The K$\ddot{a}$hler-Ricci Flow and Geometry of Open Manifolds

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Let M be a complete noncompact Kähler manifold with Kähler metric $g_{\alpha\bar{\beta}}$. We will consider the Ricci flow for the Kähler metric

$$\begin{cases} \dfrac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -R_{\alpha\bar{\beta}}(x,t), \quad x \in M, \ t > 0 \\ \\ g_{\alpha\bar{\beta}}(x,0) = g_{\alpha\bar{\beta}}(x), \quad x \in M. \end{cases}$$

- Short time existence (Shi)
- Preserving the Kählerity (Hamilton)
- Preserving the nonnegativity of holomorphic bisectional curvature (Bando, Mok)

Thus the Ricci flow is a useful tool to study complete Kähler manifolds with nonnegative holomorphic bisectional curvature.

§1. Motivations

Motivation I

The classical uniformization theorem

$$\text{simply connected, Riemann surface} \overset{\text{biholomorphic}}{\cong} \begin{cases} S^2 \\ \mathbb{C} \\ D \text{ (unit disc)} \end{cases}$$

• It gives the characterization for the standard complex structures of one-dimensional Kähler manifolds.

• There is a vast variety of biholomorphically distinct complex structures on $\mathbb{R}^{2n}$ for $n > 1$.

Differential geometry version of classical uniformization theorem

$$\text{positive curvature, Riemann surface} \overset{\text{biholomorphic}}{\cong} \begin{cases} S^2 \\ \mathbb{C} \end{cases}$$

On a higher dimensional Kähler manifold

$$\overset{\text{scalar}}{\text{curvature}} < \overset{\text{Ricci}}{\text{curvature}} < \overset{\text{holomorphic}}{\text{bisectional curvature}} < \overset{\text{sectional}}{\text{curvature}}$$

Higher dimensional uniformization problems

• **Frankel conjecture:**

$$compact\ n-dim\ Kähler\ manifold\ M,\ bisect > 0 \Rightarrow M \overset{\text{biholo.}}{\cong} \mathbb{C}P^n$$

Completely solved by Mori, Siu-Yau.

• **Greene-Wu, Yau conjecture:**

$$noncompact\ n-dim\ K\ddot{a}hler\ manifold\ M,\ bisect > 0 \Rightarrow M \overset{\text{biholo.}}{\cong} \mathbb{C}^n$$

Motivation II

Soul theorem (Cheeger-Gromoll-Meyer)

$$\begin{array}{l} noncompact,\ Riemann \\ manifold,\ sect. \geq 0 \end{array} \overset{diffeo.}{\cong} \begin{array}{l} the\ normal\ bundle\ \nu(S)\ of\ a \\ compact\ totally\ geodesic\ submanifold \\ S\ (\ of\ M\ ) \end{array}$$

Yau conjecture

$$\begin{array}{l} noncompact,\ K\ddot{a}hler \\ manifold,\ bisect. \geq 0 \end{array} \overset{biholo.}{\cong} \begin{array}{l} a\ holomorphic\ vector\ bundle \\ over\ a\ compact\ Hermitian \\ symmetric\ space \end{array}$$

§2. Long Time Existence

Let $(M, g_{\alpha\bar{\beta}}(x))$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Consider the Ricci flow with the Kähler metric $g_{\alpha\bar{\beta}}$ as initial data. Its scalar curvature evolves by

$$\frac{\partial}{\partial t}R(x,t) = \Delta_t R(x,t) + 2|Ric(x,t)|^2_{g_{\alpha\bar{\beta}}(t)}$$

•$R|_{t=0} \geq \delta_0 > 0 \ \Rightarrow$ the solution blows up in finite time

Example: $\mathbb{C}P^k \times \mathbb{C}^{n-k}$

Long time existence result (Shi)

Suppose the initial metric $g_{\alpha\bar{\beta}}$ has bounded and nonnegative bisectional curvature, and suppose $\exists\ C > 0$ and $0 < \theta < 2$ such that

$$\fint_{B_0(x_0,r)} R(x,0)dx \triangleq \frac{1}{Vol(B_0(x_0,r))} \int_{B_0(x_0,r)} R(x,0)dx \le \frac{C}{(1+r)^{\theta}},$$

for all $x_0 \in M$ and $0 \le r < +\infty$.

Then the Ricci flow has a smooth solution for all $t \in [0, +\infty)$.

Decay estimate (Chen-Zhu)

Let M be a complete Kähler manifold with bisect> 0. Then for any $x_0 \in M$, there exists a positive constant C such that

$$\fint_{B_0(x_0,r)} R(x)d \le \frac{C}{1+r}, \quad \text{for all } 0 \le \text{r} < +\infty.$$

Conjecture: Let (M,$g_{\alpha\bar{\beta}}$) be a complete noncompact Kähler manifold with bisect> 0. Then the Ricci flow with $g_{\alpha\bar{\beta}}$ as initial metric has a solution for all $t \in [0, +\infty)$.

§3. Ancient Solutions

Consider $g_{\alpha\bar{\beta}}(t)$, $-\infty < t < T$ (with $T < +\infty$), an ancient solution to the Ricci flow on a Kähler manifold, with bounded and nonnegative holomorphic bisectional curvature.

- Asymptotic scalar curvature ratio

$$A(t) = \limsup_{d_t(x,x_0)\to+\infty} R(x,t)d_t^2(x,x_0)$$

• Asymptotic volume ratio

$$\nu(t) = \lim_{r \to +\infty} \frac{Vol_t(B_t(x_0, r))}{r^{2n}}$$

Proposition: Any ancient solution to the Ricci flow on a Kähler manifold, with bounded and nonnegative bisectional curvature, has

$$A(t) = \limsup_{d_t(x,x_0) \to +\infty} R(x,t) d_t^2(x, x_0) = +\infty, \quad \text{for each } t.$$

Proposition: (Chen-Zhu)

Let M be a (complex) two-dimensional complete noncompact Kähler manifold. Suppose $g_{\alpha\bar{\beta}}(t),\ -\infty < t \leq T$, is a nonflat ancient solution to the Ricci flow with bounded and nonnegative holomorphic bisectional curvature. Then

$$\nu(t) = \lim_{r \to +\infty} \frac{Vol_t(B_t(x_0, r))}{r^{2n}} = 0$$

for each $t \in (-\infty, T]$.

• (Perelman) Any complete noncompact nonflat ancient solution to the Ricci flow with bounded and nonnegative curvature operator has

$$\nu(t) = 0, \quad \text{for each } t$$

• **Conjecture:** The previous proposition should hold for all dimensions.

Application (Chen-Zhu)

Let M be a (complex) two-dimensional complete noncompact Kähler manifold with the metric $g_{\alpha\bar{\beta}}(x)$, which has bounded and nonnegative bisectional curvature, and has maximal volume growth, i.e.,

$$Vol(B(x_0, r)) \geq wr^{2n}, \quad \textit{for all } 0 \leq r < +\infty, \textit{ and some } w > 0.$$

Then the Ricci flow with $g_{\alpha\bar{\beta}}(x)$ as initial data has a smooth solution $g_{\alpha\bar{\beta}}(x,t)$ for all $t \in [0,+\infty)$ and satisfies

$$|R_m(x,t)| \leq B/t, \quad everywhere\ for\ some\ B = B(w).$$

• (Perelman) For every $w > 0$, $\exists B = B(w), C = C(w), \tau_0 = \tau_0(w)$ with the following property. Suppose we have a solution $g_{ij}(t)$, $-r_0^2 \leq t \leq 0$, to the Ricci flow satisfying

$$R_m(x,t) \geq -r_0^{-2} \quad for - r_0^2 \leq t \leq 0\ and\ x \in B_t(x_0, r_0),$$

and

$$Vol_0(B_0(x_0, r_0)) \geq w r_0^n.$$

Then

$$|R_m(x,t)| \leq C r_0^{-2} + B(t + \tau_0 r_0^2)^{-1}$$

whenever $-r_0^2 \leq t \leq 0$ and $d_t(x, x_0) \leq \frac{1}{4} r_0$.

§4. Volume Growth and Curvature Decay

Consider a complete noncompact Kähler manifold M with bisect≥ 0.

Examples

• Klembeck, Cao :

$$n - dim\ Kähler\ manifolds\ with\ bisect > 0$$

$$Vol(B(x_0, r)) \approx c r^n$$

$$|R_m(x)| \leq C/d(x, x_0)$$

- Cao :

$$n - dim\ K\ddot{a}hler\ manifolds\ with\ bisect > 0$$

$$Vol(B(x_0, r)) \approx cr^{2n}$$

$$|R_m(x)| \leq C/d(x, x_0)^2$$

Volume growth

- Bishop-Gromov : $Vol(B(x_0, r)) \leq Cr^{2n}, \quad for\ all\ r > 0.$
- Calabi-Yau : $Vol(B(x_0, r)) \geq Cr, \quad for\ all\ r \geq 1.$

Proposition (Chen-Zhu)

Let M be a (complex) n-dimensional complete noncompact Kähler manifold with bisect≥ 0 everywhere and bisect> 0 at least at one point. Then

$$Vol(B(x_0, r)) \geq Cr^n, \quad for\ all\ r \geq 1.$$

Curvature decay

- Chen-Zhu :

$$\oint_{B(x_0,r)} R(x)dx \leq \frac{C}{1+r}, \quad for\ all\ r > 0.$$

$$(assuming\ bisect > 0).$$

- Mok-Siu-Yau : $n \geq 2$

$$Vol(B(x_0, r)) \geq cr^{2n}, \quad for\ all\ r \geq 0$$

$$R(x) \leq C/1 + d(x_0, x)^{2+\epsilon}, \quad x \in M, \quad for\ some\ \epsilon > 0$$

$$\Rightarrow M \overset{holo.\ isom.}{\cong} \mathbb{C}^n$$

• Chen-Zhu :

$$\fint_{B(x_0,r)} R(x)dx \leq \frac{\epsilon(r)}{r^2}, \quad for\ all\ x_0 \in M\ and\ r > 0$$

$$(\ where\ \epsilon(r) \to 0\ as\ r \to +\infty\ )\ and\ R(x)\ is\ bounded$$

$$\Rightarrow M\ is\ a\ complete\ flat\ Kähler\ manifold$$

• Ni-Tam :

$$\int_0^r s(\fint_{B(x_0,s)} R(y)dy)ds = o(\log r), \quad for\ all\ x_0 \in M$$

$$\liminf_{r\to+\infty}[\exp(-ar^2)\int_{B(x_0,r)} R^2] < +\infty, \quad (a > 0)$$

$$\Rightarrow M\ is\ a\ complete\ flat\ Kähler\ manifold.$$

noncompact Kähler manifolds with bisect > 0

$Vol(B(x_0,r)) \approx r^\alpha$	$\alpha < n$	$\alpha = n$	$n < \alpha < 2n$	$\alpha = 2n$	$\alpha > 2n$
	No				No
$\fint_{B(x_0,r)} R = O(r^{-\beta})$	$\beta < 1$	$\beta = 1$	$1 < \beta < 2$	$\beta = 2$	$\beta > 2$
	No				No

Yau prediction: bisect> 0, maximal volume growth (i.e. $\alpha = 2n$)

$\Rightarrow$ quadratic decay in average sense (i.e. $\beta = 2$)

Theorem (Chen-Zhu)

Suppose bisect> 0, bounded curvature, maximal volume. Suppose also $R_m \geq 0$ if $n \geq 3$. Then

$$\fint_{B(x,r)} R(y)dy \leq C\frac{\log(2+r)}{r^2}, \quad for\ all\ x \in M\ and\ r > 0.$$

Moreover $\exists$ polynomial growth holomorphic functions $f_1, \cdots, f_n$ algebraically independent.

Theorem (Ni)

Suppose bisect> 0 and admit a nonconstant holomorphic function of polynomial growth. Then

$$\fint_{B(x,r)} R(y)dy \leq \frac{C}{r^2}, \quad for\ all\ r > 0.$$

Question: For a complete noncompact (complex) $n - dim$ Kähler manifold and $\theta \in [1, 2]$,

$$Vol(B(x_0, r)) \approx r^{\theta n} \quad \overset{?}{\Longleftrightarrow} \quad \fint_{B(x_0,r)} R \approx \frac{const.}{r^{\theta}}$$

§5. Liouville Properties

Classical Liouville theorem

- f holomorphic on $\mathbb{C}$, $|f(z)| \leq const. \;\Rightarrow\; f(z) \equiv const.$,
- f holomorphic on $\mathbb{C}$, $|f(z)| \leq const.(1 + |z|^d)$

$$\Rightarrow\; f(z) \text{is a holomorphic polynomial of degree} \leq d.$$

Yau's Liouville theorem:

On a complete noncompact Riemannian manifold with $Ric \geq 0$, there hold

- u harmonic, $u(x) \geq 0 \Rightarrow\; u(x) \equiv const.$,
- (Cheng-Yau) u harmonic, $|u(x)| \leq o(d(x, x_0))$

$$\Rightarrow\; u(x) \equiv const.$$

Notations

M^m $-$ $--$ Riemannian manifold of real dimension m

M^n $-$ $--$ Kähler manifold of complex dimension n

$\mathcal{H}_d(M^m)$=the space of harmonic functions $u(x)$ with $|u(x)| \leq O(d(x, x_0)^d)$

$\mathcal{O}_d(M^n)$=the space of holomorphic functions $f(z)$ with $|f(z)| \leq O(d(x, x_0)^d)$

Yau conjecture

(I) (real case) Suppose $Ric(M^m) \geq 0$, then

(I.0) (finiteness): $dim_{\mathbb{R}}\mathcal{H}_d(M^m) < +\infty$, for each $d > 0$;

(I.1) (Sharp estimate): $dim_{\mathbb{R}}\mathcal{H}_d(M^m) < dim_{\mathbb{R}}\mathcal{H}_d(\mathbb{R}^m)$, for each $d \in \mathbb{Z}^+$;

(I.2) (rigidity): if $\exists$ a positive integer d such that

$$dim_{\mathbb{R}}\mathcal{H}_d(M^m) = dim_{\mathbb{R}}\mathcal{H}_d(\mathbb{R}^m)$$

$$\text{then} \quad \mathrm{M^m} \overset{\text{isom.}}{\cong} \mathbb{R}^{\mathrm{m}}.$$

(II) (complex case) Suppose bisect$(M^n) \geq 0$, then

(II.1) (sharp estimate): $dim_{\mathbb{C}}\mathcal{O}_d(M^n) \leq dim_{\mathbb{C}}\mathcal{O}_d(\mathbb{C}^n)$, for each $d \in \mathbb{Z}^+$;

(II.2) (rigidity): if $\exists$ a positive integer d such that

$$dim_{\mathbb{C}}\mathcal{O}_d(M^n) = dim_{\mathbb{C}}\mathcal{O}_d(\mathbb{C}^n),$$

$$\text{then} \quad \mathrm{M^n} \overset{\text{isom. biholo.}}{\cong} \mathbb{C}^{\mathrm{n}}.$$

Real case

(I.0) (finiteness): $''dim_{\mathbb{R}}\mathcal{H}_d(M^m) < +\infty''$

$$\text{completely affirmative answer} \begin{cases} \bullet\ \text{Colding} - \text{Minicozzi} \\ \\ \bullet\ \text{P.Li} \end{cases}$$

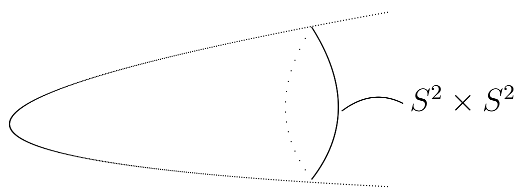
(I.1) (Sharp estimate): $''dim_{\mathbb{R}}\mathcal{H}_d(M^m) \leq dim_{\mathbb{R}}\mathcal{H}_d(\mathbb{R}^m)''$

$$\text{Only known for m} = 2 \text{ or d} = 1 \begin{cases} \bullet\ \text{Li} - \text{Tam} \\ \\ \bullet\ \text{Kasue} \end{cases}$$

(I.2) (rigidity): $''$equality holds for some $d > 0 \Longleftrightarrow M^m \overset{isom.}{\cong} \mathbb{R}^m\ ''$

$$\text{Only known for d} = 1 \begin{cases} \bullet\ \text{Li} \\ \\ \bullet\ \text{Cheeger} - \text{Colding} - \text{Minicozzi} \end{cases}$$

Donnelly's example



One can not expect a strong version of Yau's conjecture:

$$dim_{\mathbb{R}}\mathcal{H}_\alpha(M^m) \leq dim_{\mathbb{R}}\mathcal{H}_{[\alpha]}(\mathbb{R}^m)$$

for all positive constant $\alpha > 0$.

Complex case

(II.1) (Sharp estimate)

- Ni: Under an additional assumption

$$Vol(B(x_0, r)) \geq cr^{2n}, \quad for\ all\ r > 0\ and\ some\ c > 0$$

$$\Rightarrow dim_{\mathbb{C}}\mathcal{O}_d(M^n) \leq dim_{\mathbb{C}}\mathcal{O}_{[d]}(\mathbb{C}^n), \quad for\ each\ d > 0;$$

- Chen-Fu-Yin-Zhu: (completely affirmative answer)

$$dim_{\mathbb{C}}\mathcal{O}_d(M^n) \leq dim_{\mathbb{C}}\mathcal{O}_{[d]}(\mathbb{C}^n), \quad for\ each\ d > 0.$$

(II.2) (rigidity)

- Chen-Fu-Yin-Zhu: (completely affirmative answer)

$$dim_{\mathbb{C}}\mathcal{O}_d(M^n) = dim_{\mathbb{C}}\mathcal{O}_d(\mathbb{C}^n), \quad for\ some\ integer\ d > 0$$

$$\Longleftrightarrow M^n \overset{isom.\ biholo.}{\cong} \mathbb{C}^n.$$

Generalization (Chen-Fu-Yin-Zhu)

Suppose bisect$(M^n) \geq 0$ and

$$Vol(B(x_0, r)) \leq C(1 + r)^{2k},\ \forall r > 0,$$

Then

- $dim_{\mathbb{C}}\mathcal{O}_d(M^n) \leq dim_{\mathbb{C}}\mathcal{O}_{[d]}(\mathbb{C}^{[k]}), \quad for\ each\ d > 0;$
- $'' ='' $ holds for some positive integer d

$$\Longleftrightarrow M^n \overset{isom.\ biholo.}{\cong} \mathbb{C}^{[k]} \times M_2^{n-[k]}$$

§6. Uniformization Theorems

Greene-Wu-Yau conjecture:

A complete noncompact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to a complex Euclidean space.

• **Mok-Siu-Yau**

Suppose (complex) dim.≥ 2 and

(i) $Vol(B(x_0, r)) \geq C_1 r^{2n}, \quad for\ all\ r > 0,$

(ii) $R(x) \leq C_2/1 + d(x_0, x)^{2+\epsilon}, \quad for\ some\ \epsilon > 0.$

Then

$$M^n \overset{isom.\ biholo.}{\cong} \mathbb{C}^n.$$

• **Mok**

Suppose

(i) $Vol(B(x_0, r)) \geq C_1 r^{2n}, \quad for\ all\ r > 0,$

(ii) $R(x) \leq C_2/1 + d(x_0, x)^2.$

Then M^n is biholomorphic to an affine algebraic variety.

Moreover, if in addition the complex dimension $n = 2$ and

(iii) $\mathrm{sect}(M^n) > 0$

then

$$M^2 \overset{biholo.}{\cong} \mathbb{C}^2.$$

• **Chen-Tang-Zhu**

Suppose $n = 2$, bounded curvature and

$Vol(B(x_0, r)) \geq C_1 r^{2n}, \quad for\ all\ r > 0.$

Then M^2 is biholomorphic to $\mathbb{C}^2$.

Ideas of proof

Step 1 Evolve the K$\ddot{a}$hler metric by the Ricci flow

- Time-decay estimate
- Stein
- homeomorphic to $\mathbb{R}^4$

Step 2 Construct algebraically independent holomorphic functions

- Space-decay estimate
- Poincar$\acute{e}$-Lelong equation
- H$\ddot{o}$rmander's $\bar{\partial}$-theory

Step 3 Algebraic embedding

- multiplicity estimaties
- desingularizaiton