Minimum Cuts, Maximum Area, and Duality

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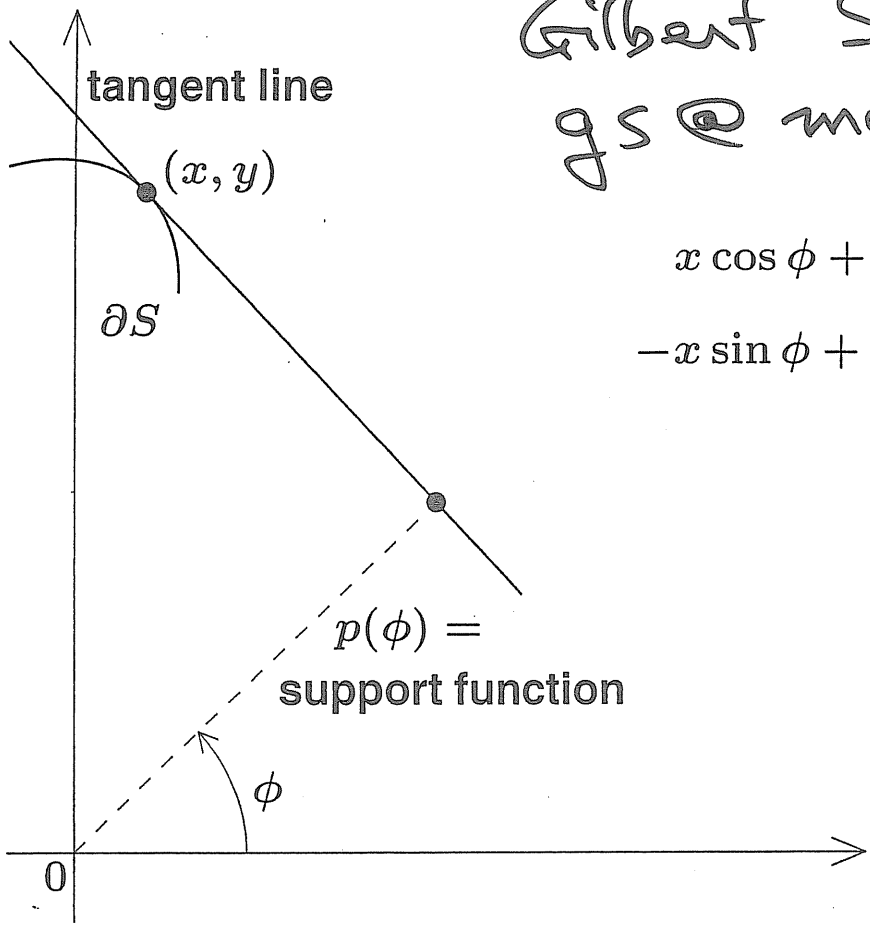

$$x \cos\phi + y \sin\phi = p(\phi)$$

$$-x \sin\phi + y \cos\phi = p'(\phi)$$

Figure 1: [Santalo]. The line at distance p is tangent to ∂S at (x, y).

Integral geometry

$$x = p\cos\phi - p'\sin\phi, \qquad y = p\sin\phi + p'\cos\phi\,.$$

$$dx = -(p+p'')\,\sin\phi\,d\phi, \qquad dy = (p+p'')\,\cos\phi\,d\phi\,.$$

$$ds = \sqrt{dx^2 + dy^2} = (p + p'')\,d\phi$$

The convexity of S is equivalent to $p + p'' \geq 0$.

$$N(\phi) = \|(-\sin\phi, \cos\phi)\|$$

Perimeter of $S = \displaystyle\int_0^{2\pi} \|(dx, dy)\| = \int_0^{2\pi} (p + p'')\, N(\phi)\, d\phi$

Area of $S = \displaystyle\frac{1}{2}\int_0^{2\pi} p(p + p'')\, d\phi = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\phi\,.$

Maximize $\displaystyle\frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\phi$

subject to $\displaystyle\int_0^{2\pi} (p + p'')\, N(\phi)\, d\phi = L\,.$

Euler-Lagrange

$$p(\phi) + p''(\phi) - \lambda\left[N(\phi) + N''(\phi)\right] = 0\,. \qquad (1)$$

$$p(\phi) = \lambda\,N(\phi) + A\cos\phi + B\sin\phi\,. \qquad (2)$$

Optimal shape S

$$p(\phi) = \lambda\,N(\phi) = \lambda\,\|(-\sin\phi, \cos\phi)\|\,. \qquad (3)$$

Classical case with $N = 1$ yields $p = \lambda$

Maximize $\frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\phi$ subject to $\int p\, d\phi = L\,.$ (4)

$$\int_0^{2\pi} \left[(p+q)^2 - (p'+q')^2\right] d\phi = \int (p^2 - p'^2)\, d\phi - 2\int (pq - p'q')\, d\phi + \int (q^2 - q'^2)\, d\phi\,. \tag{5}$$

First variation $\int (pq - p'q')\, d\phi = \int (p + p'')q\, d\phi = 0$ when $\int q\, d\phi = 0$ needs $p + p'' = \lambda\,.$ (6)

Wirtinger $\int_0^{2\pi} (q^2 - q'^2)\, d\phi = 2\pi\left[\sum |a_n|^2 - \sum n^2 |a_n|^2\right] \leq 0\,.$ (7)

Then $p = \lambda$ and $S =$ circle is optimal

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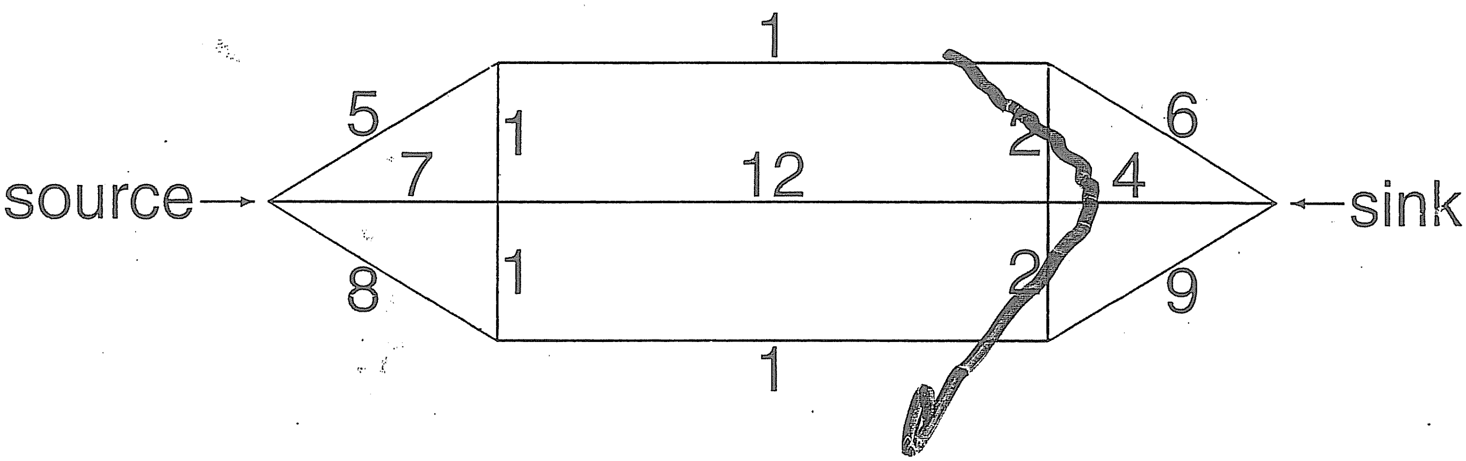
Maximum Flow = Minimum Cut



Kirchhoff's Law at each node (Flow in = Flow out)

Continuous equivalent: $\operatorname{div} v = 0$.

Continuous capacity: $\|v\| \le c(x, y)$

$v = (v_1, v_2)$

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$$

Maximize λ

$$v \cdot n = \lambda$$

$$v \cdot n = -\lambda$$

$$\operatorname{div}(v_1, v_2) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$$

$$v_1^2 + v_2^2 \leq 1$$

(capacity)

$$v \cdot n = -\lambda$$

$$v \cdot n = \lambda$$

Boundary
Source / Sink

Maximize λ

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \lambda$$

$$v_1^2 + v_2^2 \leq 1$$

(capacity)

Internal source

$$\iint \operatorname{div} v = \int_{\partial S} v \cdot n \leq |\partial S| \quad \text{so} \quad \lambda \leq \frac{|\partial S|}{|S|}$$

Linear Programming $|v_1| + |v_2| \leq 1$

Maximum Flow Out of a Square

Maximize λ with

$$\operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \lambda \quad \text{and} \quad \|v\| \le 1$$

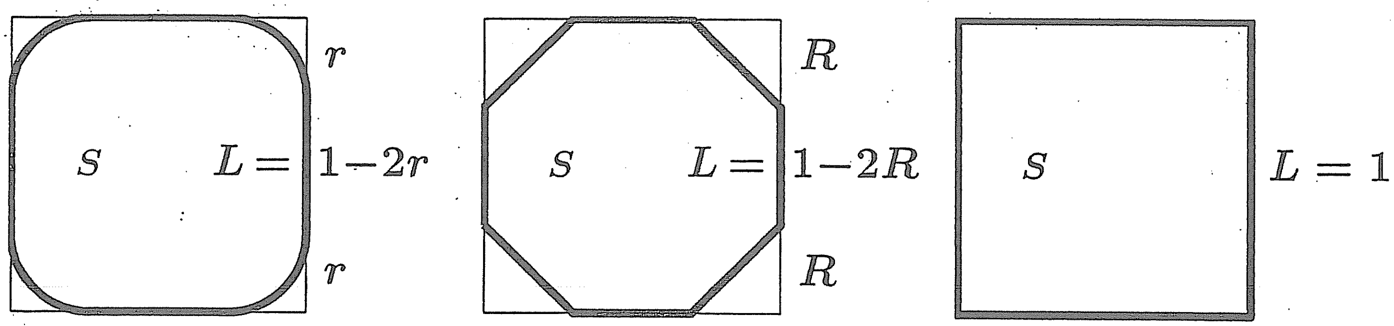
Dual problem **Minimize** $$\frac{\iint \|\operatorname{grad} u\| \, dx \, dy}{\iint u \, dx \, dy}$$

Extreme point is characteristic function of a set S

$$\underset{S \subset \Omega}{\text{Minimize}} \quad \frac{\int \|n\|_D \, ds}{\text{area of } S}$$

Solution gives minimum cut: Inside Ω it is an isoperimetrix

circular arc — diamond — square in $\ell^2 - \ell^\infty - \ell^1$



$|\partial S| = 4L + 2\pi r$ $\qquad$ $|\partial S|_\infty = 4L + 4R$ $\qquad$ $|\partial S|_1 = 4$

In ℓ^2 the maximum flow has $\operatorname{div} v = 2 + \sqrt{\pi}$. **What is v?**