow with $n < 7$.

• (Partial regularity) The parabolic Hausdorff dimension of Sing($\mathcal{K}$) is at most $n-1$. This implies that when $n = 2$, then almost every time slice is a domain with smooth boundary.

• For all $t > 0$, the Hausdorff dimension of the time t slice of the singular set, Sing($\mathcal{K}$)$\cap$ $(\mathbb{R}^{n+1} \times \{t\})$, has Hausdorff dimension at most $n - 3$.

• (Convex blow-ups) All limit flows are convex. All tangent flows are round cylindrical flows.

• Limit flows are "noncollapsed", i.e. one cannot obtain a double density plane as a limit.

• If one has a blow-up sequence $\mathcal{K}_i$ where $(0,0) \in \mathcal{K}_i$ and $H(0,0) = 1$, then the flows $\mathcal{K}_i$ converge smoothly near the time slice $\mathbb{R}^{n+1} \times \{0\}$ to a convex limit flow.

The Thick-Thin decomposition

(cf. Perelman, Huisken-Sinestrari)

For all $\epsilon > 0$ there exists $D < \infty$ such that if $\mathcal{K} \subset \mathbb{R}^3 \times \mathbb{R}$ is a mean convex flow, $t > 0$, and $\partial\mathcal{K}_t$ is smooth, then $K_t = A \cup B$ where

• For all $x \in A \cap \partial\mathcal{K}_t$, after rescaling by $\frac{D}{d(x,\partial\mathcal{K}_0)}$, the time slice $\mathcal{K}_t$ is ϵ-close to a half-space near x.

• B is a union of finitely many connected components $B = \cup_i C_i$, for each of which one of the following holds:

a. (Neck) The pair $(C_i, C_i \cap \partial\mathcal{K}_t) \overset{\text{diff}}{\approx} ([0,1] \times D^2, [0,1] \times S^1)$, and every $x \in C_i \cap \partial\mathcal{K}_t$ is an ϵ-neck.

b. (Capped neck) The pair $(C_i, C_i \cap \partial\mathcal{K}_t) \overset{\text{diff}}{\approx} (B^3, S^2_+)$, and there is an $x \in C_i \cap \partial\mathcal{K}_t$ such that every $y \in (C_i \cap \partial\mathcal{K}_t) \setminus B(x, \frac{D}{H(x)})$ is an ϵ-neck.

c. (Solid torus component) The pair $(C_i, C_i \cap \partial\mathcal{K}_t) \overset{\text{diff}}{\approx} (S^1 \times D^2, S^1 \times S^1)$, and every $x \in C_i \cap \partial\mathcal{K}_t$ is an ϵ-neck.

d. (3-ball component) The pair $(C_i, C_i \cap \partial\mathcal{K}_t) \overset{\text{diff}}{\approx} (B^3, S^2)$, and there are $x_1, x_2 \in C_i \cap \partial\mathcal{K}_t$ such that every $y \in (C_i \cap \partial\mathcal{K}_t) \setminus (B(x_1, \frac{D}{H(x_1)}) \cup B(x_2, \frac{D}{H(x_2)}))$ is an ϵ-neck.

Properties of limit flows

Suppose $n < 7$ and $\mathcal{L}$ is a limit flow which is not a static half-space.

• If $\mathcal{L}_t = \mathbb{R}^{n-1} \times \mathcal{L}'_t$ then the factor $\mathcal{L}'_t \subset \mathbb{R}^2$ is a round circle.

• If $\mathcal{L}_t = \mathbb{R}^{n-2} \times \mathcal{L}'_t$ then $\mathcal{L}'_t \subset \mathbb{R}^3$ is a convex set whose tangent cone at infinity has dimension ≤ 1, i.e. its cone is a point, a ray, or a line.

Corollary. *There exists $R = R(\theta, \epsilon) < \infty$ such that if $n = 2$, $\mathcal{L}$ is a limit flow, $x \in \mathcal{L}_t$, $H(x) = 1$ and there are points $y, z \in \mathcal{L}_t$ with* $\min(d(x,y), d(x,z)) > R$, $\angle_x(y,z) \geq \theta$, *then x is an ϵ-neck.*

Topology of the region between time slices

If $\mathcal{K} \subset \mathbb{R}^3 \times \mathbb{R}$ is a mean convex flow, and $\mathcal{K}_{t_1}, \mathcal{K}_{t_2}$ are smooth time slices of $\mathcal{K}$, then the region $\mathcal{K}_{t_1} \setminus \mathrm{Int}(\mathcal{K}_{t_2})$ may be obtained from $\partial\mathcal{K}_{t_1}$ by attaching 2 and 3-handles.

This is precisely the assertion one would expect to make if one performed flow with surgery rather than using level set flow.

Structure of the singular set

Conjecture. (??) When $n = 2$, the singular set consists of finitely many isolated points, plus a finite collection of closed curves. When $n = n$, Brian White has conjectured that the singular set is $(n-1)$-rectifiable with respect to the parabolic distance.

There is a function $\alpha = \alpha(\rho)$ with

$$\lim_{\rho \to 0} \rho(\alpha) = 0$$

with the following property. If $n < 7$, $\mathcal{K} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is a mean convex flow, $(x_0, t_0) \in \mathrm{Sing}(\mathcal{K})$, $t \geq t_0$, and $(x, t) \in \mathcal{K}$, then

$$t - t_0 \leq \alpha \left(\frac{d(x, x_0)}{d(x_0, \partial \mathcal{K}_0)} \right) d^2(x, x_0).$$

A sub-Reifenberg property for the singular set

If $n < 7$, there is a $\rho = \rho(\delta) > 0$ such that if $\mathcal{K} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is a mean convex flow, $(x,t) \in \mathrm{Sing}(K)$, and $r < \rho d(x, \partial\mathcal{K}_0)$, then there is an affine subspace $A \subset \mathbb{R}^{n+1} \times \{t\}$ such that

$$\mathrm{Sing}(K) \cap PB(x,t,r) \subset N_{\delta r}(A) \cap PB(x,t,r).$$

This implies, for instance, that when $n = 2$, if $t_0 > 0$, then the part of the singular set with $t \geq t_0$ is contained in a finite union of arcs which satisfy a δ-Reifenberg condition for all $\delta > 0$.

Special case. If $n = 2$, $\partial\mathcal{K}_0$ is a 2-torus, and $\mathcal{K}_t = \emptyset$ after the first singular time T, then $\mathcal{K}_t$ collapses to a Reifenberg curve at the singular time T.