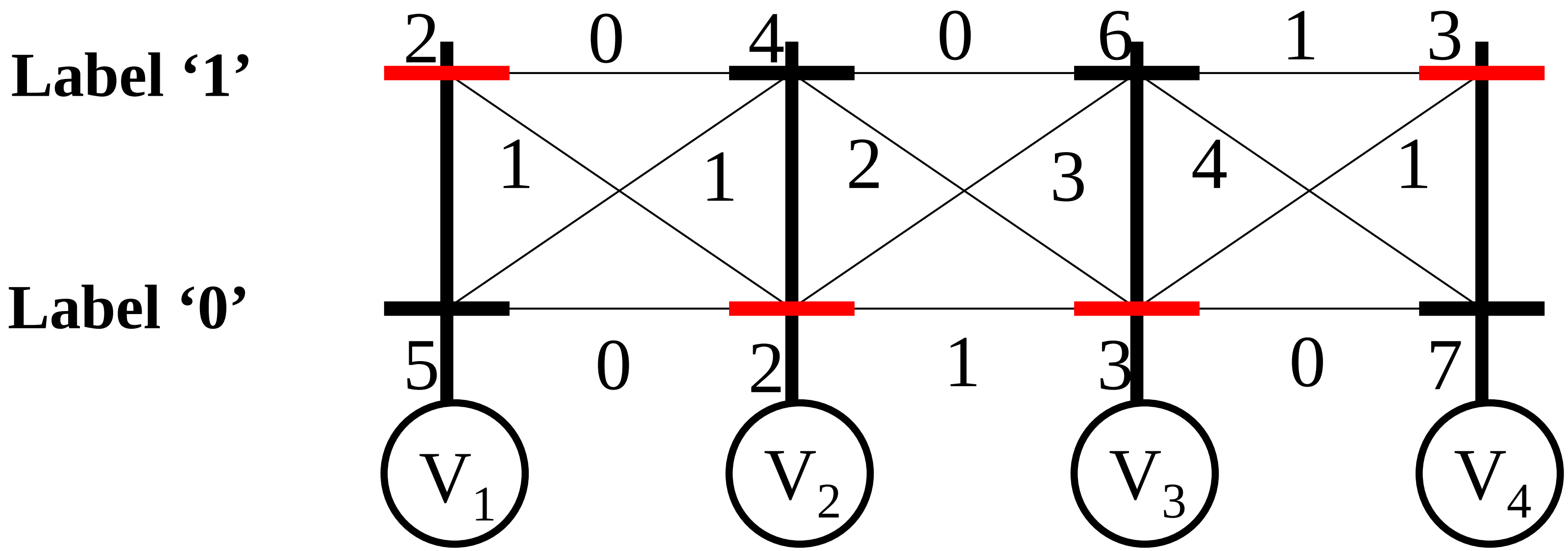
An Analysis of Convex Relaxations for MAP Estimation

M. Pawan Kumar, P. Kohli
Vladimir Kolmogorov
Philip Torr

Aim

- To analyze convex relaxations for MAP estimation



Random Variables $V = \{V_1, \ldots, V_4\}$

Label Set $L = \{0, 1\}$

Labelling $m = \{1, 0, 0, 1\}$

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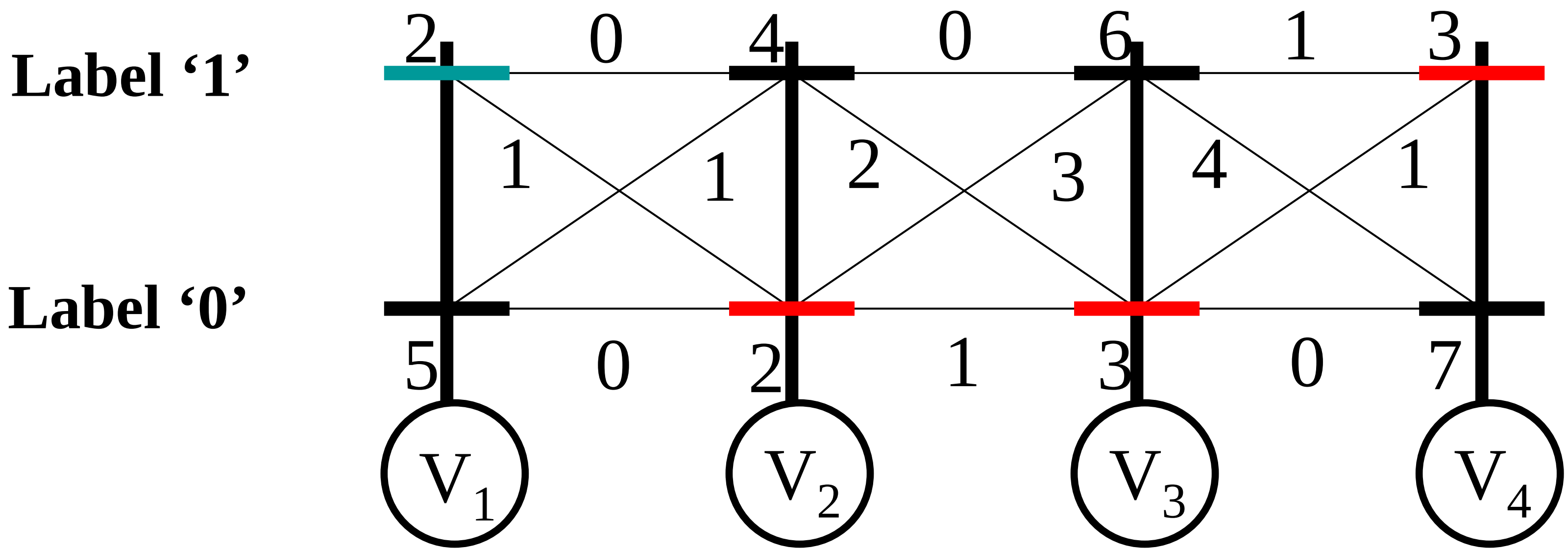
- To analyze convex relaxations for MAP estimation



Cost(m) = 2

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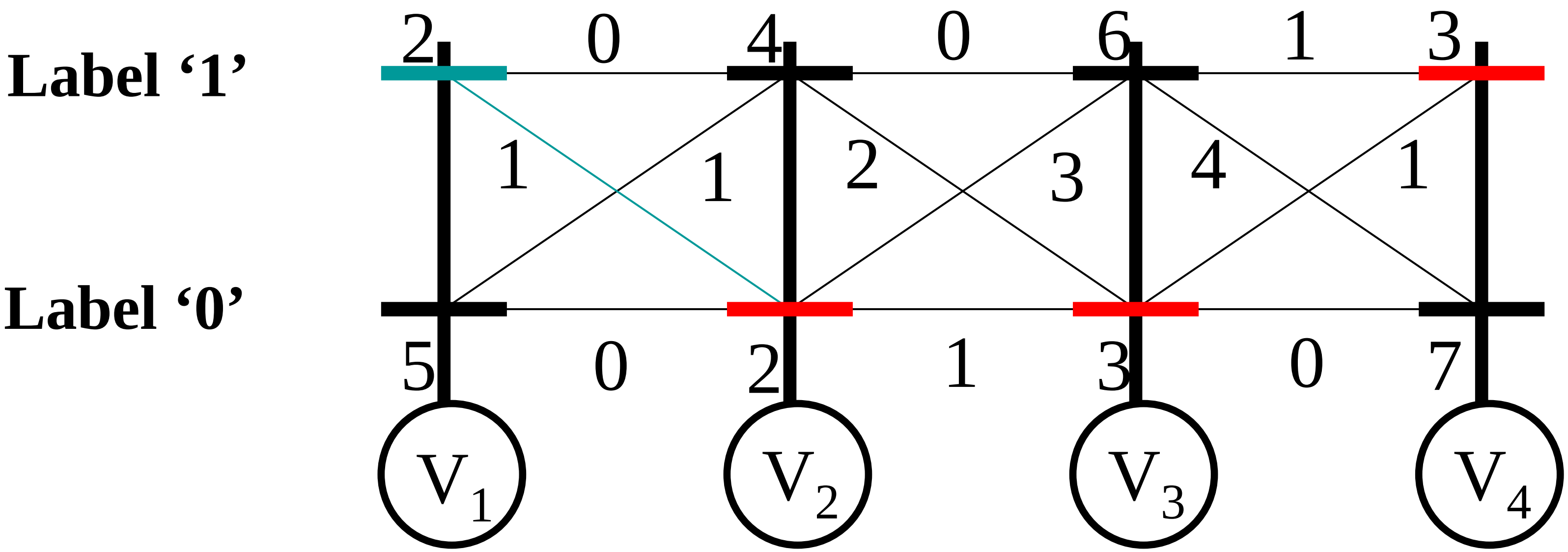
- To analyze convex relaxations for MAP estimation



Cost(m) = 2 + 1

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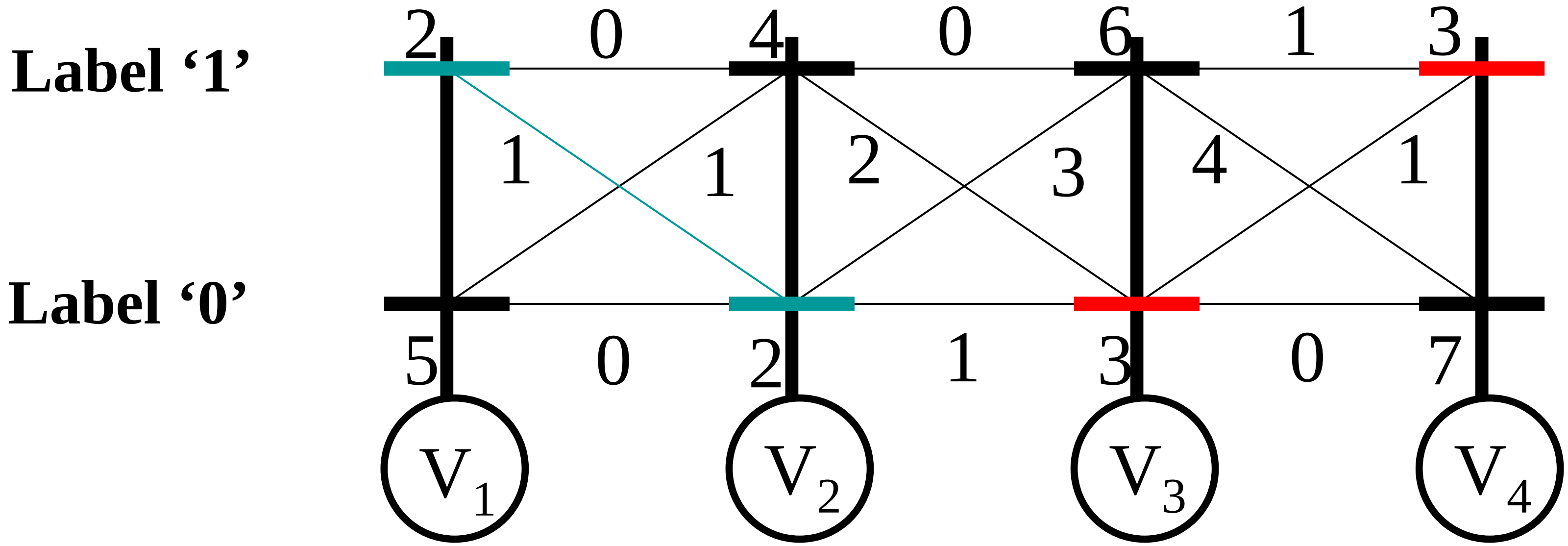
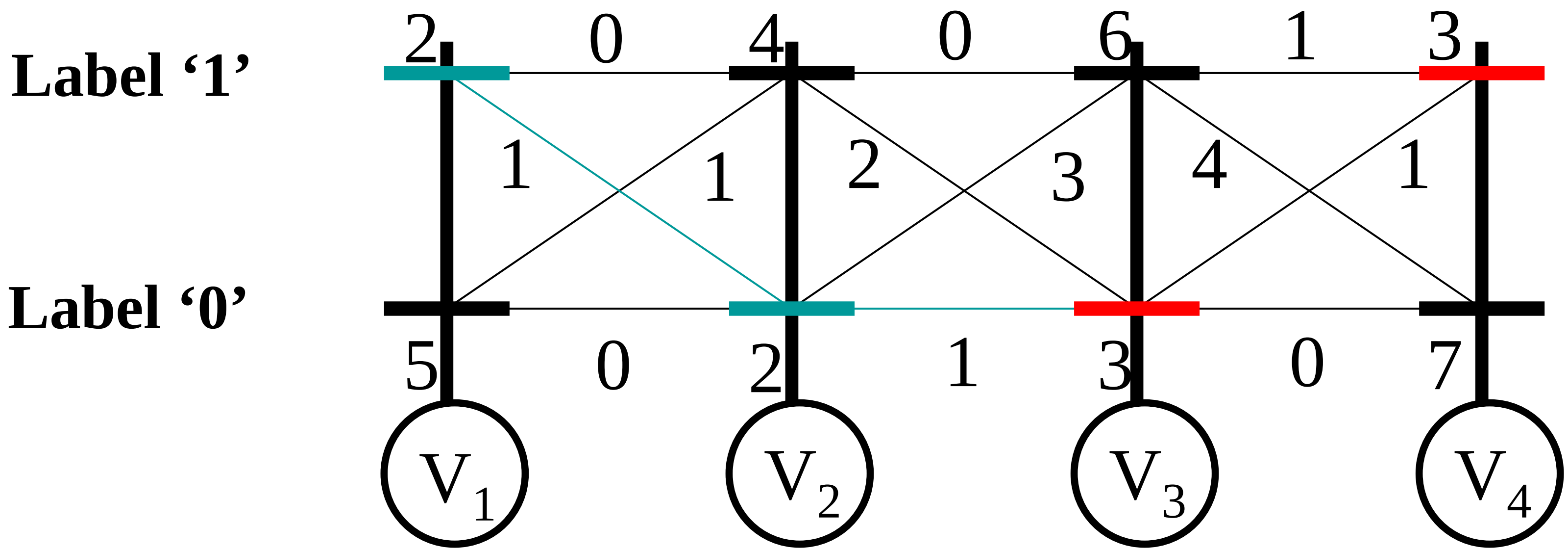
- To analyze convex relaxations for MAP estimation



Cost(m) = 2 + 1 + 2

Aim

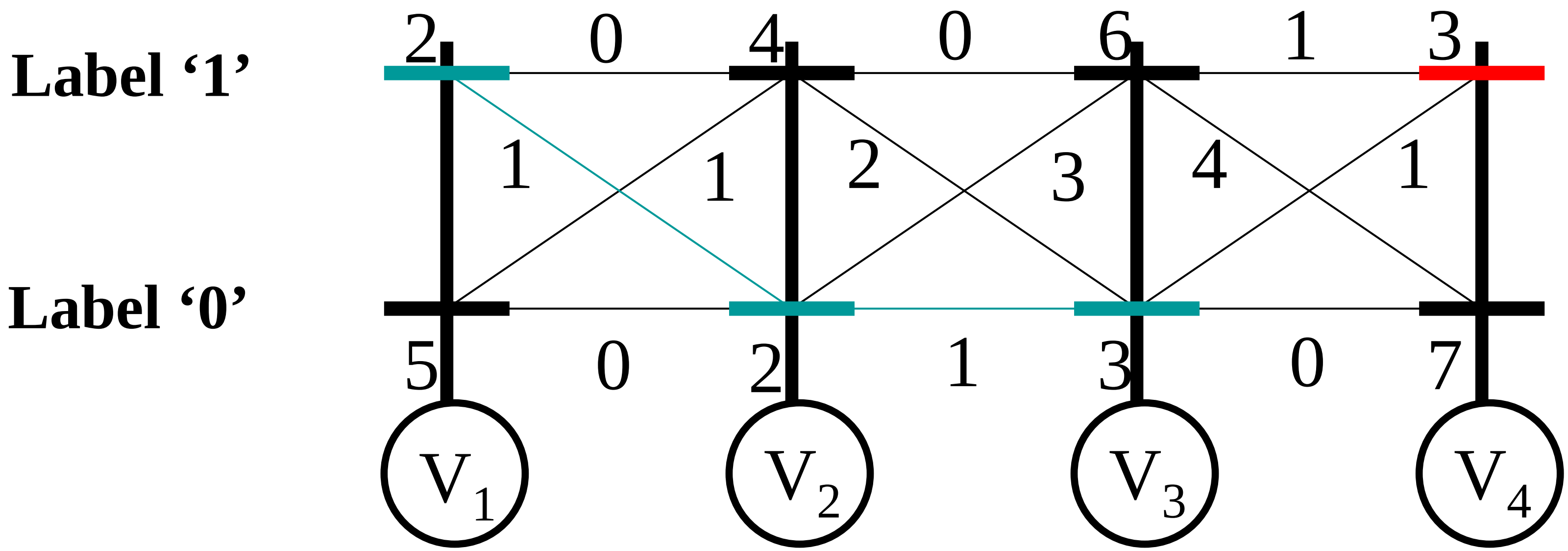
- To analyze convex relaxations for MAP estimation



Cost(m) = 2 + 1 + 2 + 1

Aim

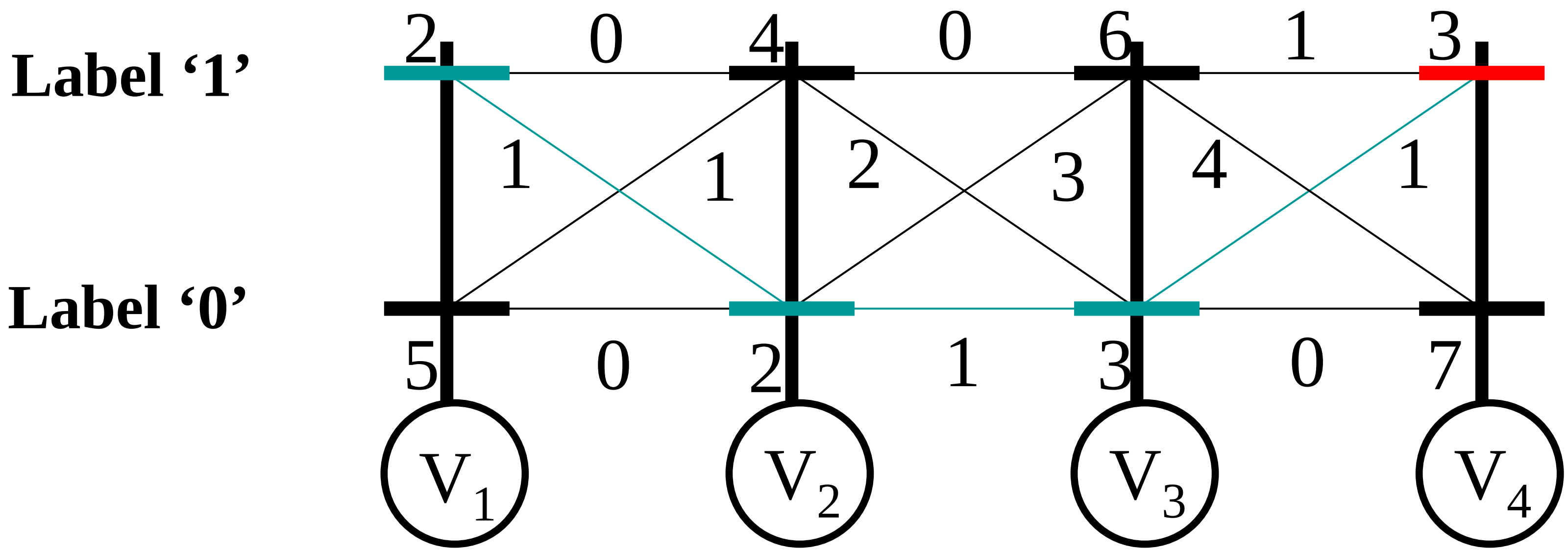
- To analyze convex relaxations for MAP estimation



Cost(m) = 2 + 1 + 2 + 1 + 3

Aim

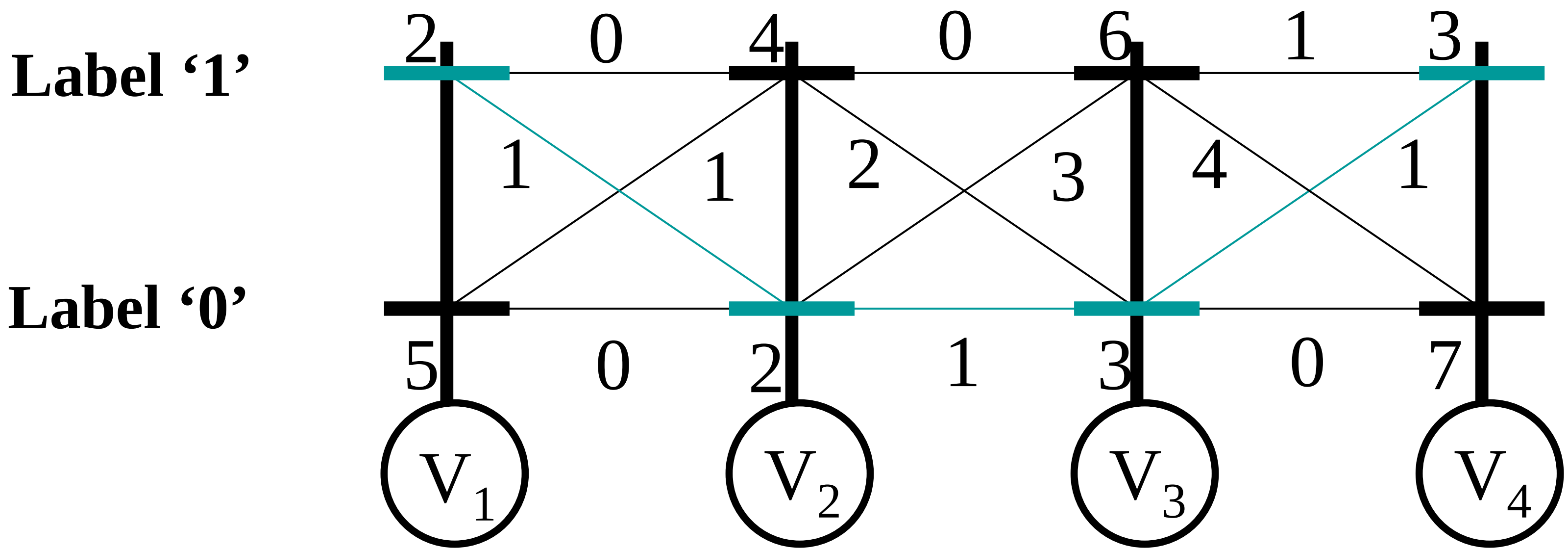
- To analyze convex relaxations for MAP estimation



Cost(m) = 2 + 1 + 2 + 1 + 3 + 1

Aim

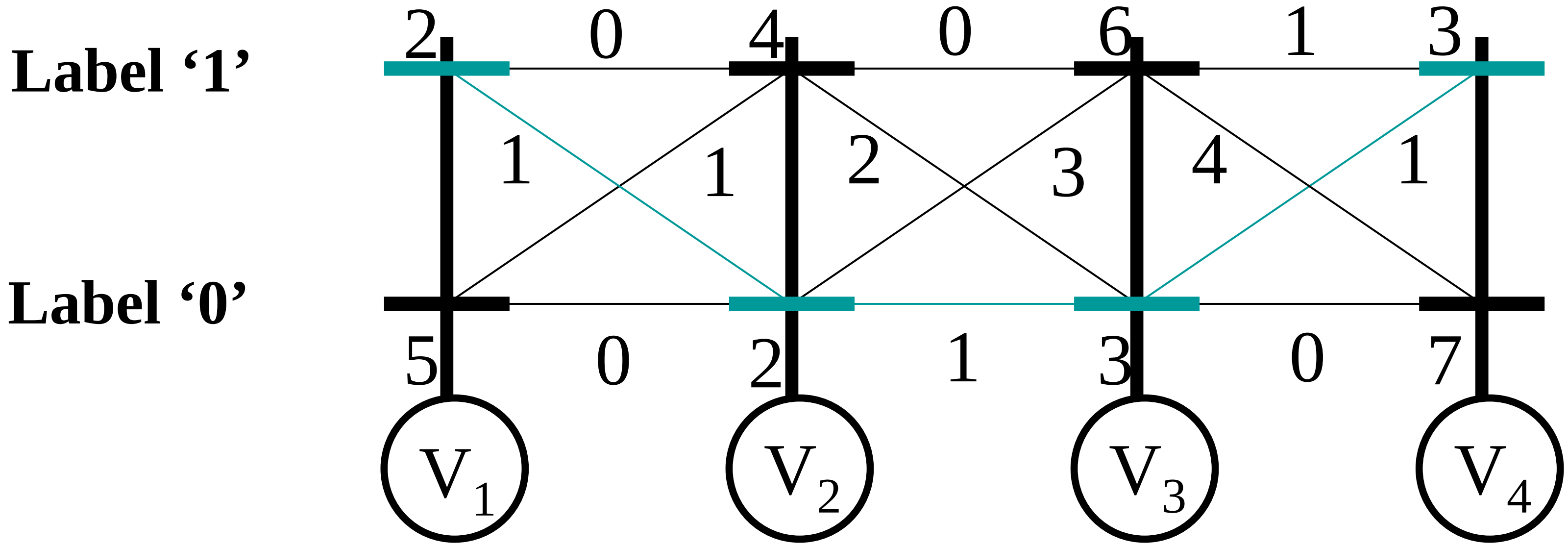
- To analyze convex relaxations for MAP estimation



Cost(m) = 2 + 1 + 2 + 1 + 3 + 1 + 3

Aim

- To analyze convex relaxations for MAP estimation



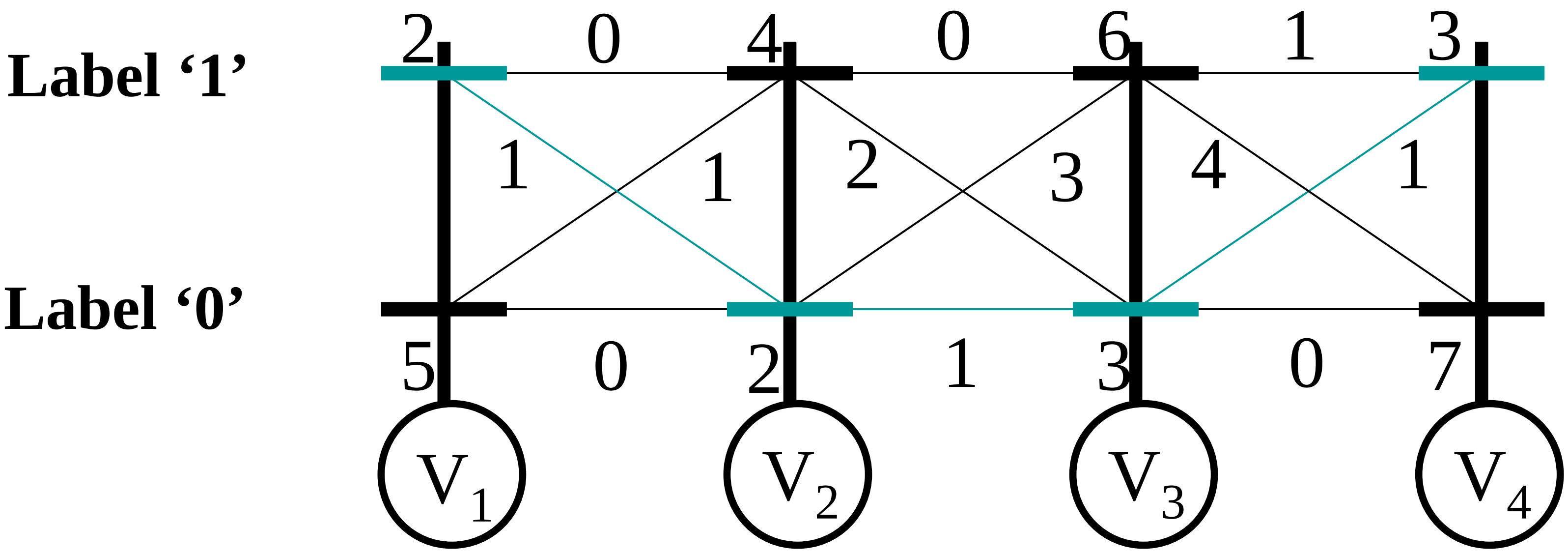
Cost(m) = 2 + 1 + 2 + 1 + 3 + 1 + 3 = 13

Pr(m) ∝ exp(-Cost(m))

Minimum Cost Labelling = MAP estimate

Aim

- To analyze convex relaxations for MAP estimation



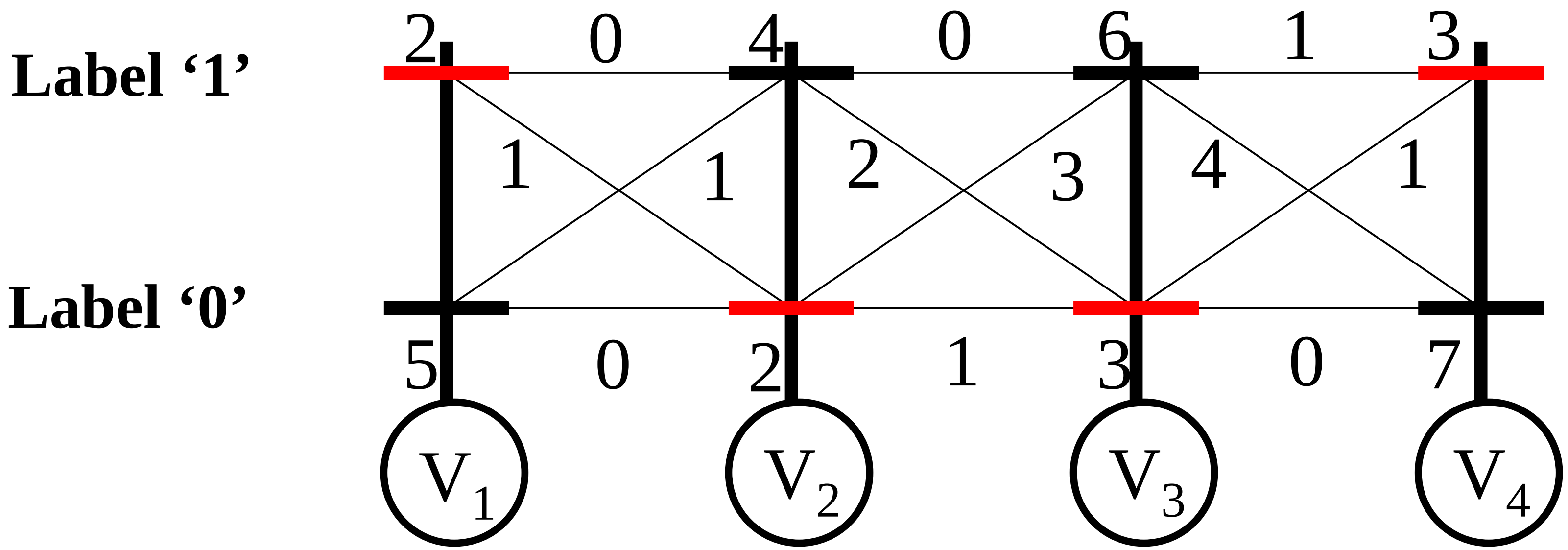
Cost(m) = 2 + 1 + 2 + 1 + 3 + 1 + 3 = 13

NP-hard problem

Which approximate algorithm is the best?

Aim

- To analyze convex relaxations for MAP estimation



Objectives

- Compare existing convex relaxations – LP, QP and SOCP

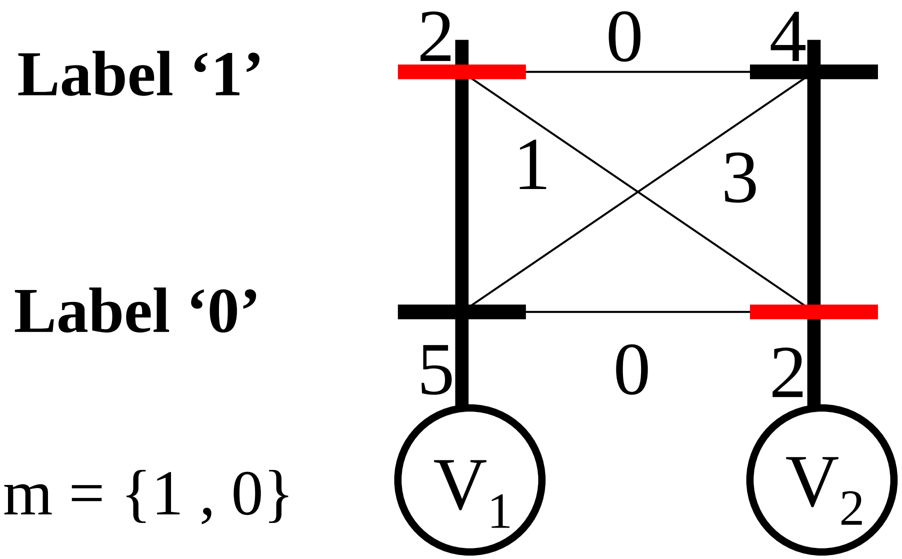
- Develop new relaxations based on the comparison

Outline

- Integer Programming Formulation
- Existing Relaxations
- Comparison
- Generalization of Results
- Two New SOCP Relaxations

Integer Programming Formulation

Unary Cost



Labelling m = {1 , 0}

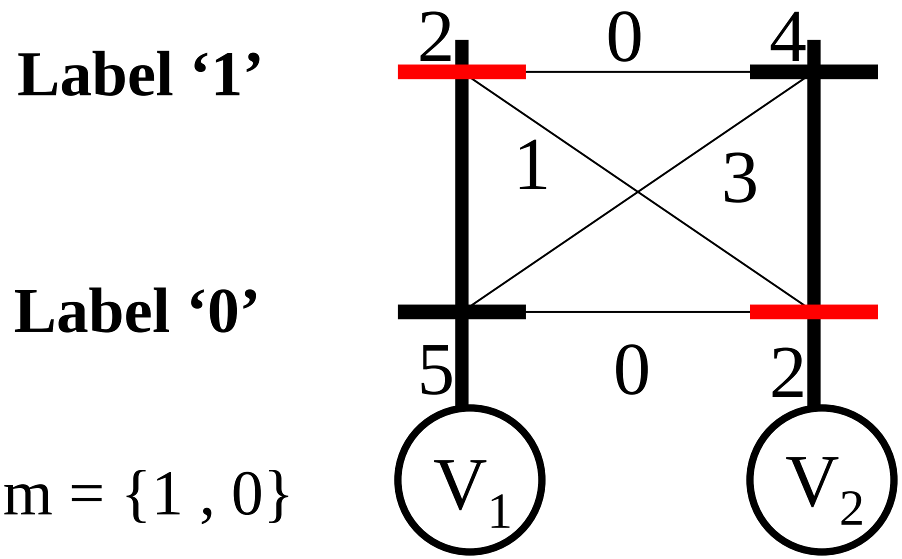
Unary Cost Vector **u** = [5 2 ; 2 4]

Cost of V_1 = 0

Cost of V_1 = 1

Integer Programming Formulation

Unary Cost



Labelling m = {1 , 0}

Unary Cost Vector $\mathbf{u}$ = [5 2 ; 2 4]^T

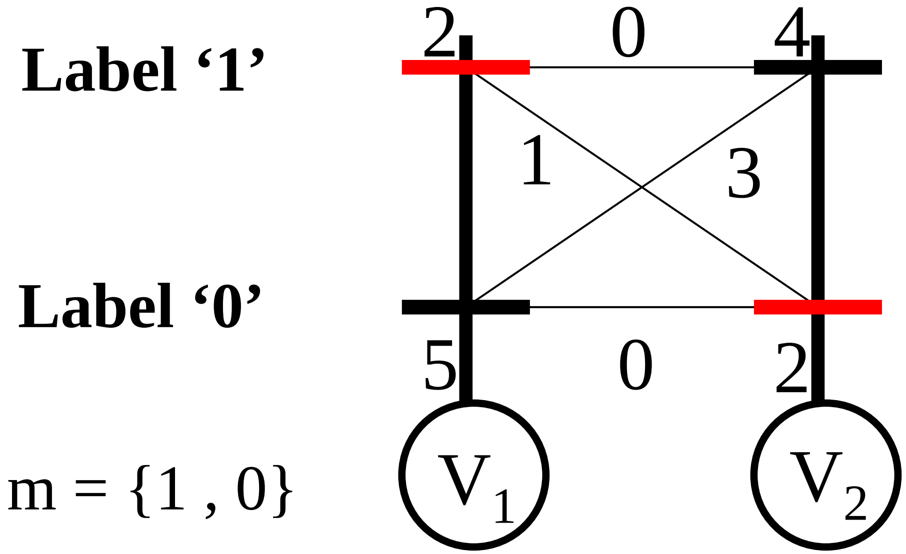
Label vector $\mathbf{x}$ = [-1 1 ; 1 -1]^T

$V_1 = 0$ $V_1 = 1$

Recall that the aim is to find the optimal **x**

Integer Programming Formulation

Unary Cost



Labelling m = {1 , 0}

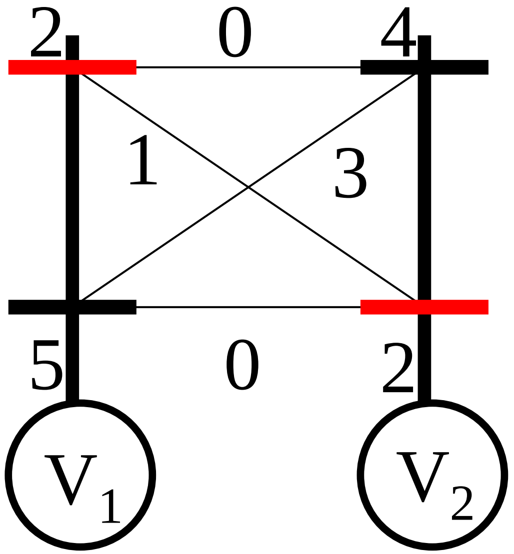
Unary Cost Vector **u** = [5 2 ; 2 4]^T

Label vector **x** = [-1 1 ; 1 -1]^T

Sum of Unary Costs = $\frac{1}{2} \sum_i \mathbf{u}_i (1 + \mathbf{x}_i)$

Integer Programming Formulation

Pairwise Cost



Labelling m = {1 , 0}

Pairwise Cost Matrix **P**

0	0	0	3
0	0	1	0
0	1	0	0
3	0	0	0

Cost of $V_1 = 0$ and $V_1 = 0$

Cost of $V_1 = 0$ and $V_2 = 0$

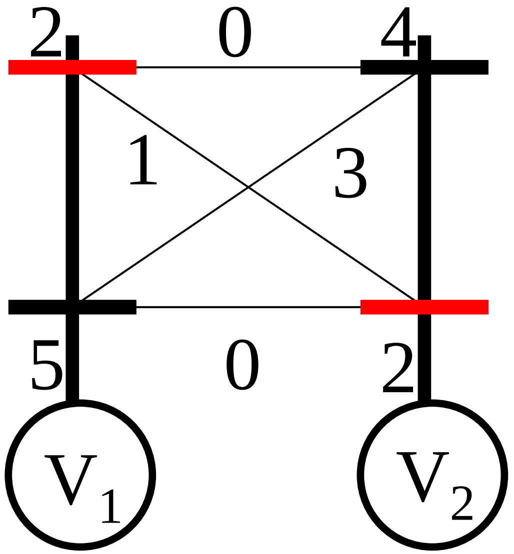
Cost of $V_1 = 0$ and $V_2 = 1$

Integer Programming Formulation

Pairwise Cost

Label '1'

Label '0'



Labelling m = {1 , 0}

Pairwise Cost Matrix **P**

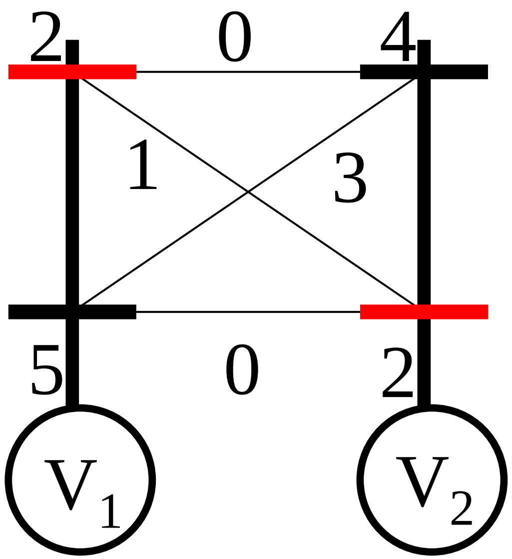
0	0	0	3
0	0	1	0
0	1	0	0
3	0	0	0

Sum of Pairwise Costs

$$\frac{1}{4} \sum_{ij} \mathbf{P}_{ij} (1 + \mathbf{x}_i)(1+\mathbf{x}_j)$$

Integer Programming Formulation

Pairwise Cost



Labelling m = {1 , 0}

Pairwise Cost Matrix **P**

0	0	0	3
0	0	1	0
0	1	0	0
3	0	0	0

Sum of Pairwise Costs

$$\frac{1}{4} \sum_{ij} \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{x}_i \mathbf{x}_j)$$

$$= \frac{1}{4} \sum_{ij} \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$ $\quad$ $\mathbf{X}_{ij} = \mathbf{x}_i\,\mathbf{x}_j$

Integer Programming Formulation

Constraints

- Integer Constraints

$$\mathbf{x}_i \in \{-1,1\}$$

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

- Uniqueness Constraint

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

Integer Programming Formulation

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

Convex

$$\mathbf{x}_i \in \{-1,1\}$$

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

Non-Convex

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  - Semidefinite Programming (SDP-L)
  - Second Order Cone Programming (SOCP-MS)
- Comparison
- Generalization of Results
- Two New SOCP Relaxations

LP-S

Schlesinger, 1976

Retain Convex Part

$$\mathbf{x}^* = \operatorname{argmin} \ \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in \{-1,1\}$$

Relax Non-Convex Constraint

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

LP-S

Schlesinger, 1976

Retain Convex Part

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

Relax Non-Convex Constraint

LP-S

Schlesinger, 1976

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

$$\mathbf{X}_{ij} \in [-1,1]$$

$$1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij} \geq 0$$

$$\sum_{j \in V_b} \mathbf{X}_{ij} = (2 - |L|)\,\mathbf{x}_i$$

LP-S

Schlesinger, 1976

Retain Convex Part

$$\mathbf{x}^* = \operatorname{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

Relax Non-Convex Constraint

LP-S

Schlesinger, 1976

Retain Convex Part

$$\mathbf{x}^* = \operatorname{argmin} \ \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1], \quad \mathbf{X}_{ij} \in [-1,1]$$

$$1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij} \geq 0$$

$$\sum_{j \in V_b} \mathbf{X}_{ij} = (2 - |L|)\, \mathbf{x}_i$$

LP-S

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- Experiments

SDP-L

Lasserre, 2000

Retain Convex Part

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in \{-1,1\}$$

Relax Non-Convex Constraint

$$\mathbf{X} = \mathbf{x}\, \mathbf{x}^T$$

SDP-L

Lasserre, 2000

Retain Convex Part

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

Relax Non-Convex Constraint

SDP-L

$$\begin{pmatrix}1\\ \mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_n\end{pmatrix}\begin{bmatrix}1 & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n\end{bmatrix} = \begin{pmatrix}1 & \mathbf{x}^T\\ \mathbf{x} & \mathbf{X}\end{pmatrix}$$

Convex

$\mathbf{X}_{ii} = 1$

Positive Semidefinite

Non-Convex

Rank = 1

SDP-L

$$\begin{pmatrix} 1 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} \begin{bmatrix} 1 & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} = \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{pmatrix}$$

Convex

$\mathbf{X}_{ii} = 1$

Positive Semidefinite

Schur's Complement

$$\begin{pmatrix} A & B \\ B^T & \mathbf{C} \end{pmatrix} \succeq 0$$

$$= \begin{pmatrix} I & \mathbf{0} \\ B^T A^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \boxed{A} & \mathbf{0} \\ \mathbf{0} & \boxed{\mathbf{C} - B^T A^{-1} B} \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & \mathbf{I} \end{pmatrix}$$

$$A \succeq 0 \qquad C - B^T A^{-1} B \succeq 0$$

SDP-L

$$\begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{x} & \mathbf{I} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boxed{\mathbf{X} - \mathbf{x}\mathbf{x}^T} \end{pmatrix} \begin{pmatrix} \mathrm{I} & \mathbf{x}^T \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Schur's Complement

$$\mathbf{X} - \mathbf{x}\mathbf{x}^T \succeq 0$$

SDP-L

Lasserre, 2000

Retain Convex Part

$$\mathbf{x}^* = \operatorname{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

$$\mathbf{X} = \mathbf{x}\,\mathbf{x}^T$$

Relax Non-Convex Constraint

SDP-L

Lasserre, 2000

Retain Convex Part

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

SDP-L

$$\mathbf{X}_{ii} = 1 \qquad \mathbf{X} - \mathbf{x}\mathbf{x}^T \succeq 0$$

Accurate

Inefficient

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- Two New SOCP Relaxations

SOCP Relaxation

Derive SOCP relaxation from the SDP relaxation

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

$$\mathbf{X}_{ii} = 1 \qquad \mathbf{X} - \mathbf{x}\mathbf{x}^T \succeq 0$$

Further Relaxation

1-D Example

$$\mathbf{X} - \mathbf{x}\mathbf{x}^T \succeq 0$$

For two semidefinite matrices,
Frobenius inner product is non-negative

$$A \bullet \mathbf{X} - \mathbf{x}^2 \geq 0 \qquad A \geq 0$$

$$\mathbf{x}^2 \leq \mathbf{X} = 1$$

SOC of the form $\|\,\mathbf{v}\,\|^2 \leq st$

2-D Example

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{21} & \mathbf{X}_{22} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{X}_{12} \\ \mathbf{X}_{12} & 1 \end{pmatrix}$$

$$\mathbf{x}\mathbf{x}^{T} = \begin{pmatrix} \mathbf{x}_1\mathbf{x}_1 & \mathbf{x}_1\mathbf{x}_2 \\ \mathbf{x}_2\mathbf{x}_1 & \mathbf{x}_2\mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1{}^2 & \mathbf{x}_1\mathbf{x}_2 \\ \mathbf{x}_1\mathbf{x}_2 & \mathbf{x}_2{}^2 \end{pmatrix}$$

2-D Example

$$\mathbf{C}_1 \bullet (\mathbf{X} - \mathbf{x}\mathbf{x}^T) \geq 0 \qquad \mathbf{C}_1 \succeq 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 1 - \mathbf{x}_1^2 & \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 \\ \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 & 1 - \mathbf{x}_2^2 \end{bmatrix} \geq 0$$

$$\mathbf{x}_1^2 \leq 1$$

$$-1 \leq \mathbf{x}_1 \leq 1$$

2-D Example

$$\mathbf{C}_2 \bullet (\mathbf{X} - \mathbf{x}\mathbf{x}^T) \geq 0 \qquad \mathbf{C}_2 \succeq 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 - \mathbf{x}_1^2 & \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 \\ \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 & 1 - \mathbf{x}_2^2 \end{bmatrix} \geq 0$$

$$\mathbf{x}_2^2 \leq 1$$

$$-1 \leq \mathbf{x}_2 \leq 1$$

2-D Example

$$\mathbf{C}_3 \bullet (\mathbf{X} - \mathbf{x}\mathbf{x}^T) \geq 0 \qquad \mathbf{C}_3 \succeq 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 - \mathbf{x}_1^2 & \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 \\ \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 & 1 - \mathbf{x}_2^2 \end{bmatrix} \geq 0$$

$$(\mathbf{x}_1 + \mathbf{x}_2)^2 \leq 2 + 2\mathbf{X}_{12}$$

SOC of the form $\| \mathbf{v} \|^2 \leq st$

2-D Example

$$\mathbf{C}_4 \bullet (\mathbf{X} - \mathbf{x}\mathbf{x}^T) \geq 0 \qquad \mathbf{C}_4 \succeq 0$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 - \mathbf{x}_1^2 & \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 \\ \mathbf{X}_{12} - \mathbf{x}_1\mathbf{x}_2 & 1 - \mathbf{x}_2^2 \end{bmatrix} \geq 0$$

$$(\mathbf{x}_1 - \mathbf{x}_2)^2 \leq 2 - 2\mathbf{X}_{12}$$

SOC of the form $\| \mathbf{v} \|^2 \leq st$

SOCP Relaxation

Kim and Kojima, 2000

Consider a matrix $\mathbf{C}_1 = \mathbf{U}\mathbf{U}^T \succeq 0$

$$\mathbf{C}_1 \bullet (\mathbf{X} - \mathbf{x}\mathbf{x}^T) \geq 0$$

$$\|\mathbf{U}^T\mathbf{x}\|^2 \leq \mathbf{X} \bullet \mathbf{C}_1$$

SOC of the form $\| \mathbf{v} \|^2 \leq st$

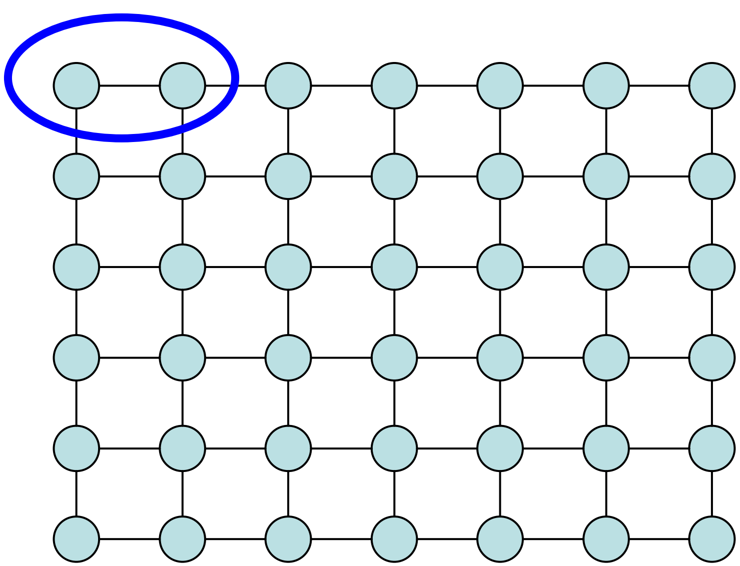
Continue for $\mathbf{C}_2, \mathbf{C}_3, \ldots, \mathbf{C}_n$

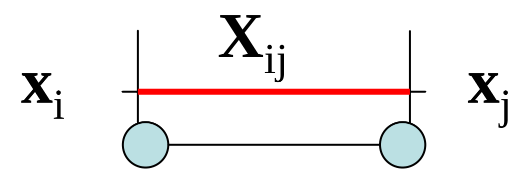
SOCP Relaxation

How many constraints for SOCP = SDP ?

Infinite. For all $C \succeq 0$

Specify constraints similar to the 2-D example





$$(\mathbf{x}_i + \mathbf{x}_j)^2 \leq 2 + 2\mathbf{X}_{ij}$$

$$(\mathbf{x}_i + \mathbf{x}_j)^2 \leq 2 - 2\mathbf{X}_{ij}$$

SOCP-MS

Muramatsu and Suzuki, 2003

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

$$\mathbf{X}_{ii} = 1 \qquad \mathbf{X} - \mathbf{x}\mathbf{x}^T \succeq 0$$

SOCP-MS

Muramatsu and Suzuki, 2003

$$\mathbf{x}^* = \text{argmin} \quad \frac{1}{2} \sum \mathbf{u}_i (1 + \mathbf{x}_i) + \frac{1}{4} \sum \mathbf{P}_{ij} (1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij})$$

$$\sum_{i \in V_a} \mathbf{x}_i = 2 - |L|$$

$$\mathbf{x}_i \in [-1,1]$$

$$(\mathbf{x}_i + \mathbf{x}_j)^2 \leq 2 + 2\mathbf{X}_{ij} \qquad (\mathbf{x}_i - \mathbf{x}_j)^2 \leq 2 - 2\mathbf{X}_{ij}$$

Specified only when $\mathbf{P}_{ij} \neq 0$

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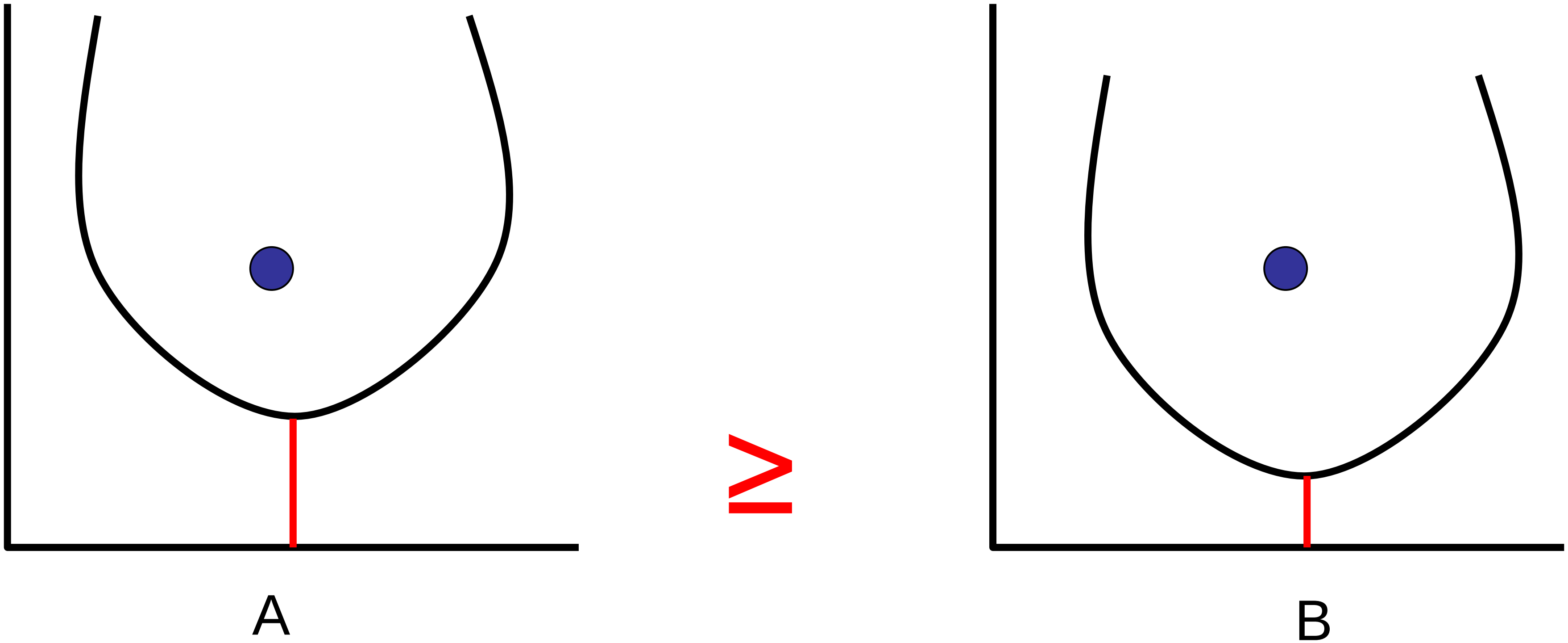
Dominating Relaxation

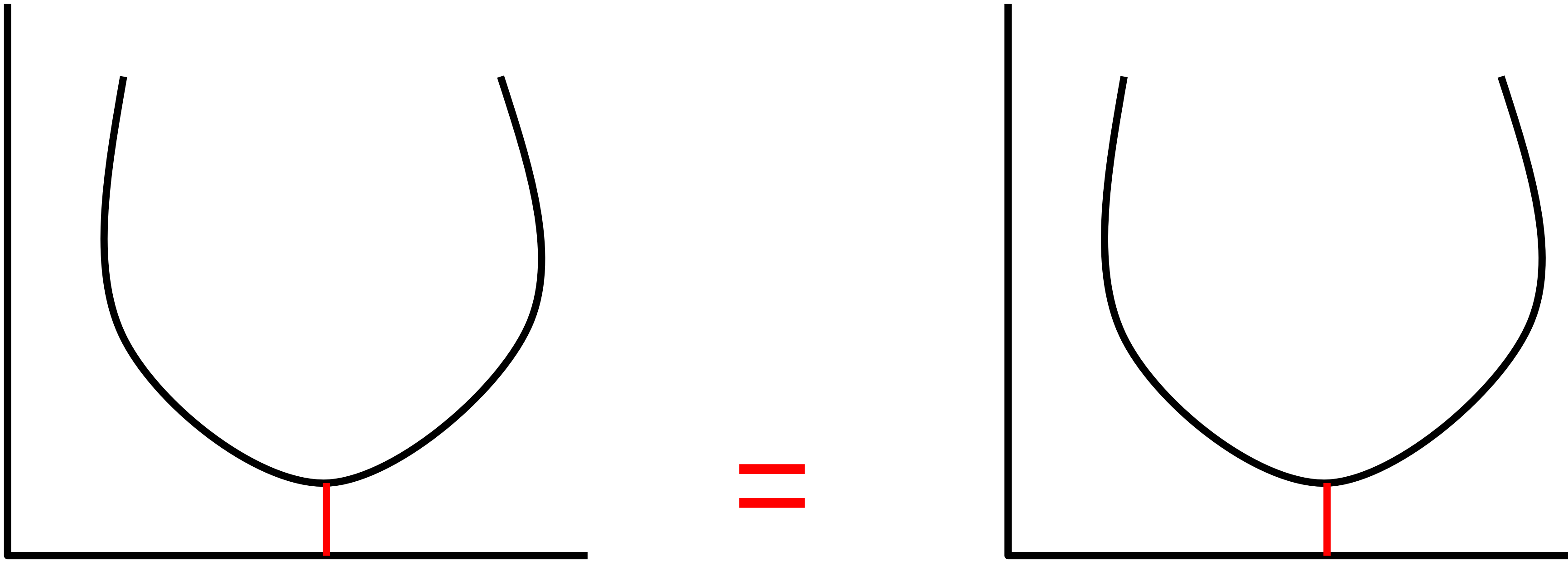


For all MAP Estimation problem (**u**, **P**)

A dominates B

Dominating relaxations are better

Equivalent Relaxations



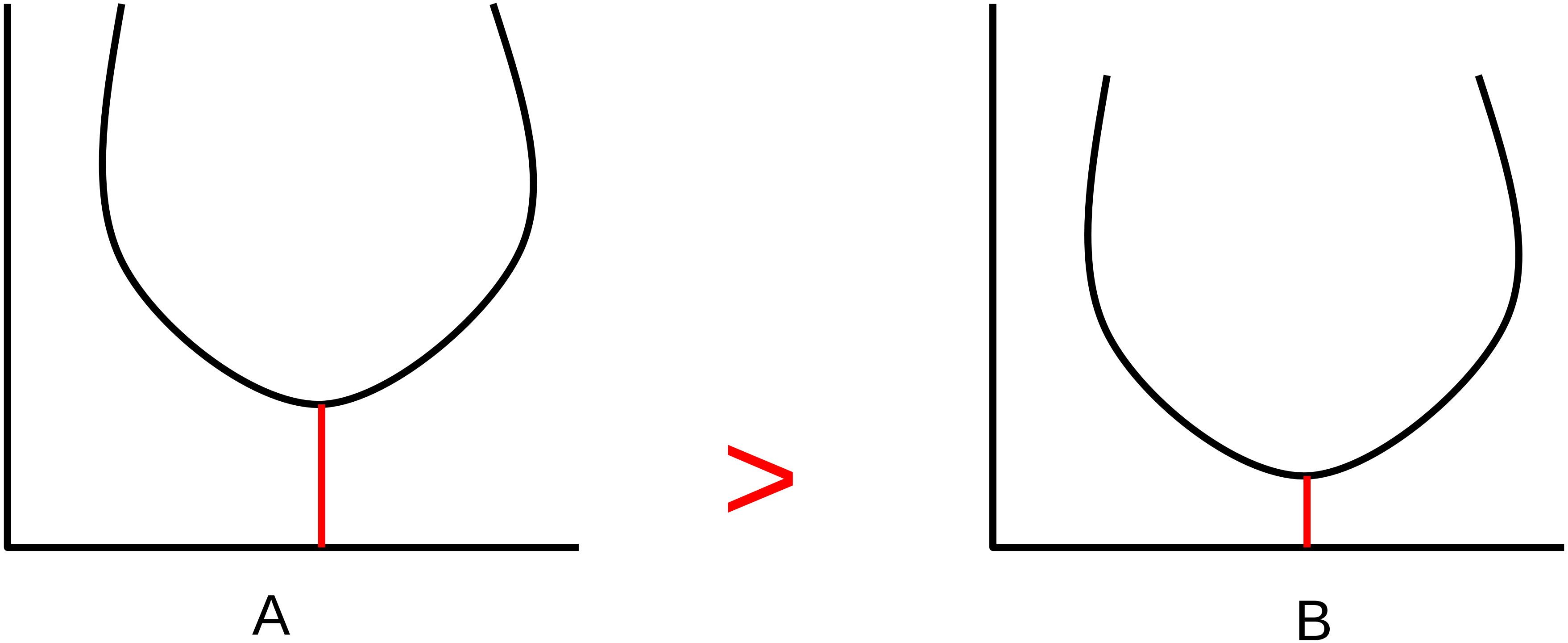
A B

For all MAP Estimation problem (**u**, **P**)

A dominates B

B dominates A

Strictly Dominating Relaxation



For at least one MAP Estimation problem (**u**, **P**)

A dominates B

B does not dominate A

SOCP-MS

Muramatsu and Suzuki, 2003

$$(\mathbf{x}_i + \mathbf{x}_j)^2 \leq 2 + 2\mathbf{X}_{ij}$$

$$(\mathbf{x}_i - \mathbf{x}_j)^2 \leq 2 - 2\mathbf{X}_{ij}$$

- $\mathbf{P}_{ij} \geq 0$ $\quad \mathbf{X}_{ij} = \frac{(\mathbf{x}_i + \mathbf{x}_j)^2}{2} - 1$
- $\mathbf{P}_{ij} < 0$ $\quad \mathbf{X}_{ij} = 1 - \frac{(\mathbf{x}_i - \mathbf{x}_j)^2}{2}$

SOCP-MS is a QP

Same as QP by Ravikumar and Lafferty, 2005

SOCP-MS ≡ QP-RL

LP-S vs. SOCP-MS

Differ in the way they relax $\mathbf{X} = \mathbf{x}\mathbf{x}^T$

$$\mathbf{X}_{ij} \in [-1,1]$$

$$1 + \mathbf{x}_i + \mathbf{x}_j + \mathbf{X}_{ij} \geq 0$$

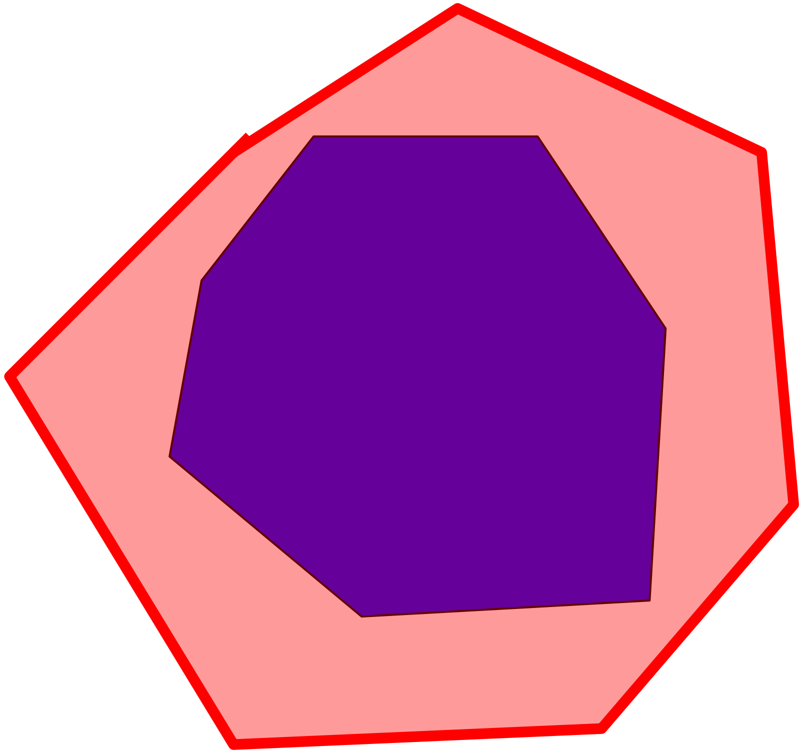
$$\sum_{j \in V_b} \mathbf{X}_{ij} = (2 - |L|)\, \mathbf{x}_i$$

LP-S

$$(\mathbf{x}_i + \mathbf{x}_j)^2 \leq 2 + 2\mathbf{X}_{ij}$$

$$(\mathbf{x}_i - \mathbf{x}_j)^2 \leq 2 - 2\mathbf{X}_{ij}$$

SOCP-MS



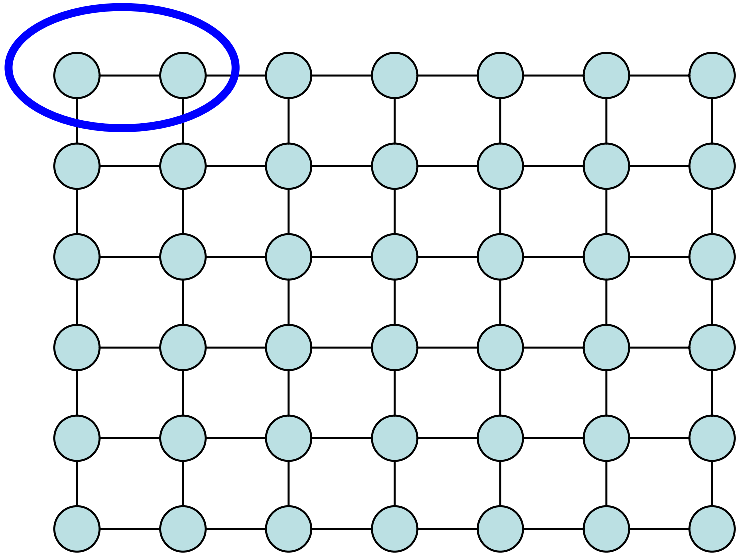
F(LP-S)

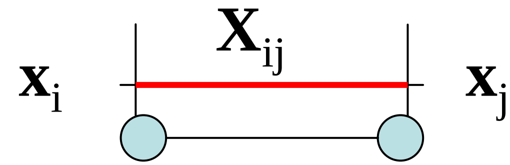
F(SOCP-MS)

LP-S vs. SOCP-MS

- LP-S strictly dominates SOCP-MS
- LP-S strictly dominates QP-RL
- Where have we gone wrong?
- A Quick Recap !

Recap of SOCP-MS

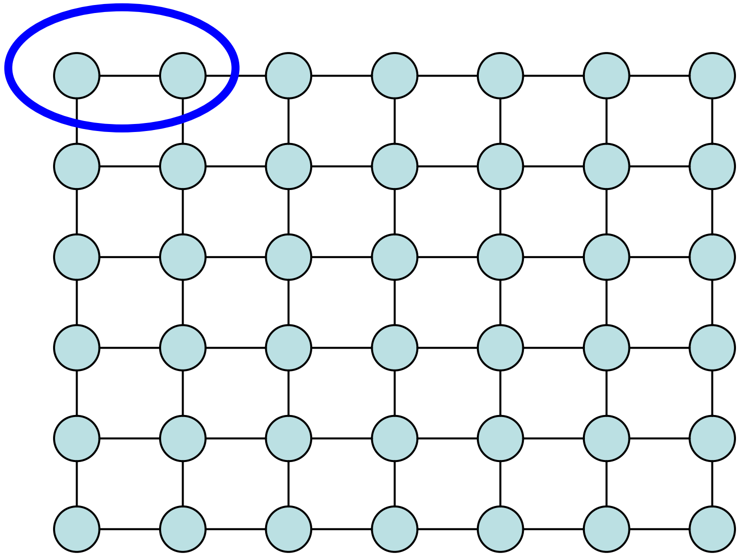


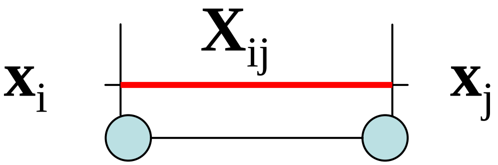


$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$(\mathbf{x}_i + \mathbf{x}_j)^2 \leq 2 + 2\mathbf{X}_{ij}$$

Recap of SOCP-MS





$$C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$(\mathbf{x}_i - \mathbf{x}_j)^2 \leq 2 - 2\mathbf{X}_{ij}$$

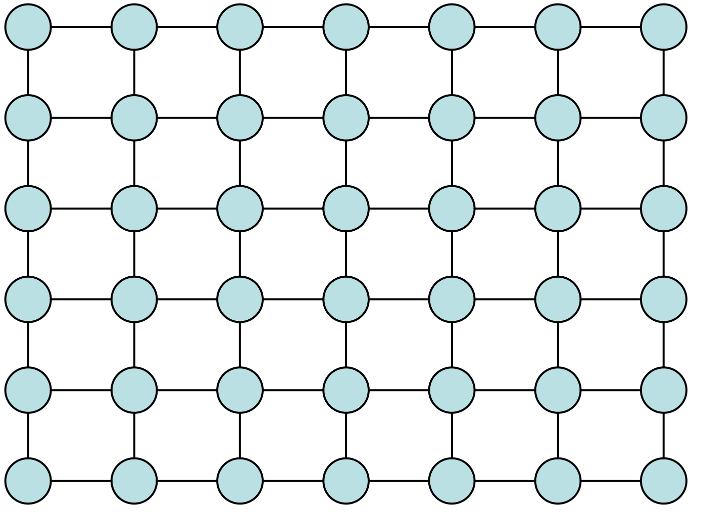
Can we use different C matrices ??

Can we use a different subgraph ??

Outline

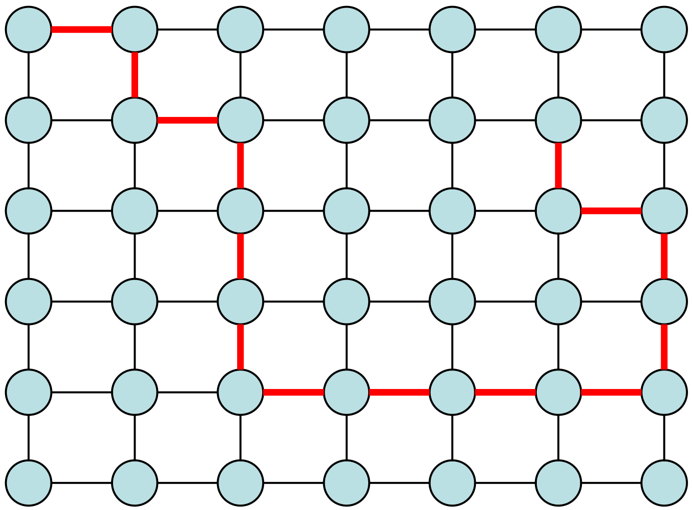
- Integer Programming Formulation
- Existing Relaxations
- Comparison
- Generalization of Results
  - SOCP Relaxations on Trees
  - SOCP Relaxations on Cycles
- Two New SOCP Relaxations

SOCP Relaxations on Trees



Choose any arbitrary tree

SOCP Relaxations on Trees



Choose any arbitrary $C \succeq 0$

Repeat over trees to get relaxation SOCP-T

LP-S strictly dominates SOCP-T

LP-S strictly dominates QP-T

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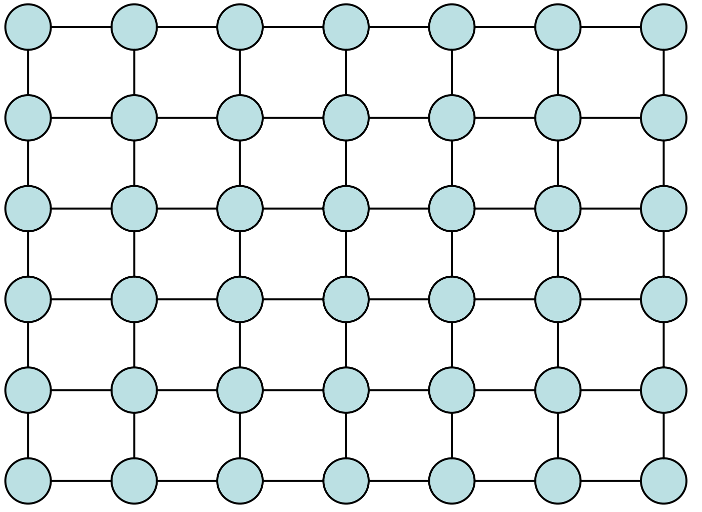
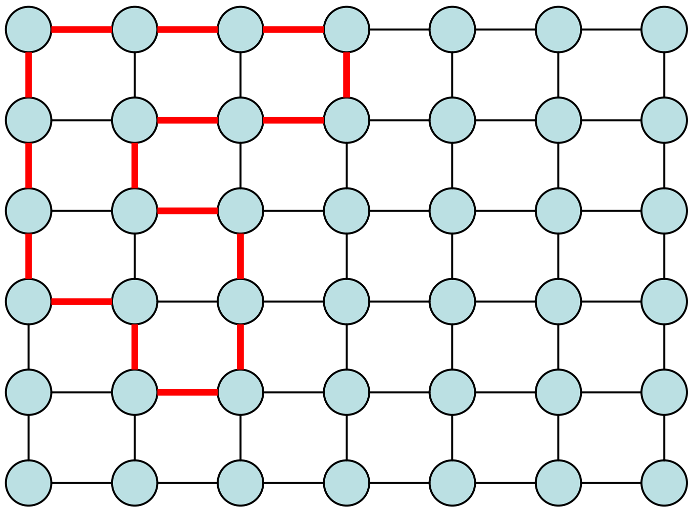
SOCP Relaxations on Cycles



Choose an arbitrary even cycle

$\mathbf{P}_{ij} \geq 0$ OR $\mathbf{P}_{ij} \leq 0$

SOCP Relaxations on Cycles



Choose any arbitrary $C \succeq 0$

Repeat over even cycles to get relaxation SOCP-E

LP-S strictly dominates SOCP-E

LP-S strictly dominates QP-E

SOCP Relaxations on Cycles

- True for odd cycles with $\mathbf{P}_{ij} \leq 0$
- True for odd cycles with $\mathbf{P}_{ij} \leq 0$ for only one edge
- True for odd cycles with $\mathbf{P}_{ij} \geq 0$ for only one edge
- True for all combinations of above cases

Outline

- Integer Programming Formulation
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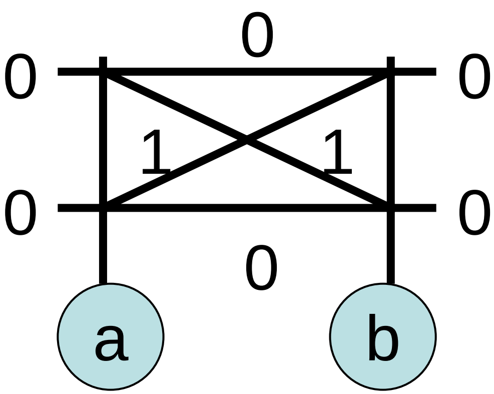
- Two New SOCP Relaxations
  - The SOCP-C Relaxation
  - The SOCP-Q Relaxation

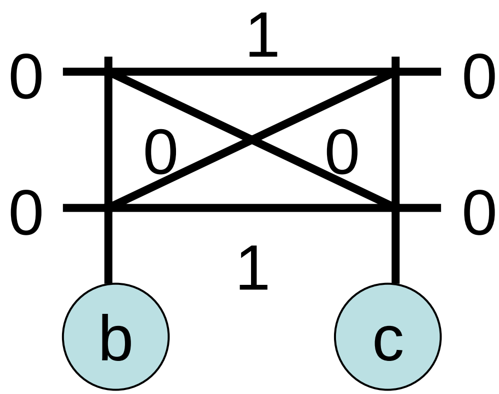
The SOCP-C Relaxation

Include all LP-S constraints

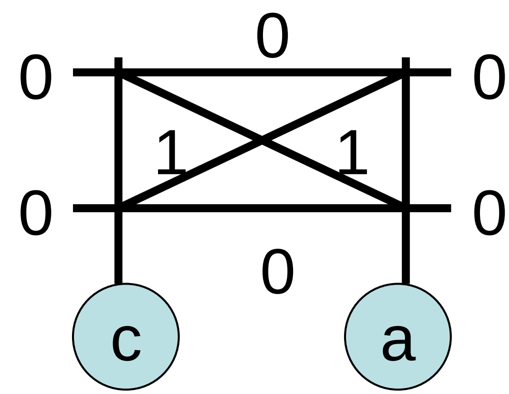
True SOCP



Submodular



Non-submodular

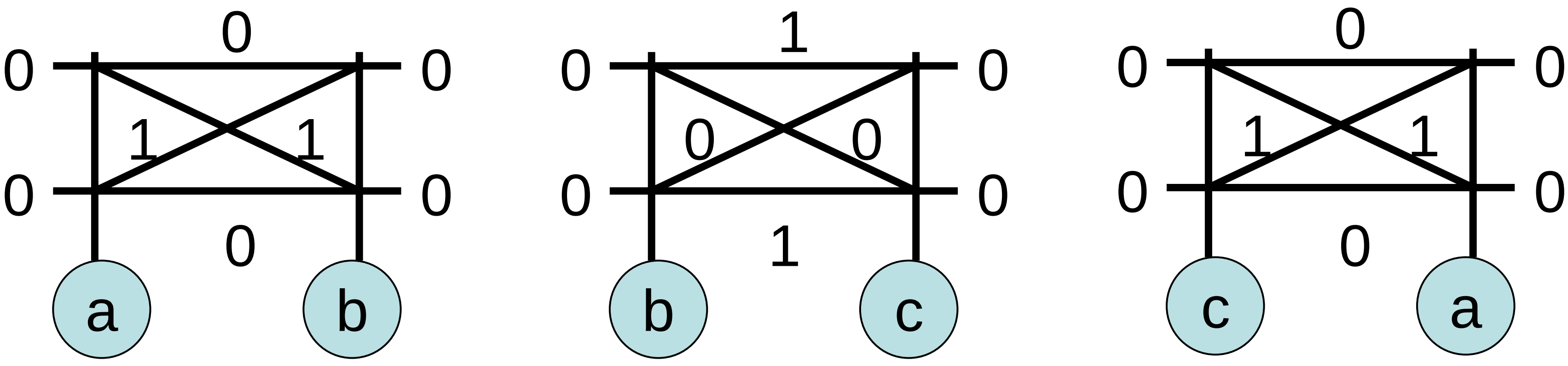


Submodular

The SOCP-C Relaxation

Include all LP-S constraints

True SOCP

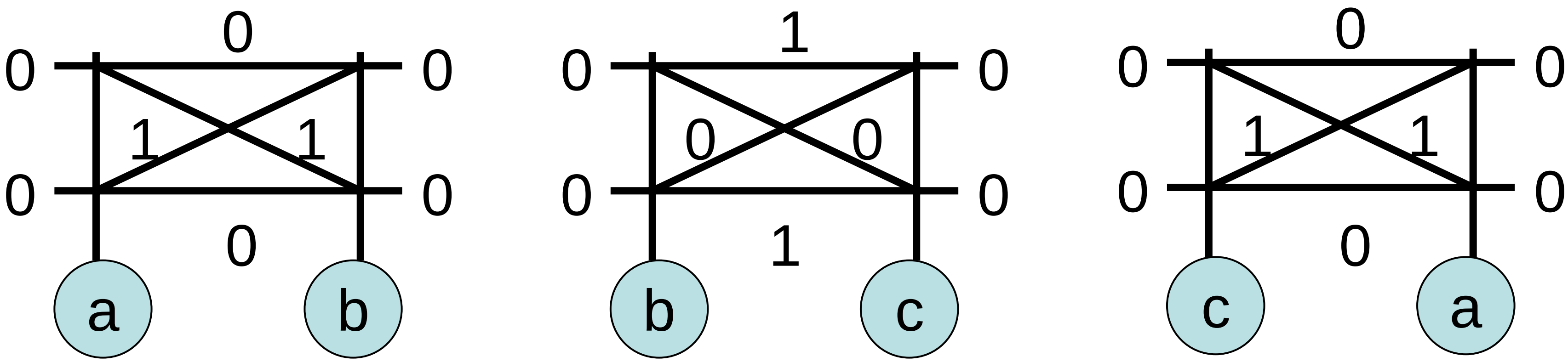


Frustrated Cycle

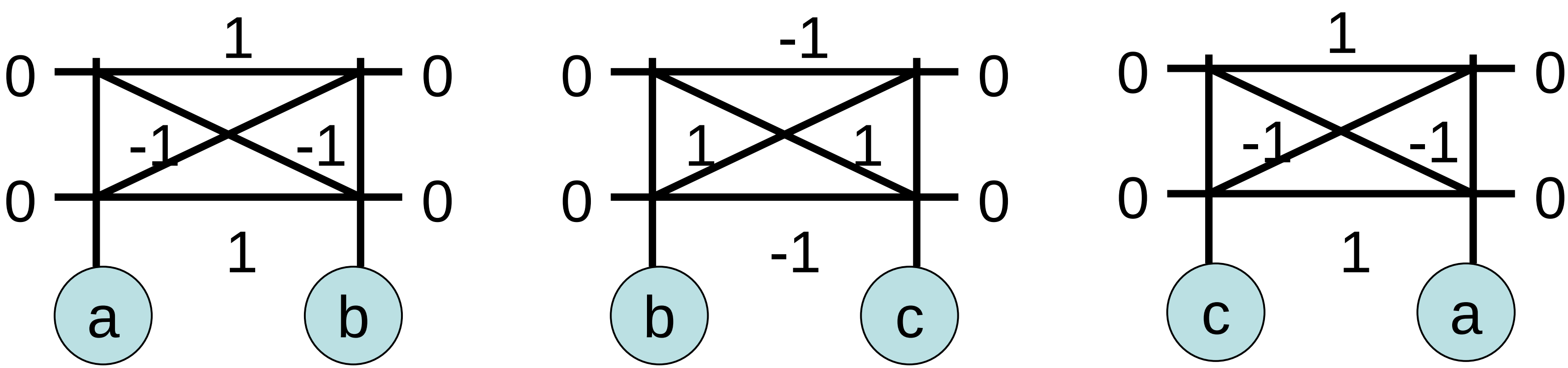
The SOCP-C Relaxation

Include all LP-S constraints

True SOCP



LP-S Solution

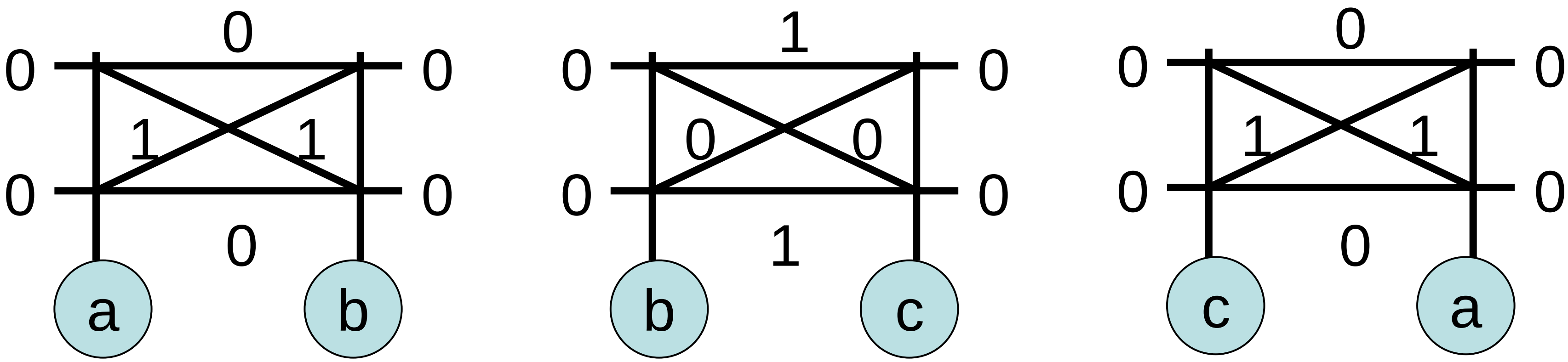


Objective Function = 0

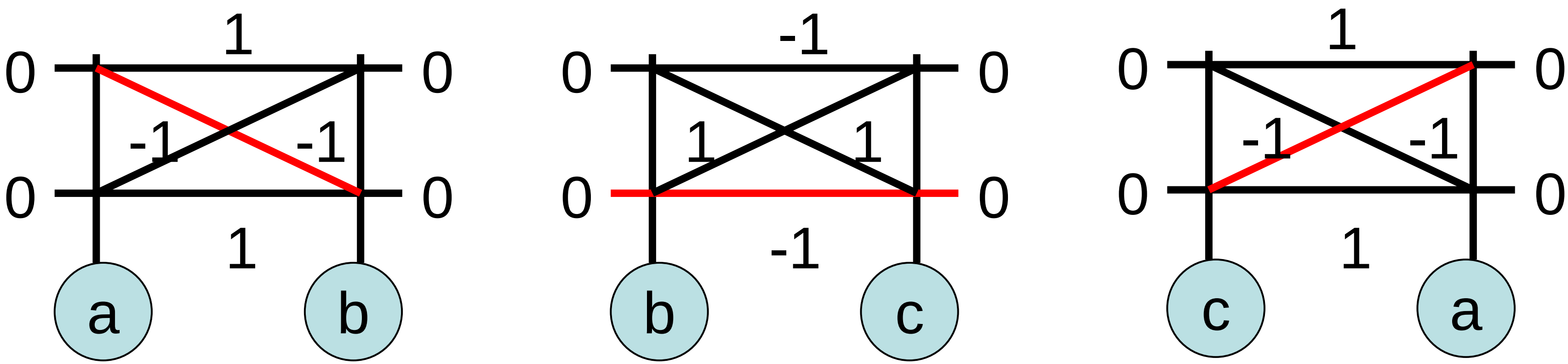
The SOCP-C Relaxation

Include all LP-S constraints

True SOCP



LP-S Solution

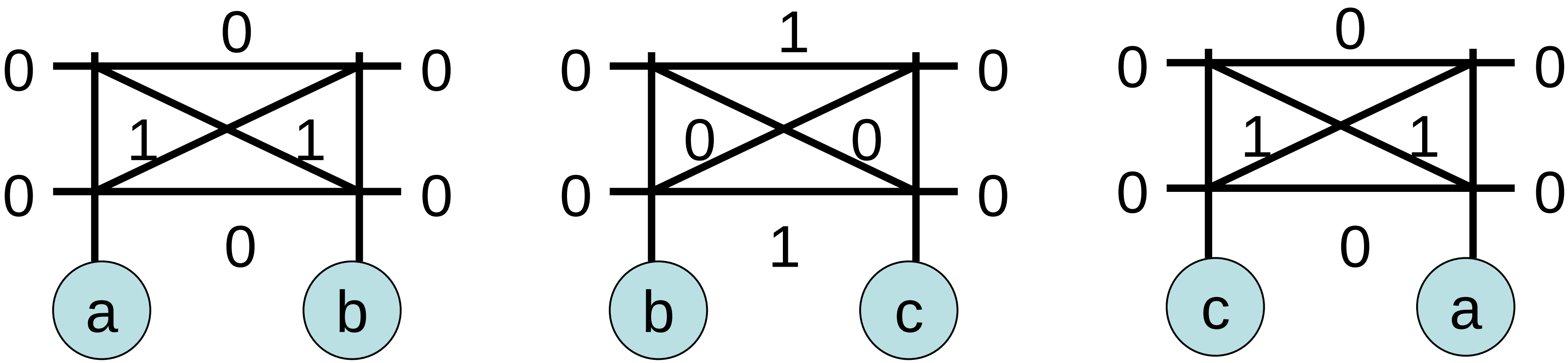


Define an SOC Constraint using C = **1**

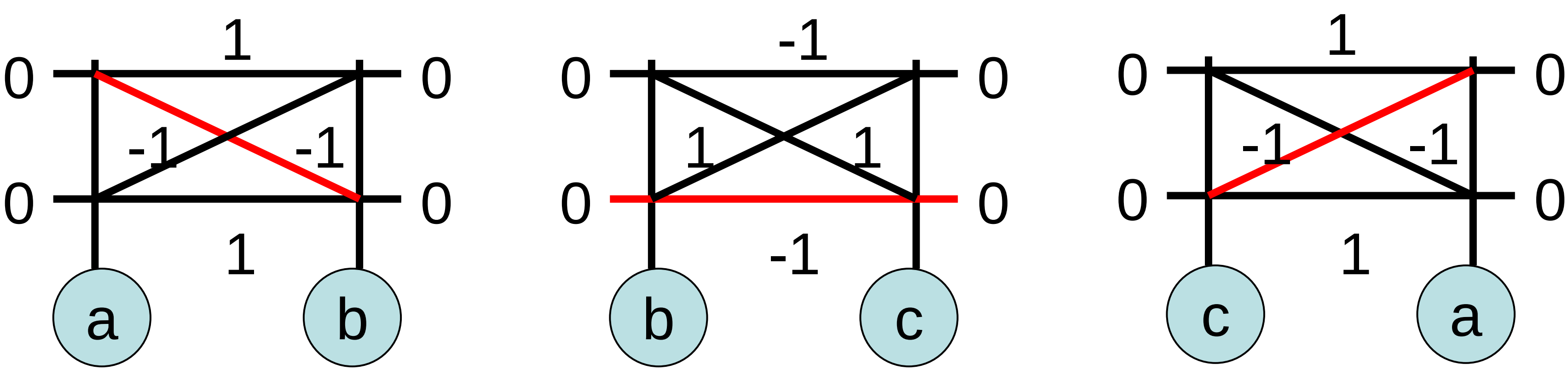
The SOCP-C Relaxation

Include all LP-S constraints

True SOCP



LP-S Solution

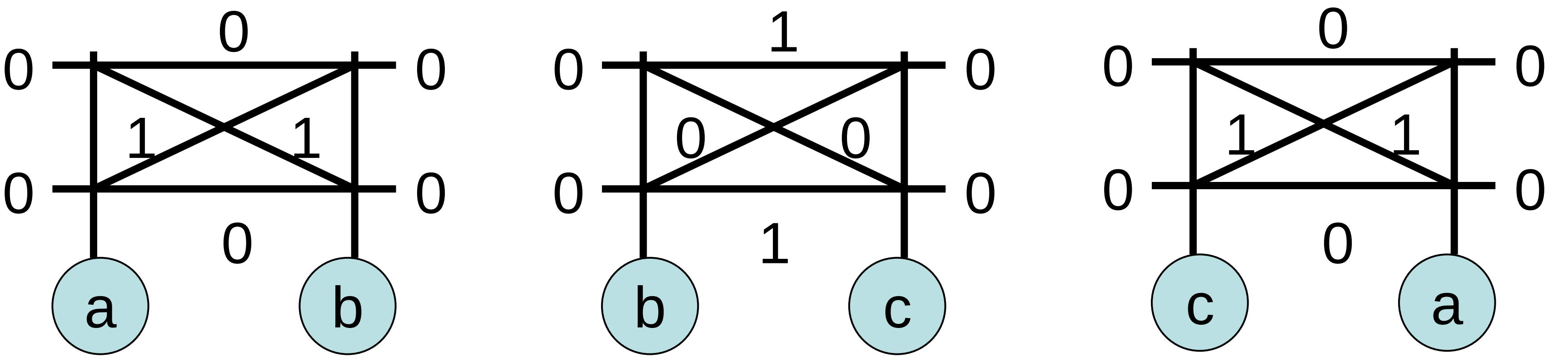


$$(\mathbf{x}_i + \mathbf{x}_j + \mathbf{x}_k)^2 \leq 3 + 2\,(\mathbf{X}_{ij} + \mathbf{X}_{jk} + \mathbf{X}_{ki})$$

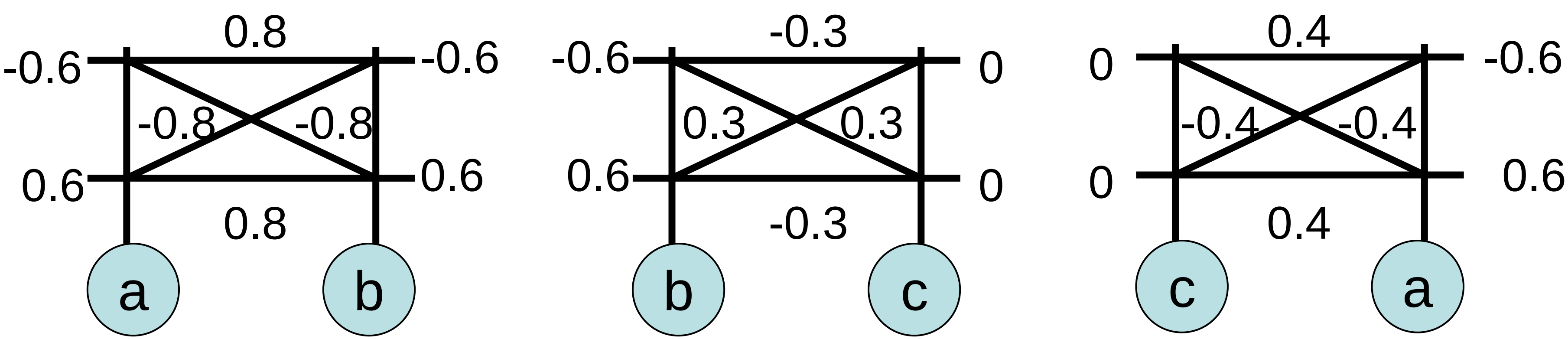
The SOCP-C Relaxation

Include all LP-S constraints

True SOCP



SOCP-C Solution



Objective Function = 0.75

SOCP-C strictly dominates LP-S

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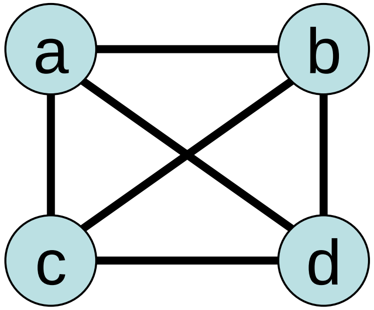
The SOCP-Q Relaxation

Include all cycle inequalities

True SOCP



Clique of size n

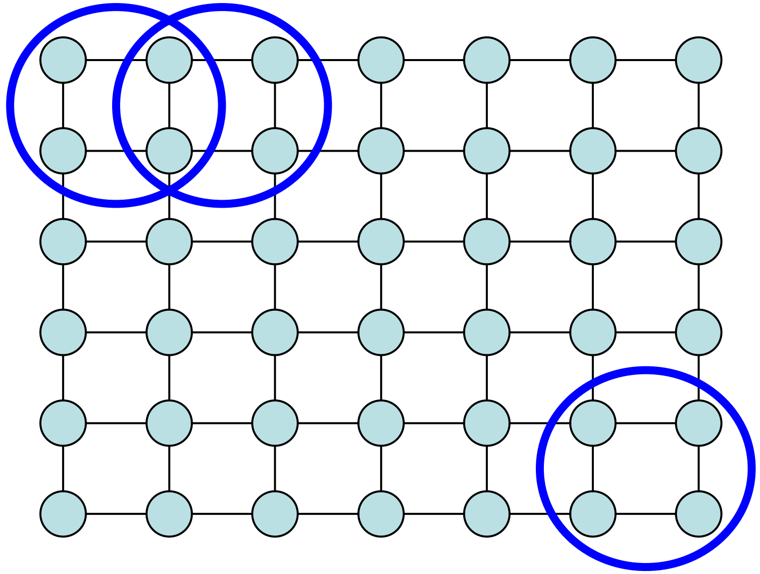
Define an SOCP Constraint using C = **1**

$$(\Sigma\, \mathbf{x}_i)^2 \leq n + (\Sigma\, \mathbf{X}_{ij})$$

SOCP-Q strictly dominates LP-S

SOCP-Q strictly dominates SOCP-C

4-Neighbourhood MRF



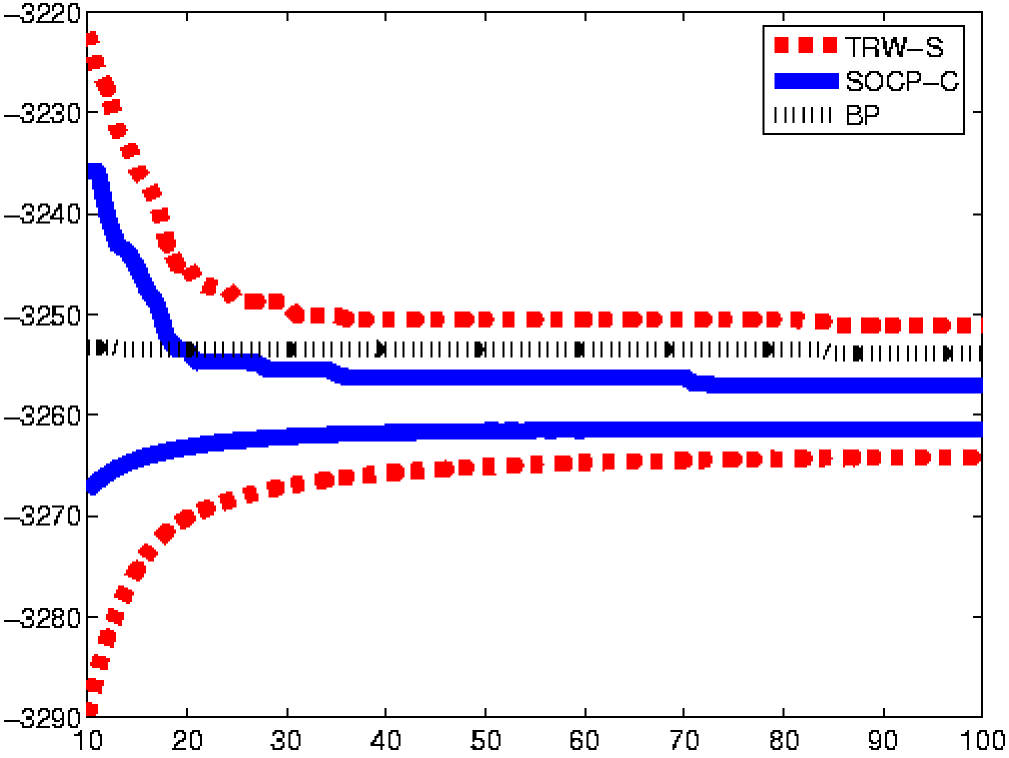
Test SOCP-C

50 binary MRFs of size 30x30

$\mathbf{u} \approx \mathcal{N}(0,1)$

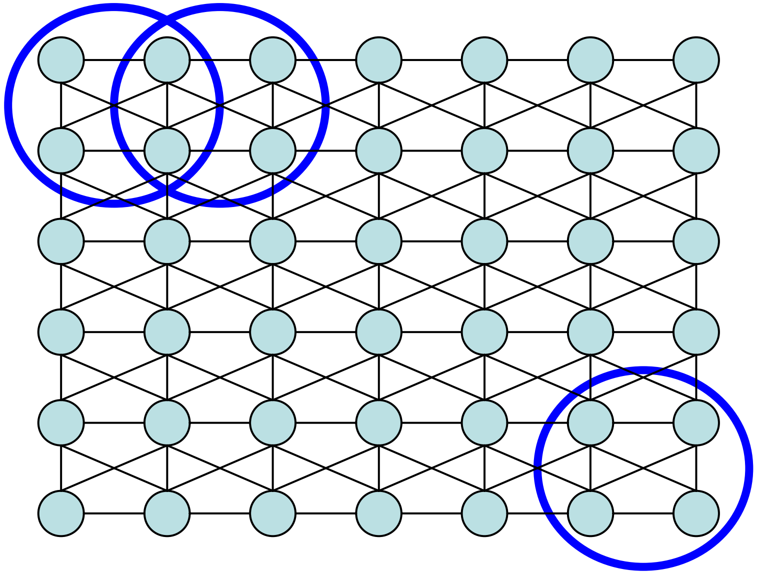
$\mathbf{P} \approx \mathcal{N}(0,\sigma^2)$

4-Neighbourhood MRF



σ = 2.5

8-Neighbourhood MRF



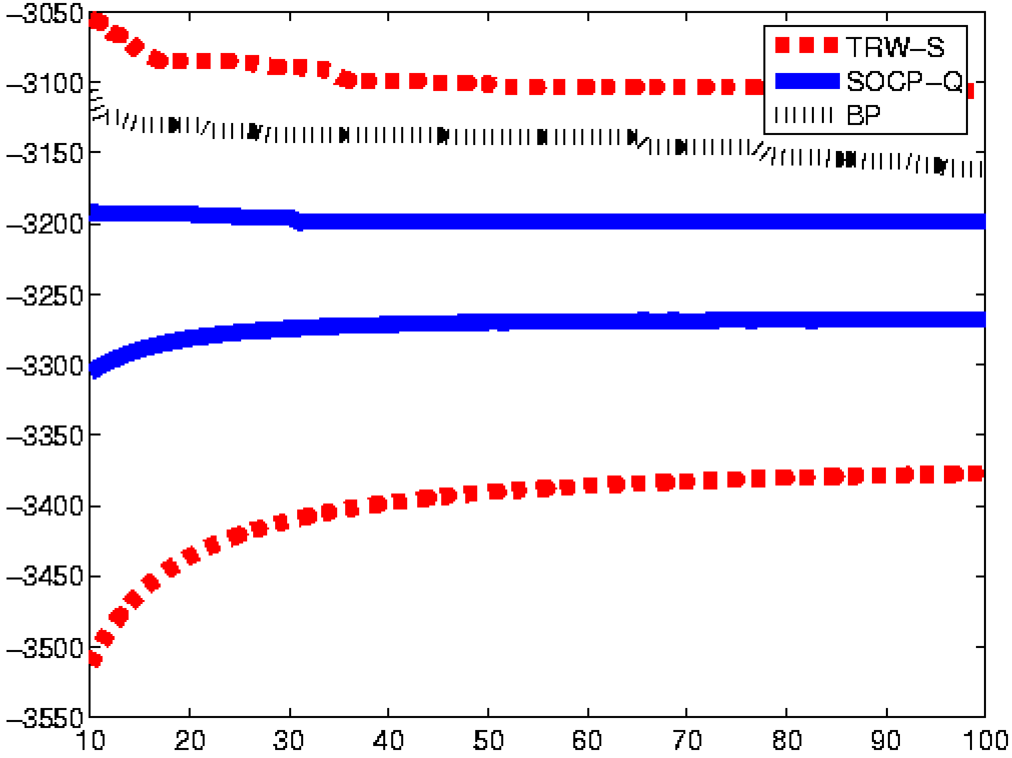
Test SOCP-Q

50 binary MRFs of size 30x30

$\mathbf{u} \approx \mathcal{N}(0,1)$

$\mathbf{P} \approx \mathcal{N}(0,\sigma^2)$

8-Neighbourhood MRF



σ = 1.125

Conclusions

- Large class of SOCP/QP dominated by LP-S
- New SOCP relaxations dominate LP-S
- More experimental results in poster

Future Work

- Comparison with cycle inequalities
- Determine best SOC constraints
- Develop efficient algorithms for new relaxations

Questions ??