P^3 & Beyond

Solving Energies with Higher Order Cliques

Pushmeet Kohli **Pawan Kumar** **Philip H. S. Torr**

Oxford Brookes University

Talk Overview

Oxford Brookes Vision Group

- **Higher Order Energy Functions**
- **Optimization Algorithms**
- **Move making Algorithms**
- **Optimal moves for Higher Order Energies**
- **P^N Potts Model**
- **Experiments**

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- Experiments

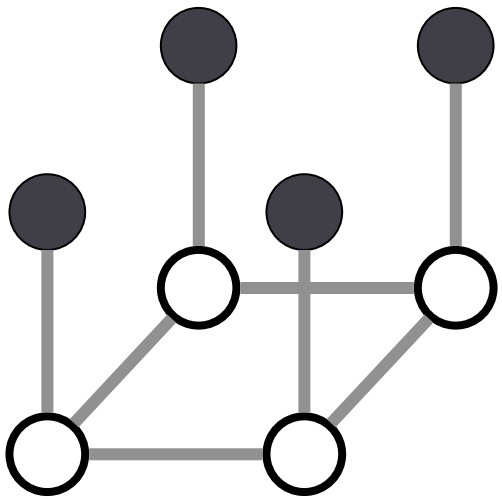
Energy Functions

- **Pairwise** Energy Functions

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j)$$

Unary **Pairwise**

Markov Random Field



Observed Variables

Hidden Variables

Energy Functions

- **Pairwise** Energy Functions

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j)$$

Unary **Pairwise**

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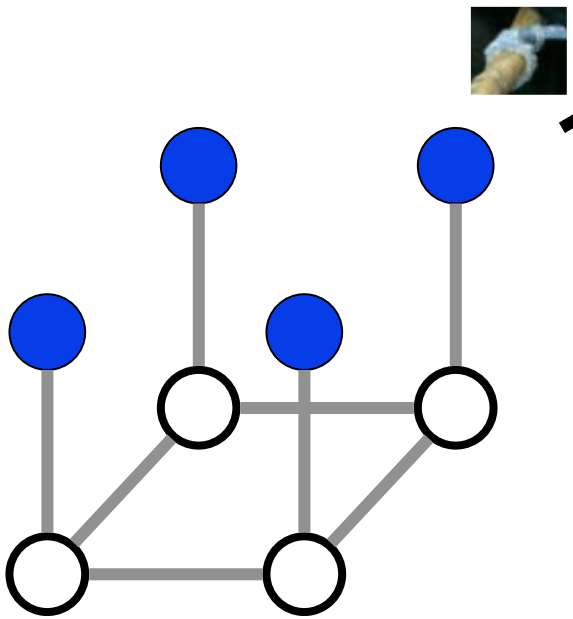

Energy Functions

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Unary **Pairwise**

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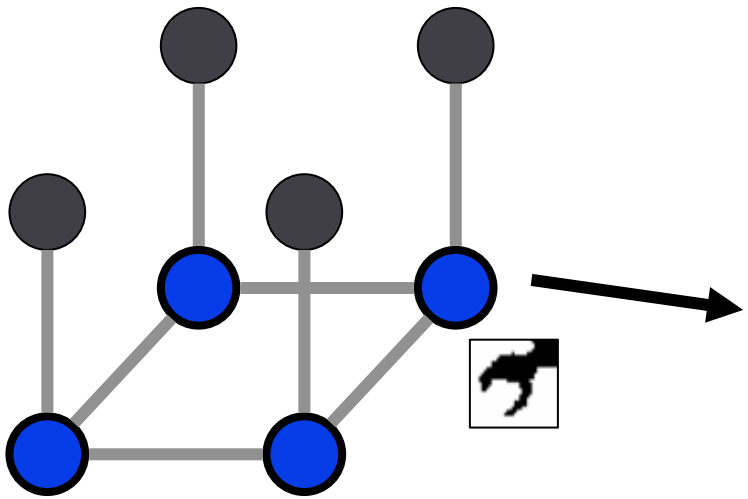
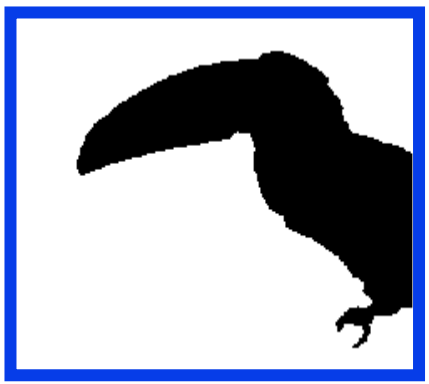




Energy Functions

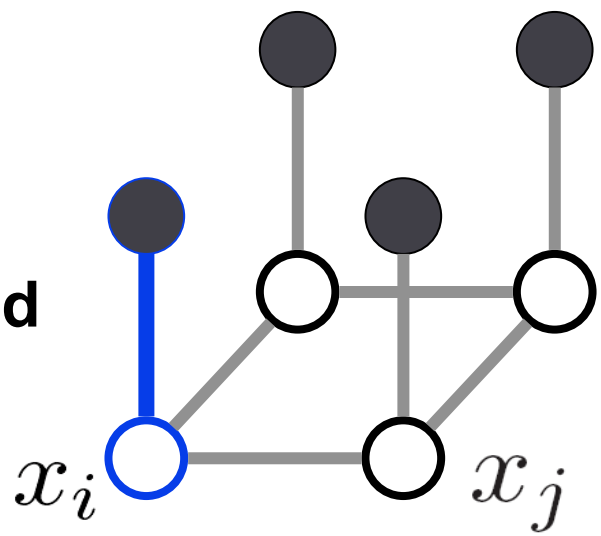
Oxford Brookes Vision Group

- **Pairwise** Energy Functions

$$E(\mathbf{x}) = \boxed{\sum_i \psi_i(x_i)} + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j)$$

Unary **Pairwise**

Markov Random Field



Energy Functions

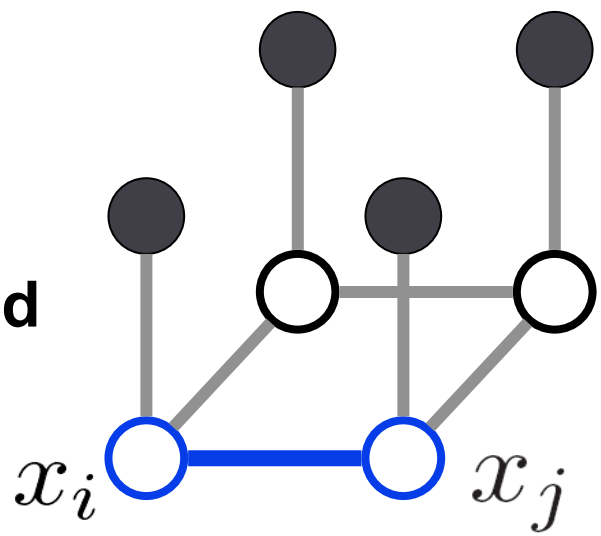
Oxford Brookes Vision Group

- **Pairwise** Energy Functions

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \boxed{\sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j)}$$

Unary **Pairwise**

Markov Random Field



- **Efficient Algorithms for Minimization**
- **Restricted Expressive Power!**

Energy Functions

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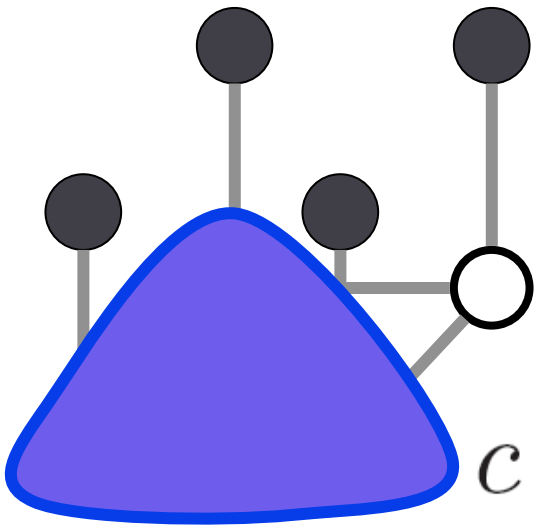
- **Higher Order** Energy Functions

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \boxed{\sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)}$$

Unary **Pairwise** **Higher order**

More expressive than pairwise

Markov Random Field



FOE: Field of Experts
(Roth & Black CVPR05)

Energy Functions

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- **Higher Order** Energy Functions

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \boxed{\sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)}$$

Unary **Pairwise** **Higher order**

Original

Pairwise MRF

Higher order MRF

MRF for Image Denoising

Images Courtesy: Lan *et al.* ECCV06

Energy Functions

Oxford Brookes Vision Group

- **Higher Order** Energy Functions

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \boxed{\sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)}$$

Unary **Pairwise** **Higher order**

- **Computationally expensive to minimize!**
- **Exponential Complexity $O(L^N)$**
  - **L = Number of Labels**
  - **N = Size of Clique**

Energy Functions

Oxford Brookes Vision Group

- **Higher Order** Energy Functions

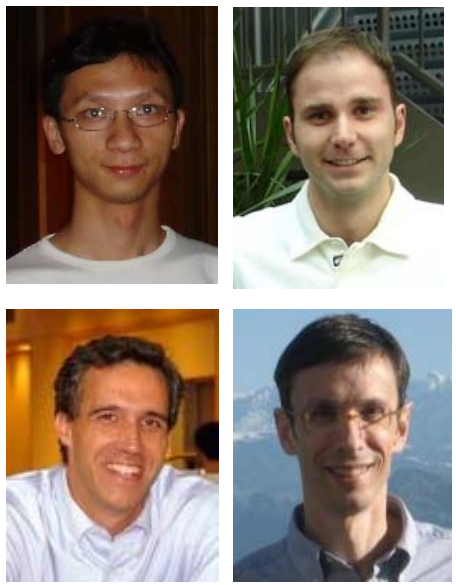
$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \boxed{\sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)}$$

Unary **Pairwise** **Higher order**

Efficient BP in Higher Order MRFs

ECCV06 (Lan, Roth, Huttenlocher, Black)

- **2x2 cliques learned using FOE model**
- **Approximation methods to make BP feasible**
- **Search a restricted state space**
- **16 minutes per iteration**



Energy Functions

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- **Higher Order** Energy Functions

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \boxed{\sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)}$$

Unary **Pairwise** **Higher order**

- **Our Method**
  - **Can handle cliques of thousand of variables**
  - **Extremely Efficient (works in seconds)**

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- **Move making Algorithms**
- **Optimal moves for Higher Order Energies**
- **P^N Potts Model**
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Optimization Algorithms

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- **General Energy Functions**
  - **NP-hard to minimize**
  - **Algorithms for Approximate Minimization**

- **Easy energy functions**
  - **Global minima in polynomial time**
  - **Tree topology**
  - **Submodular functions**

Submodular functions

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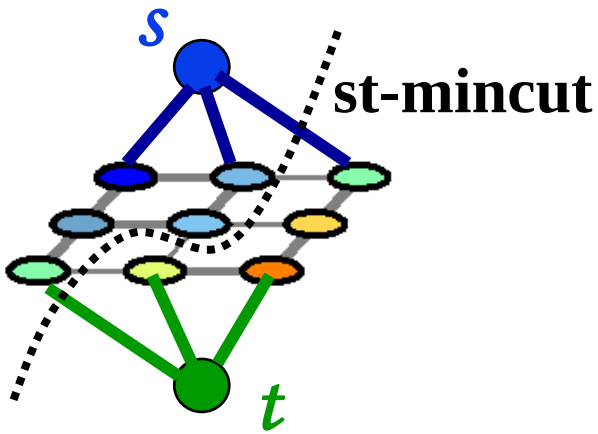
- **All projections on two variables are submodular.**
- **Any function f: {0,1}$^2 \rightarrow$ R is submodular if:**

$$f(0,0) + f(1,1) \leq f(0,1) + f(1,0)$$

- **In certain cases minimization equivalent to a st-mincut problem:**

$$\arg\min_{\mathbf{x} \in \mathbf{L}} E(\mathbf{x}) \quad \Longleftrightarrow$$



Minimization Problem

Graph Cut

Approximate Energy Functions

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Message Passing Algorithms

Belief Propagation (BP)

Tree Reweighted (TRW)

Move making Algorithms

α–Expansion

$\alpha\beta$–Swap

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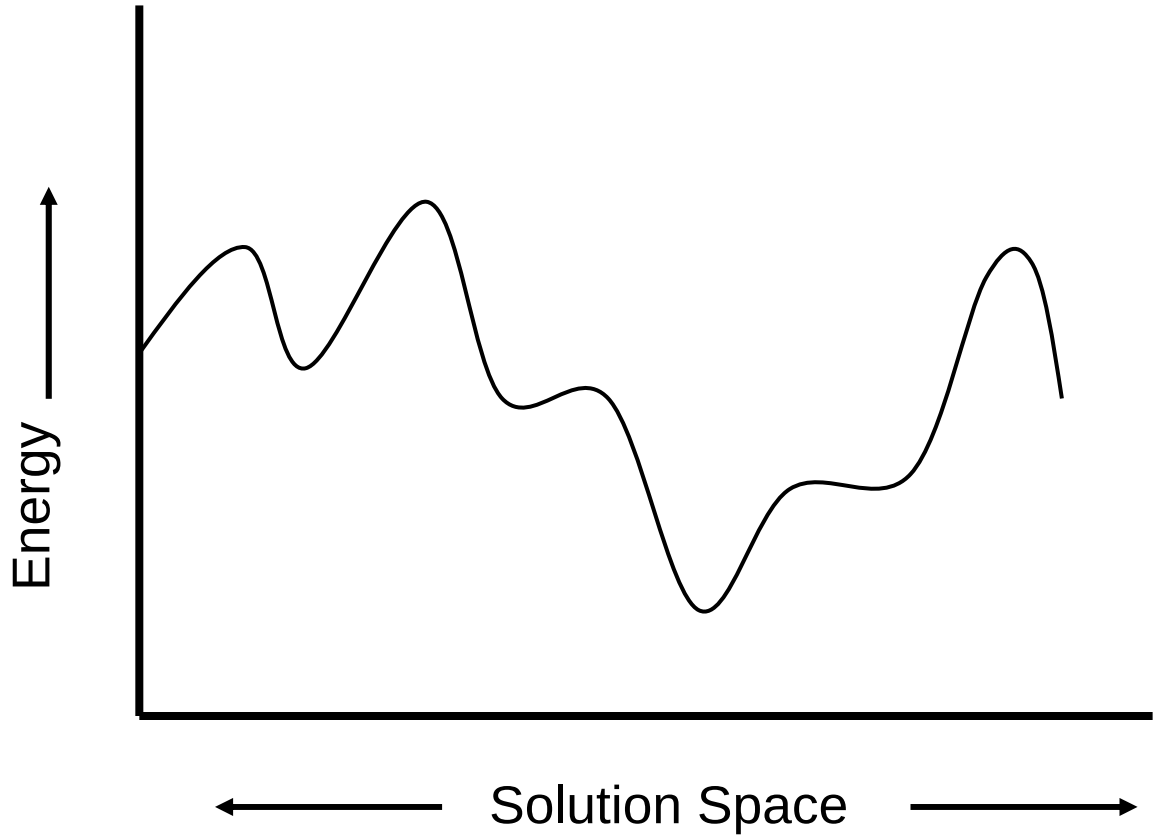
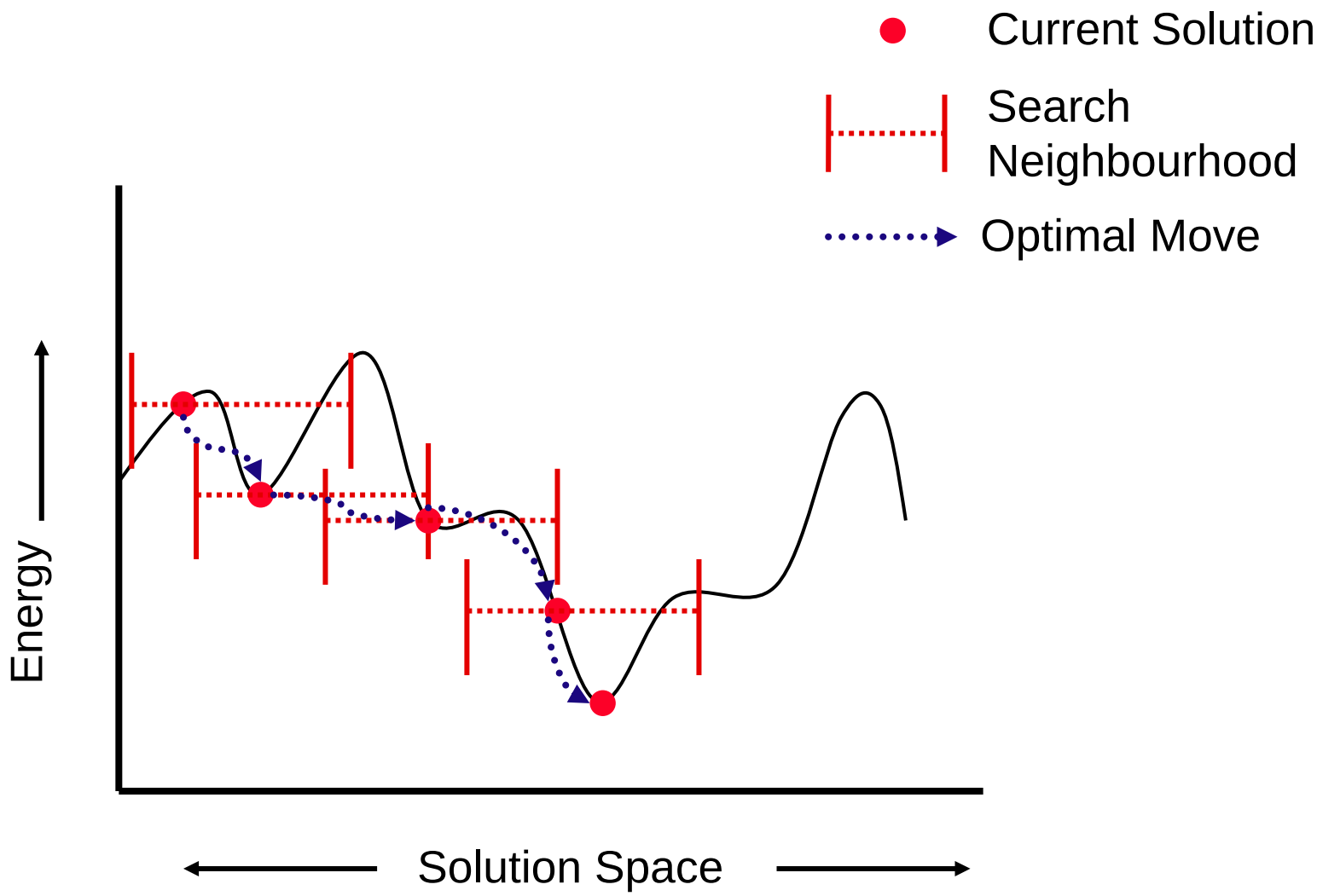
Move Making Algorithms



Move Making Algorithms



Move Making Algorithms

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- **Moves using graph cuts**

  [Boykov, Veksler, Zabih] PAMI 2001

- **Algorithm**
  - **Encode move by vector** $\mathbf{t}$
  - **Transformation function** $T(\mathbf{x},\mathbf{t}) \rightarrow \mathbf{x}`$
  - **Move Energy**

$$\boxed{E_m(\mathbf{t}) \;=\; E(T(\mathbf{x},\mathbf{t}))}$$

Submodular

  - **Optimal move t***

$$\mathbf{t}^* = \arg\min_{\mathbf{t}} E(T(\mathbf{x},\mathbf{t}))$$

Expansion Move

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Characteristics

- **Move**
  - **Variables take label α or retain current label**

$$T_\alpha(x_i, t_i) = \begin{cases} x_i & \text{if} \quad t_i = 0 \\ \alpha & \text{if} \quad t_i = 1 \end{cases}$$

- **Algorithm**
  - **Make a move for all α in $\mathcal{L}$**

Expansion Move

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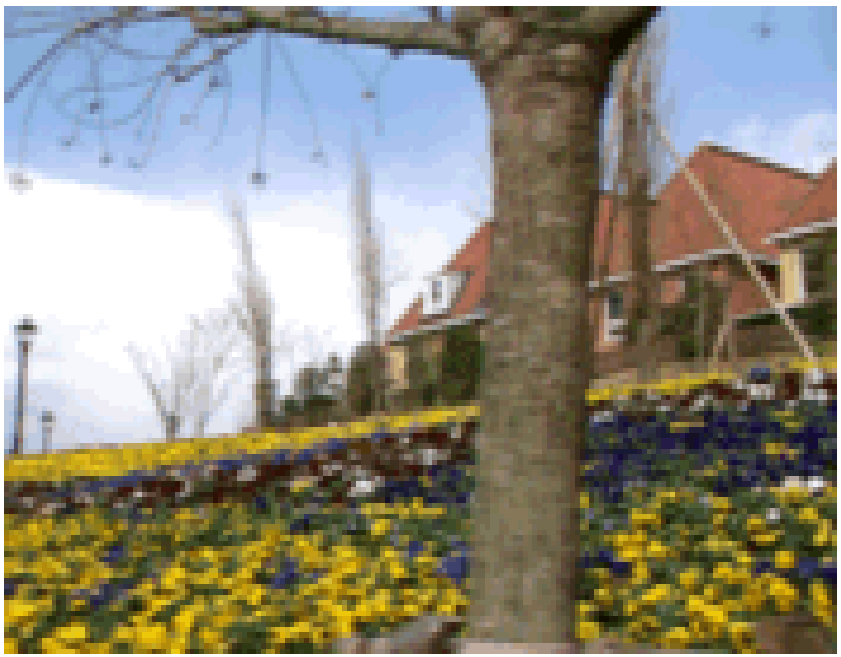
Status:

Initialize Sky to Tree

Expand House

- Tree
- Ground
- House
- Sky



Expansion Move

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Characteristics

- **Neighbourhood Size**
  - **2^N where N is the number of variables**

- **Guarantee**
  - **Move energy is submodular for all metric energy functions.** **[Boykov, Veksler, Zabih] PAMI 2001**

$$\begin{aligned} \psi_{ij}(a,b) &= 0 \iff a = b \\ \psi_{ij}(a,b) &= \psi_{ij}(b,a) \geq 0 \\ \psi_{ij}(a,d) &\leq \psi_{ij}(a,b) + \psi_{ij}(b,d) \end{aligned}$$

Swap Move

Characteristics

- **Move**
  - **Variables labeled α, β can swap their labels**

$$T_{\alpha\beta}(x_i, t_i) = \begin{cases} \alpha & \text{if} \quad x_i = \alpha \text{ or } \beta \text{ and } t_i = 0 \\ \beta & \text{if} \quad x_i = \alpha \text{ or } \beta \text{ and } t_i = 1 \end{cases}$$

- **Algorithm**
  - **Make a move for all α, β in $\mathcal{L}$**

Swap Move

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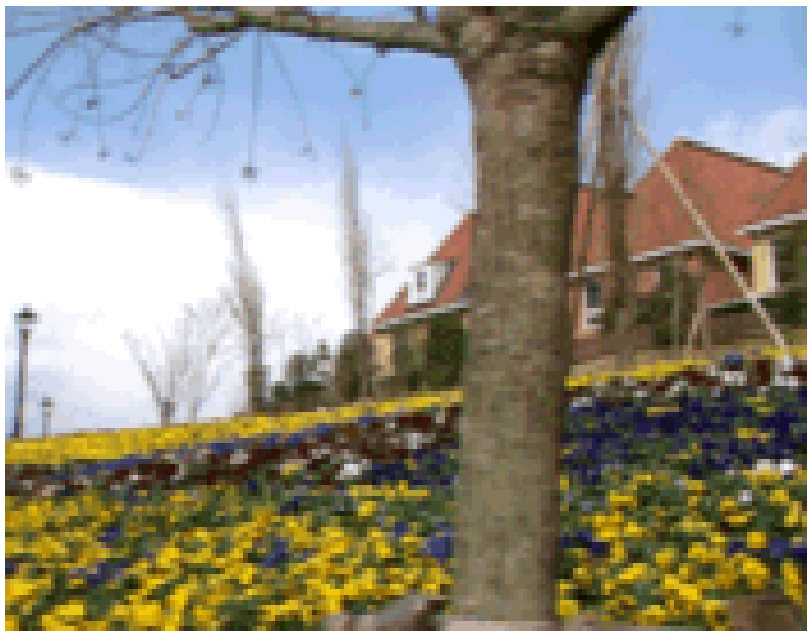
Swap Sky, House



Swap Move

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Characteristics

- **Neighbourhood Size**
  - **2^N where N is the number of variables**
- **Guarantee**
  - **Move energy is submodular for all semi-metric energy functions.** **[Boykov, Veksler, Zabih] PAMI 2001**

$$\boxed{\begin{aligned} \psi_{ij}(a,b) &= 0 \iff a = b \\ \psi_{ij}(a,b) &= \psi_{ij}(b,a) \geq 0 \end{aligned}}$$

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Moves for Higher Order Cliques

- **Form of the Higher Order Potentials**

$$\psi_c(\mathbf{x}_c) = f_c(\mathcal{Q}_c(\mathbf{x}_c))$$

- **Sum Form**

$$\mathcal{Q}_c(\mathbf{x}_c) = \sum_{i,j \in c} \phi_c(x_i, x_j)$$

- **Max Form**

$$\mathcal{Q}_c(\mathbf{x}_c) = \max_{i,j \in c} \phi_c(x_i, x_j)$$

Theoretical Results: Swap

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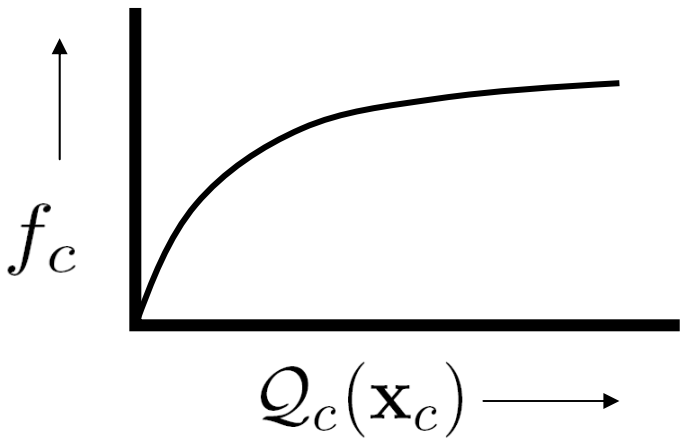
- **Move energy is always submodular if**

$$\psi_c(\mathbf{x}_c) = f_c(\mathcal{Q}_c(\mathbf{x}_c))$$

non-decreasing concave.



$$\mathcal{Q}_c(\mathbf{x}_c) = \sum_{i,j \in c} \phi_c(x_i, x_j)$$

$$\phi_c(a,b) = \phi_c(b,a) \quad \forall a,b \in \mathcal{L}$$
$$\phi_c(a,b) \geq \phi_c(d,d) \quad \forall a,b,d \in \mathcal{L}$$

See paper for proofs

Theoretical Results: Expansion

Oxford Brookes Vision Group

- **Move energy is always submodular if**

$$\psi_c(\mathbf{x}_c) = f_c(\mathcal{Q}_c(\mathbf{x}_c))$$

increasing linear

$$\mathcal{Q}_c(\mathbf{x}_c) = \max_{i,j \in c} \phi_c(x_i, x_j)$$

$$\phi_c(x_i, x_j) = \begin{cases} \gamma_k & \text{if} \quad x_i = x_j = l_k \\ \gamma_{\max} & \text{otherwise.} \end{cases}$$

See paper for proofs

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P^N Potts Model

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$$\psi_c(\mathbf{x}_c) = \begin{cases} \gamma_k & \text{if} \quad x_i = l_k, \forall i \in c \\ \gamma_{\max} & \text{otherwise.} \end{cases}$$

$$\gamma_{\max} > \gamma_k$$

- **Generalization of the Potts Model:**

$$\phi_c(x_i, x_j) = \begin{cases} \gamma_k & \text{if} \quad x_i = x_j = l_k \\ \gamma_{\max} & \text{otherwise.} \end{cases}$$

$$\gamma_k = 0$$

Solving the P^N Potts Model

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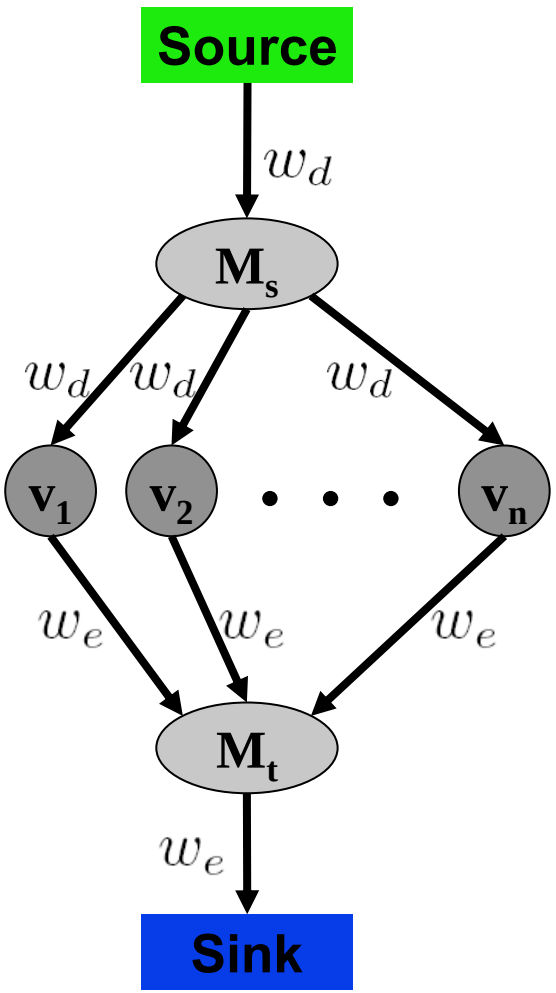
- Computing the optimal **swap** move

$$\psi_c(T_{\alpha\beta}(\mathbf{x}_c, \mathbf{t}_c)) = \begin{cases} \gamma_\alpha & \text{if } t_i = 0, \forall i \in c \\ \gamma_\beta & \text{if } t_i = 1, \forall i \in c \\ \gamma_{\max} & \text{otherwise.} \end{cases}$$

$$\kappa = \gamma_{\max} - \gamma_\alpha - \gamma_\beta$$
$$w_e = \gamma_\alpha + \kappa$$
$$w_d = \gamma_\beta + \kappa$$



$t_i = 0 \longleftrightarrow v_i \in$ **Source**

$t_j = 1 \longleftrightarrow v_j \in$ **Sink**

Solving the $\mathbf{P^N}$ Potts Model

Oxford Brookes Vision Group

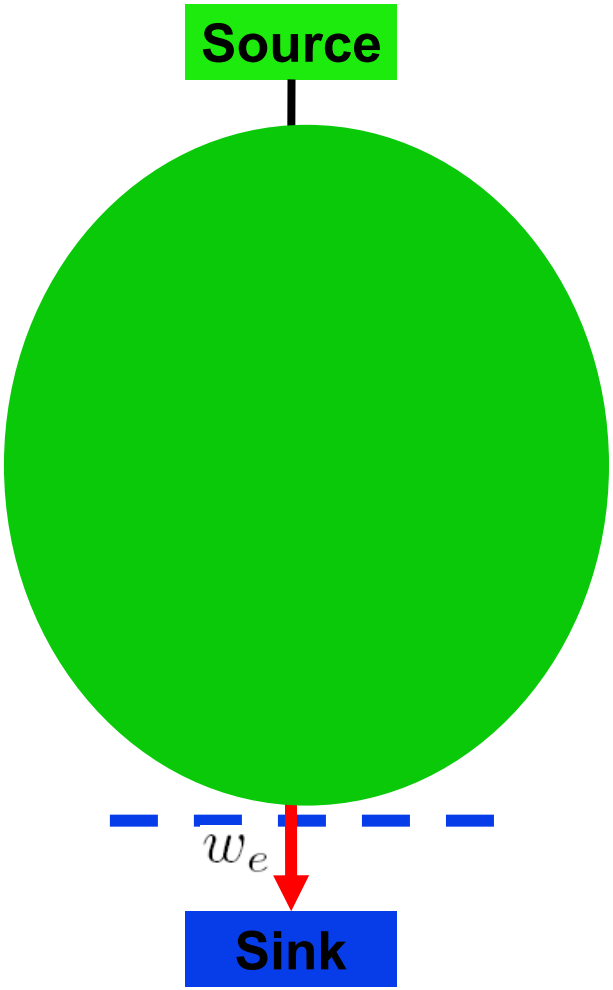
- Computing the optimal **swap** move

$$\psi_c(T_{\alpha\beta}(\mathbf{x}_c, \mathbf{t}_c)) = \begin{cases} \text{[redacted]} \\ \gamma_\beta \quad \text{if } t_i = 1, \forall i \in c \\ \gamma_{\max} \text{ otherwise.} \end{cases}$$

$$\kappa = \gamma_{\max} - \gamma_\alpha - \gamma_\beta$$
$$w_e = \gamma_\alpha + \kappa$$
$$w_d = \gamma_\beta + \kappa$$

Case 1: all $t_i = 0$ ($x_i = \alpha$)

Cost: [redacted]



Solving the $\mathbf{P^N}$ Potts Model

Oxford Brookes Vision Group

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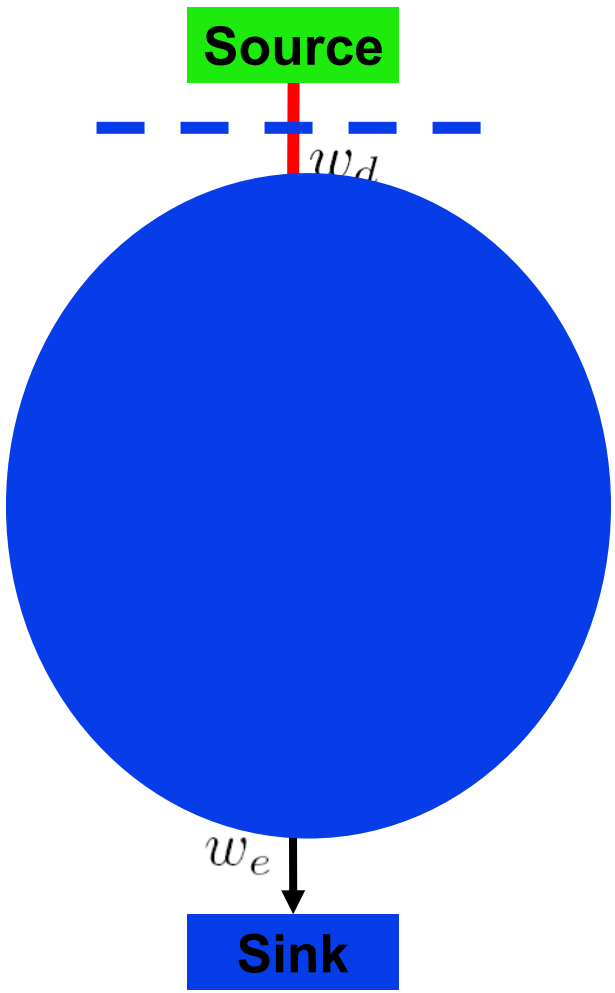

Case 2: all t_i = 1 ($x_i = \beta$)

Cost:

Solving the $\mathbf{P^N}$ Potts Model

Oxford Brookes Vision Group

- Computing the optimal **swap** move

$$\psi_c(T_{\alpha\beta}(\mathbf{x}_c, \mathbf{t}_c)) = \begin{cases} \gamma_\alpha & \text{if } t_i = 0, \forall i \in c \\ \gamma_\beta & \text{if } t_i = 1, \forall i \in c \\ \text{[redacted]} \end{cases}$$

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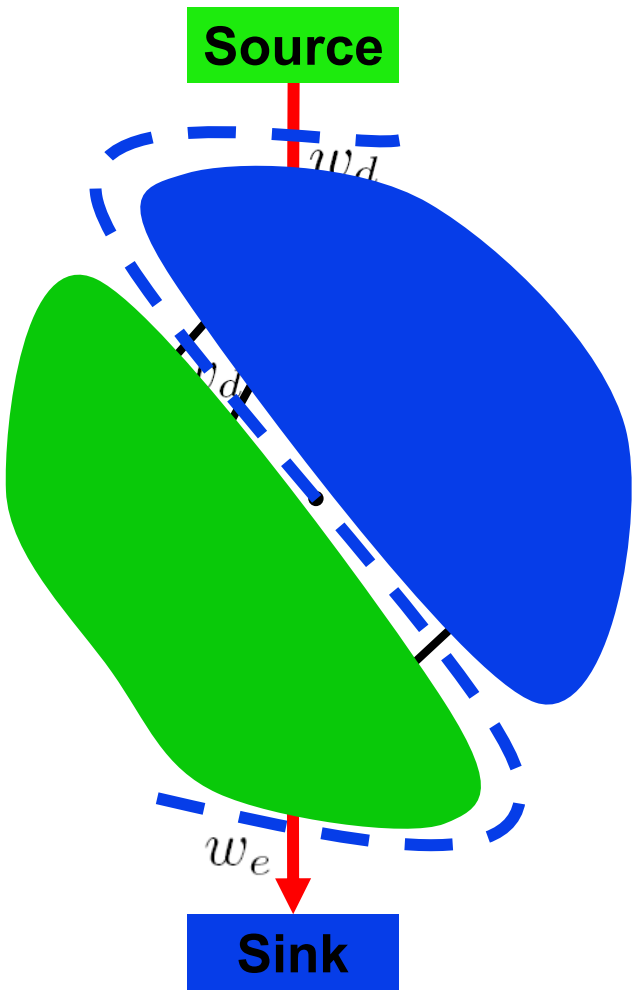


Case 3: t_i = 0,1 ($x_i = \alpha,\beta$)

Cost: [redacted]

Solving the $\mathbf{P^N}$ Potts Model

Oxford Brookes Vision Group

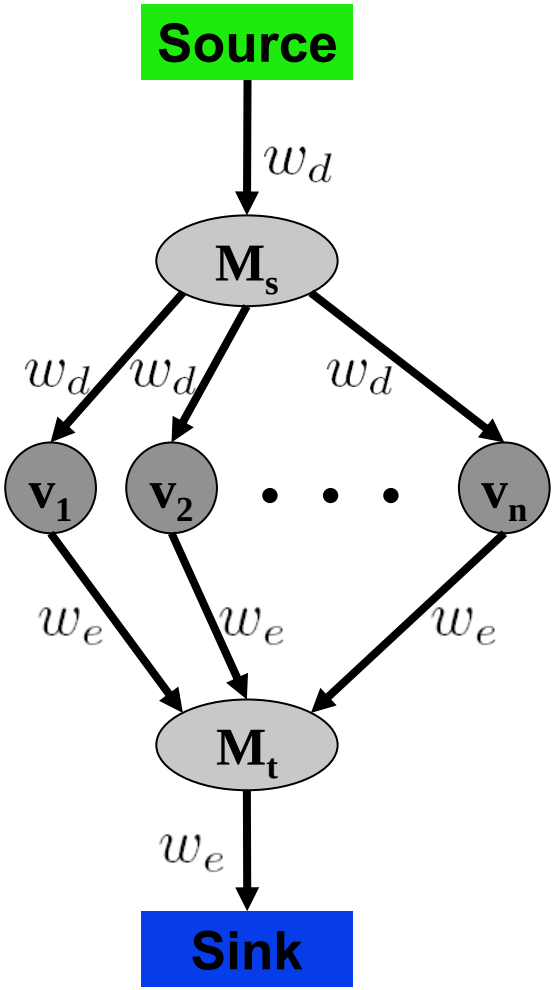
- Computing the optimal **expansion** move

$$\psi_c(T_\alpha(\mathbf{x}_c, \mathbf{t}_c)) = \begin{cases} \gamma & \text{if } t_i = 0, \forall i \in c \\ \gamma_\alpha & \text{if } t_i = 1, \forall i \in c \\ \gamma_{\max} & \text{otherwise,} \end{cases}$$

$$\kappa = \gamma_{\max} - \gamma_\alpha - \gamma$$

$$w_d = \gamma_\alpha + \kappa$$

$$w_e = \gamma + \kappa$$



Solving the $\mathbf{P^N}$ Potts Model

Oxford Brookes Vision Group

- Computing the optimal **expansion** move

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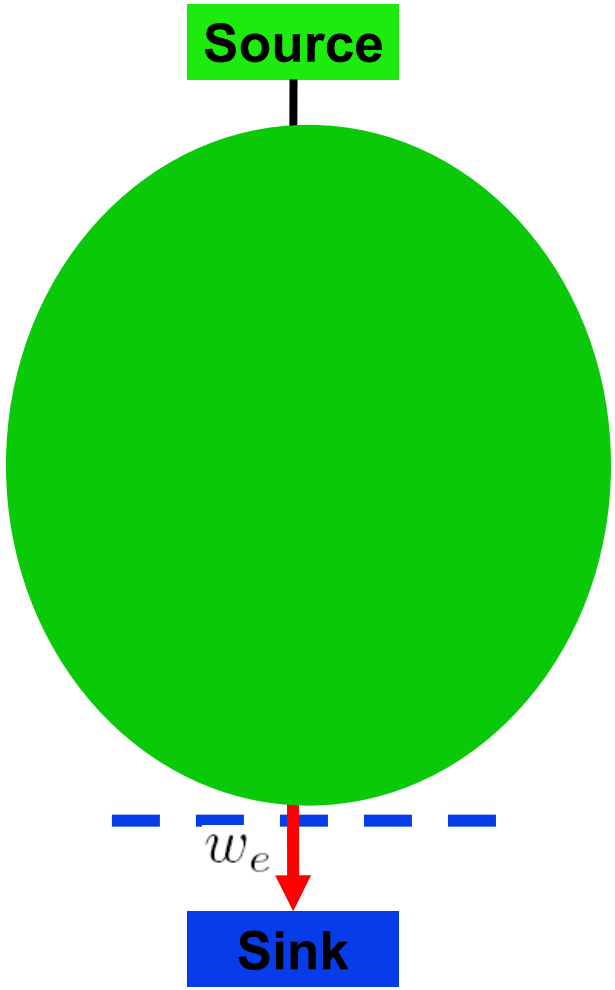
Case 1: all t_i = 0 ($x_i = x_i$)

Cost:



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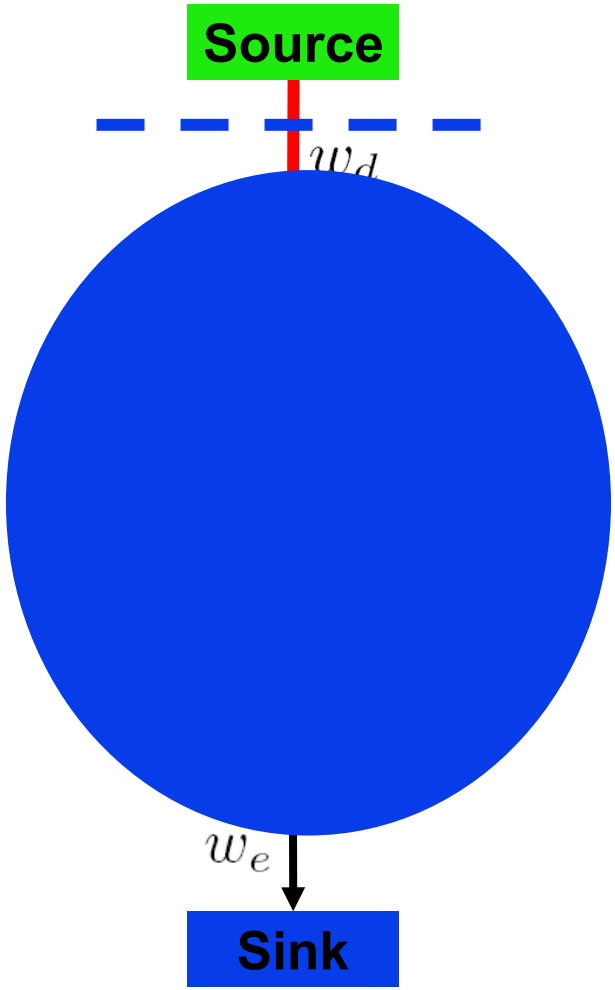
$$w_d = \gamma_\alpha + \kappa$$

$$w_e = \gamma + \kappa$$

Case 2: all t_i = 1 ($x_i = \alpha$)

Cost: [redacted]



Solving the P^N Potts Model

Oxford Brookes Vision Group

- Computing the optimal **expansion** move

$$\psi_c(T_\alpha(\mathbf{x}_c, \mathbf{t}_c)) = \begin{cases} \gamma & \text{if } t_i = 0, \forall i \in c \\ \gamma_\alpha & \text{if } t_i = 1, \forall i \in c \\ \text{[redacted]} \end{cases}$$

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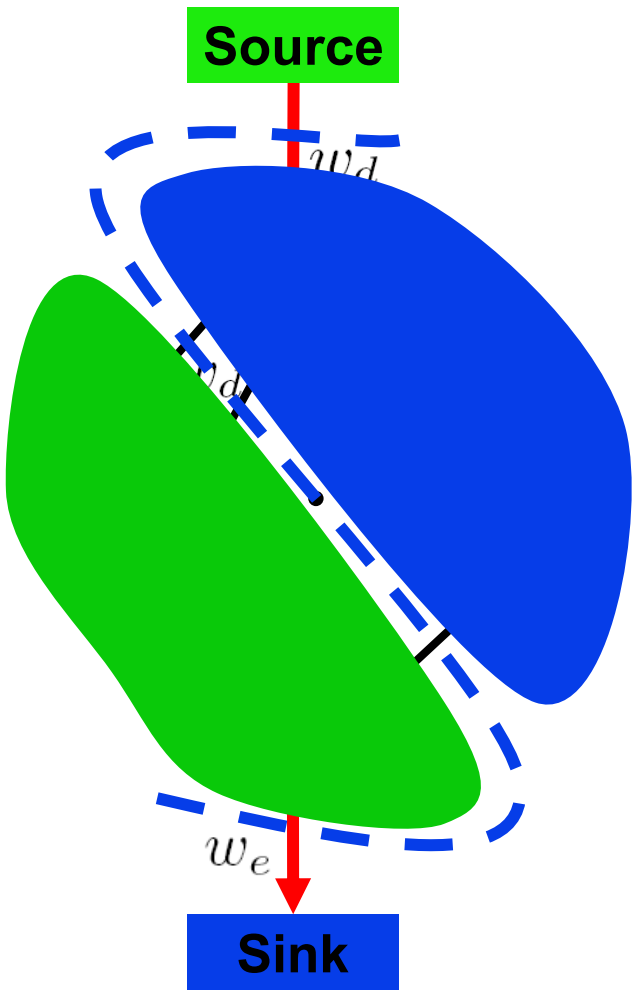
$$w_d = \gamma_\alpha + \kappa$$

$$w_e = \gamma + \kappa$$

Case 3: t_i = 0,1 ($x_i = x_i$, α)

Cost: [redacted]



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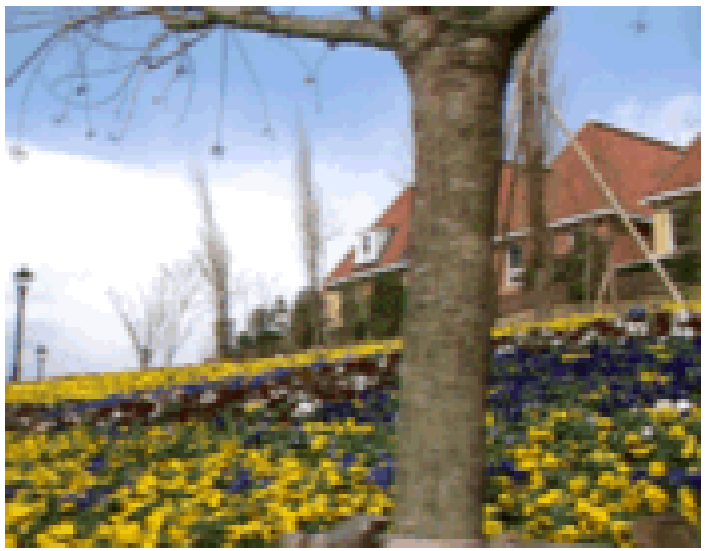
Experimental Results

Oxford Brookes Vision Group

- **Exemplar based Texture Segmentation**

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)$$

Unary (Colour) | **Pairwise (Smoothness)** | **Higher Order (Texture)**



Original Image

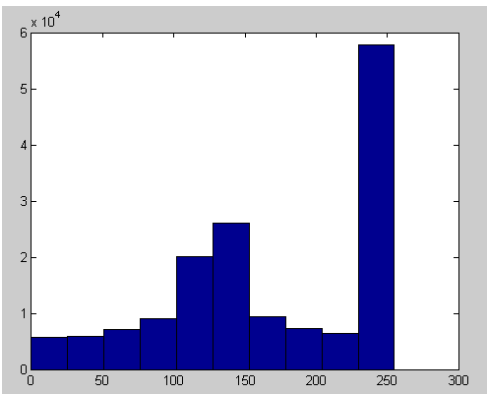
Experimental Results

Oxford Brookes Vision Group

- **Exemplar based Texture Segmentation**

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)$$

Unary (Colour) | **Pairwise (Smoothness)** | **Higher Order (Texture)**



Colour Histograms

Unary Cost: Tree

Experimental Results

Oxford Brookes Vision Group

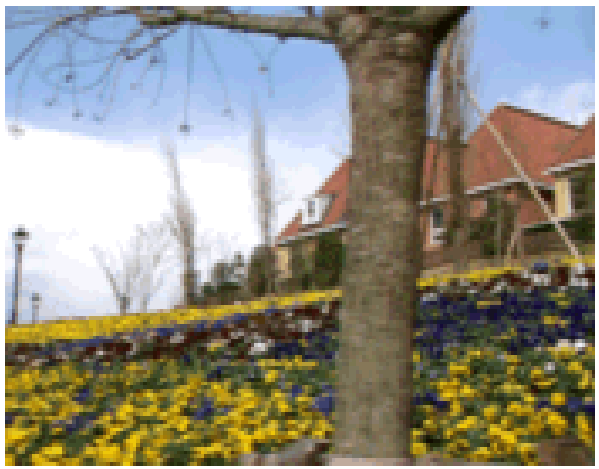
- **Exemplar based Texture Segmentation**

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)$$

Unary (Colour)

Pairwise (Smoothness)

Higher Order (Texture)



Edge Sensitive Smoothness Cost

Experimental Results

Oxford Brookes Vision Group

- **Exemplar based Texture Segmentation**

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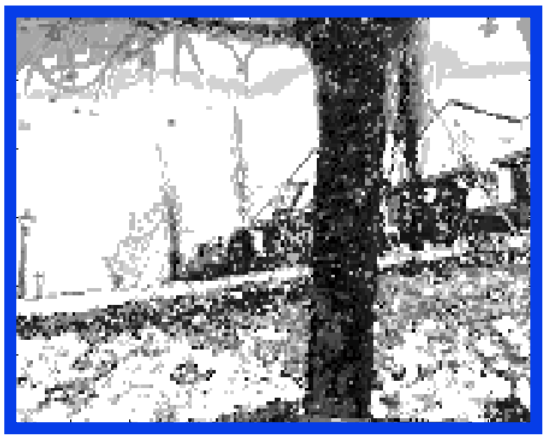
Unary (Colour)

Pairwise (Smoothness)

Higher Order (Texture)



MAP Solution

Experimental Results

Oxford Brookes Vision Group

- **Exemplar based Texture Segmentation**

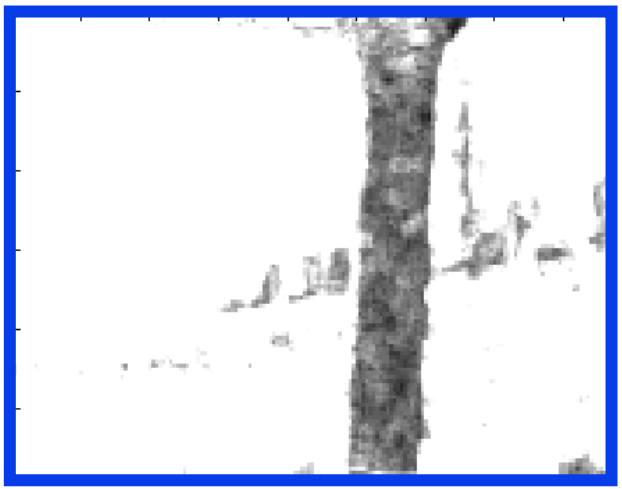
$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)$$

Unary (Colour)

Pairwise (Smoothness)

Higher Order (Texture)

Patch Dictionary



Higher Order Cost: Tree

Experimental Results

Oxford Brookes Vision Group

- **Exemplar based Texture Segmentation**

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)$$

Unary (Colour) | **Pairwise (Smoothness)** | **Higher Order (Texture)**

Unary Cost: Tree

Higher Order Cost: Tree

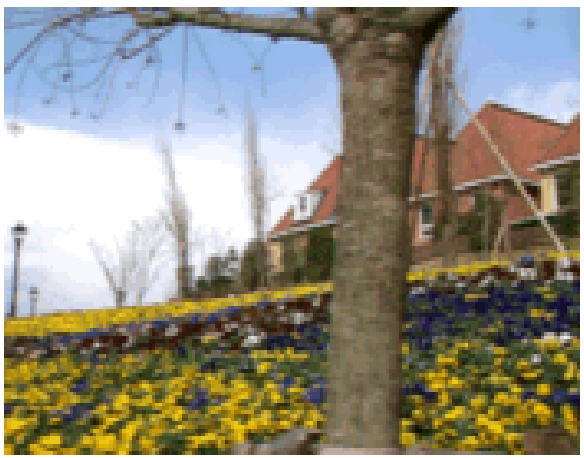
Experimental Results

Oxford Brookes Vision Group

- **Exemplar based Texture Segmentation**

$$E(\mathbf{x}) = \sum_{i} \psi_i(x_i) + \sum_{i \in \mathcal{V}, j \in \mathcal{N}_i} \psi_{ij}(x_i, x_j) + \sum_{c \in \mathcal{C}} \psi_c(\mathbf{x}_c)$$

Unary (Colour) | **Pairwise (Smoothness)** | **Higher Order (Texture)**



Original

Pairwise

Higher order

Experimental Results

Oxford Brookes Vision Group

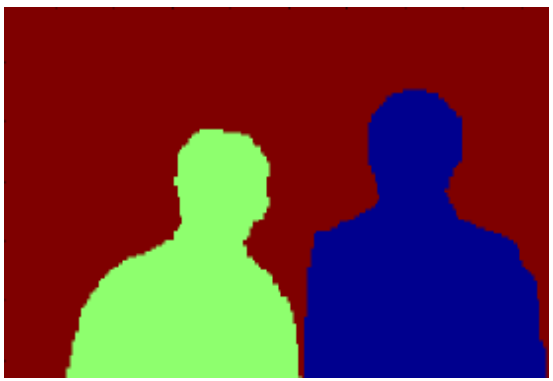
Pairwise

Higher Order

Original

Swap (3.2 sec)

Swap (4.2 sec)



Ground Truth

Expansion (2.5 sec)

Expansion (3.0 sec)

Experimental Results

Oxford Brookes Vision Group

Pairwise

Higher Order

Original

Swap (4.7 sec)

Swap (5.0 sec)

Ground Truth

Expansion (3.7sec)

Expansion (4.4 sec)

Conclusions & Future Work

Oxford Brookes Vision Group

- **Efficient minimization of certain higher order energies**
- **Can handle very large cliques**
- **Useful for many Computer Vision problems**
- **Explore other interesting family of potential functions**

Thanks

Oxford Brookes Vision Group

- Questions?