A Strongly Polynomial Preprocessing for QUBO

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Joint work with P.L. Hammer[1] and G. Tavares

[1] (1936-2006)

Outline

Quadratic Unconstrained Binary Optimization (QUBO)

Variables and Literals

- **Variables**: $x_1, x_2, \ldots, x_n \in \{0,1\}$.
- **Negations**: $\overline{x}_i = 1 - x_i \in \{0,1\}$ for $i = 1, \ldots, n$
- **Literals**: $x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n$

Quadratic Pseudo-Boolean Function (QPBF): $f : \{0,1\}^n \to \mathbb{R}$

$$f(x_1, \ldots, x_n) = c_0 + \sum_{j=1}^{n} c_j x_j + \sum_{1 \le i < j \le n} c_{ij} x_i x_j$$

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Posiforms: Nonnegative (except maybe the constant terms) multi-linear polynomials in $2n$ literals x_1, $\overline{x}_1$, ..., x_n, $\overline{x}_n$

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f &= -2 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 && \text{QPBF} \\
&= -5 + \overline{x}_1 + \overline{x}_2 + \overline{x}_3 + x_1x_2 + x_1x_3 + x_2x_3 && \text{quadratic posiform}
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Roof Dual Bound: $C_2(f) \leq f$ (Hammer, Hansen and Simeone, 1984)

$\mathbf{C_2}(f)$ = largest C s.t. $f = C + \phi$ for some **quadratic posiform** ϕ.

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$$\mathbf{C_2}(f) \leq \mathbf{C_3}(f) \leq \cdots \leq \mathbf{C_n}(f) = \min f$$

Outline

Network Model for Submodular QUBO (Hammer, 1965)

- A QPBF is submodular IFF all quadratic coefficients are nonpositive. (*Doit Yourself, anytime*)
- To a submodular QPBF f associate a network G_f as follows

  $=$

- There is a one-to-one correspondence between values of f and $s-t$ cut values of G_f. (*Hammer, 1965*)

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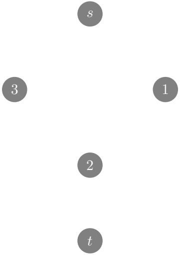

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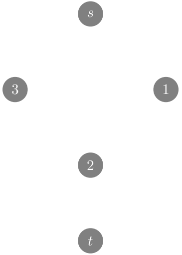

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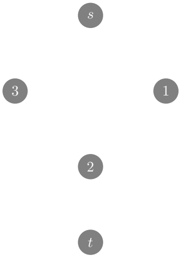

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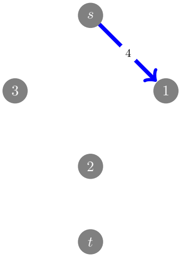

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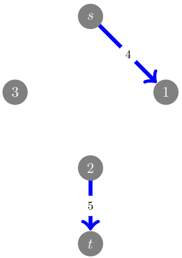

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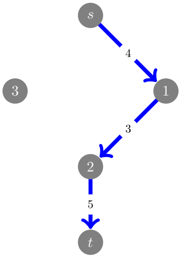

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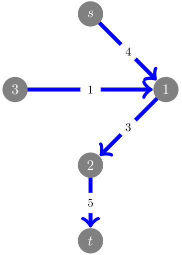

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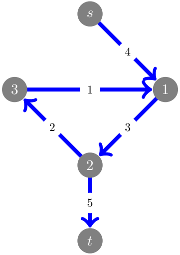

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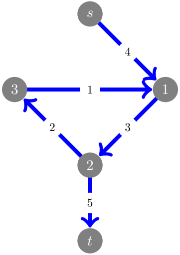

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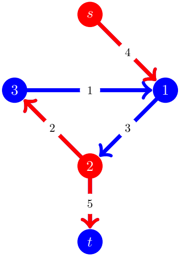

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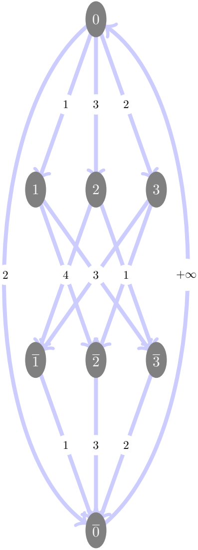
$$\mathbf{f(0,1,0) = C(\{s,2\},\{1,3,t\}) = 11}$$

Outline

Implication Networks (Boros, Hammer, Sun, 1989, 1992)



To a quadratic posiform

$$\phi \ = \ \mathbf{2} x_0 x_0 + \mathbf{2\overline{x}_1} x_0 + \mathbf{6\overline{x}_2} x_0 + \mathbf{4\overline{x}_3} x_0 + \mathbf{8x_1x_2} + \mathbf{6x_1x_3} + \mathbf{2x_2x_3}$$

we associate a directed network N_ϕ on vertex set

$$V(N_\phi) = \{x_0, \overline{x}_0, x_1, \overline{x}_1, ..., x_n, \overline{x}_n\} \qquad (x_0 \equiv 1)$$

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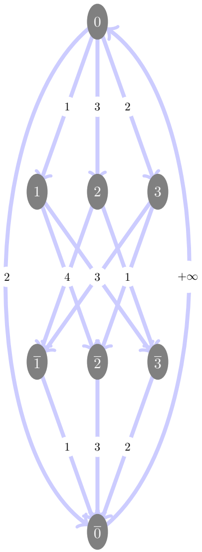

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- Homogenize it by x_0.
- Associate to each term $\alpha\, uv$ $(u \neq v)$ two arcs $(u, \overline{v})$ and $(v, \overline{u})$ with capacities $c(u, \overline{v}) = c(v, \overline{u}) = \alpha/2$.
- Associate to $\gamma\, x_0 x_0$ one arc $(x_0, \overline{x}_0)$ with capacity $c(x_0, \overline{x}_0) = \gamma$ and add arc $(\overline{x}_0, x_0)$ with capacity $c(\overline{x}_0, x_0) = +\infty$.

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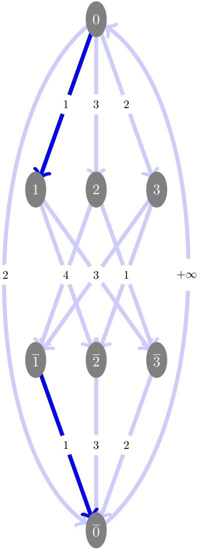

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$$V(N_\phi) = \{x_0, \overline{x}_0, x_1, \overline{x}_1, ..., x_n, \overline{x}_n\} \qquad (x_0 \equiv 1)$$

- Homogenize it by x_0.
- Associate to each term $\alpha\, uv$ $(u \neq v)$ two arcs $(u, \overline{v})$ and $(v, \overline{u})$ with capacities $c(u, \overline{v}) = c(v, \overline{u}) = \alpha/2$.
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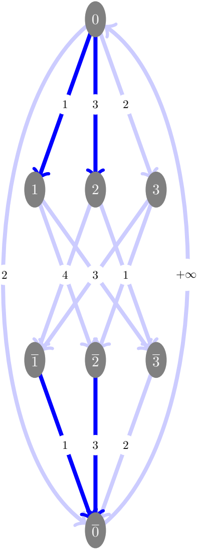

To a quadratic posiform

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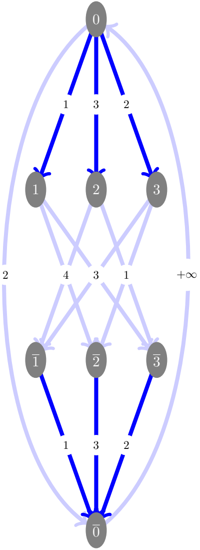

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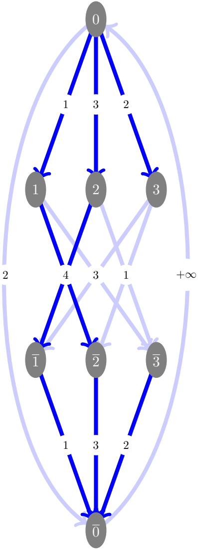

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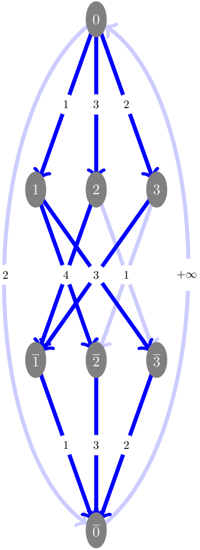

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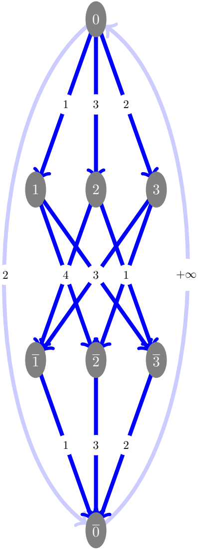

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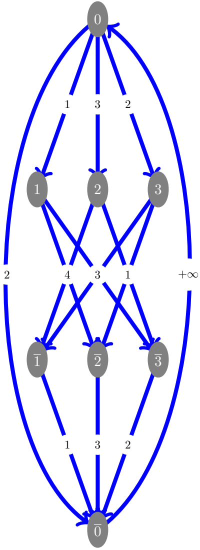

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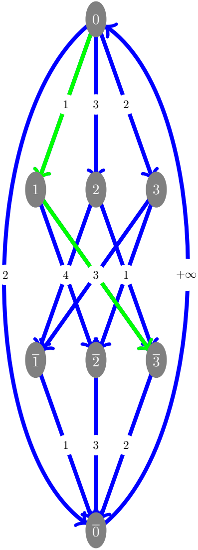

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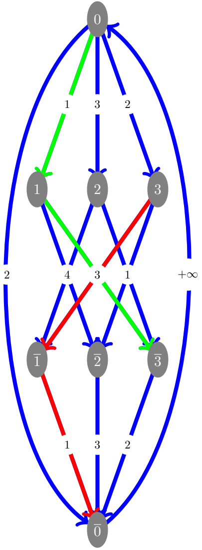

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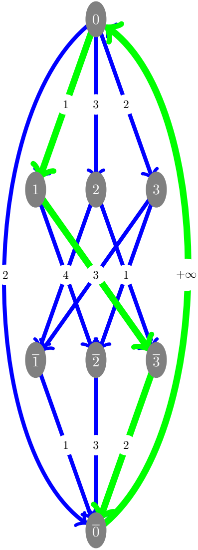

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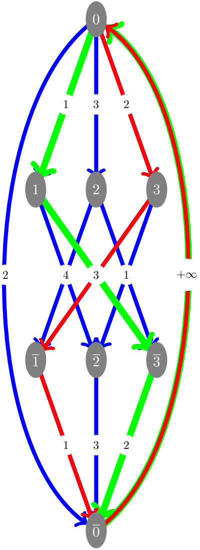

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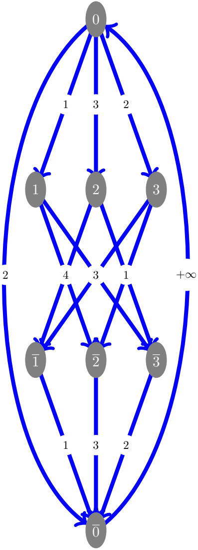

Claims

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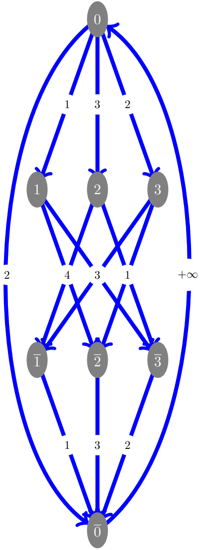

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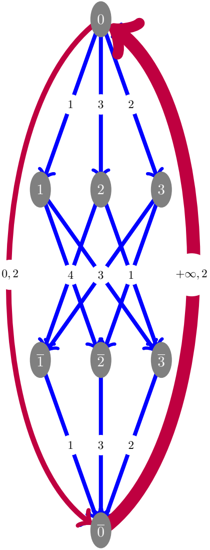

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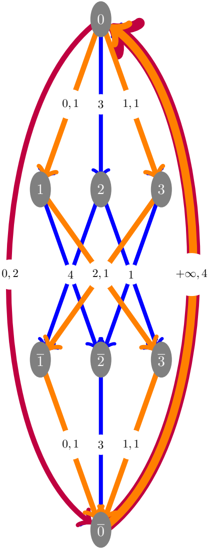

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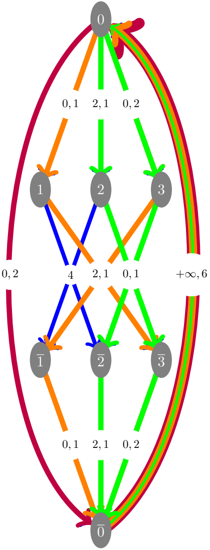

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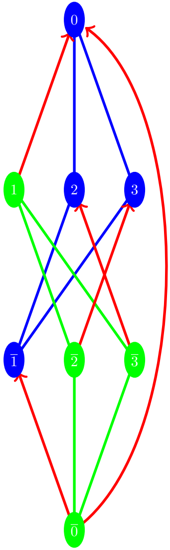

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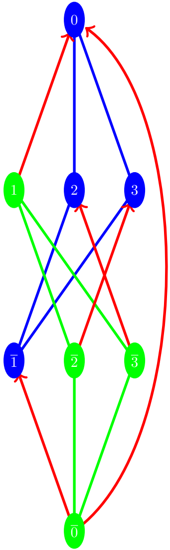

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Outline

Components of the Algorithm

The **purpose** of the preprocessing algorithm is to **fix** some of the variables at their optimum values and **decompose** the remaining problem into several smaller problems which do not share variables.

- Build implication network
- Compute maximum flow; fix variables by persistency (increase capacities of some arcs)
- Probe remaining variables and repeat all of the above as long as there is some change.
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If the input QPBF is submodular, then the above procedure will fix all the variables at their optimal values in the first round, without any probing.

Outline

Via Minimization in VLSI Design

Problem[2]	n	Percentage of Variables Fixed by					Time (sec)
		Roof Duality (strong)	Roof Duality (weak)	Probing (forcing)	Probing (equalities)	**ALL TOOLS**	
via.c1y	829	93.6%	6.4%	0%	0%	**100%**	**0.03**
via.c2y	981	94.7%	5.3%	0%	0%	**100%**	**0.06**
via.c3y	1328	94.6%	5.4%	0%	0%	**100%**	**0.09**
via.c4y	1367	96.4%	3.6%	0%	0%	**100%**	**0.09**
via.c5y	1203	93.1%	6.9%	0%	0%	**100%**	**0.08**
via.c1n	828	57.4%	9.6%	32.4%	0.6%	**100%**	**0.49**
via.c2n	980	12.4%	4.4%	83.1%	0.1%	**100%**	**7.14**
via.c3n	1327	6.8%	5.7%	87.3%	0.2%	**100%**	**18.17**
via.c4n	1366	11.1%	1.3%	87.6%	0%	**100%**	**23.08**
via.c5n	1202	3.4%	1.4%	95.0%	0.2%	**100%**	**17.13**

[2] S. Homer and M. Peinado. Design and performance of parallel and distributed approximation algorithms for maxcut. Journal of Parallel and Distributed Computing 46 (1997) 48-61.

Vertex Cover in Planar Graphs

	Averages for 100 graphs in each of the 4 groups			
	Variables Fixed (%)		Time (sec)	
n	A. D. N.[3]	**QUBO**[4]	A. D. N.[2]	**QUBO**[3]
1000	68.4	**100**	4.06	**0.05**
2000	67.4	**100**	12.24	**0.16**
3000	65.5	**100**	30.90	**0.27**
4000	62.7	**100**	60.45	**0.53**

[3] Alber, Dorn, Niedermeier. Experimental evaluation of a tree decomposition based algorithm for vertex cover on planar graphs. Discrete Applied Mathematics 145 (2005) 219-231; 750 GHz, Linux PC, 720 MB

[4] Pentium 4, 2.8 GHz, Windows XP, 512 MB

Jumbo Vertex Cover in Planar Graphs

Vertices	Computing Times (min)[5]		
	Planar Density		
	10%	50%	90%
50,000	0.7	2.3	0.9
100,000	2.9	10.2	3.9
250,000	19.5	69.8	26.3
500,000	79.3	277.3	106.9

[5] Averages over 3 experiments on a Xeon 3.06 GHz, XP, 3.5 GB RAM; ALL problems had 100% of their variables fixed.

One Dimensional Ising Models

σ	Number of Spins	Average Computing Time (s)		
		Branch, Cut & Price[6]	Biq Maq[5]	**QUBO[7]**
2.5	100	699	68	**1**
	150	92 079	388	**3**
	200	N/A	993	**9**
	250	N/A	6 567	**14**
	300	N/A	34 572	**21**
3.0	100	256	59	**1**
	150	13 491	293	**2**
	200	61 271	1 034	**3**
	250	55 795	3 594	**4**
	300	55 528	8 496	**5**

[6] F. Rendl, G. Rinaldi, A. Wiegele. (2007). Solving max-cut to optimality by intersecting semidefinite and polyhedral relaxations.

[7] ALL problems were solved by QUBO.

Larger One Dimensional Ising Models

σ	n	Average of 3 Problems: Variables not fixed	Average of 3 Problems: QUBO Time (s)[8]
2.5	500	**5**	**13**
	750	**22**	**30**
	1000	**24**	**53**
	1250	**20**	**81**
	1500	**32**	**124**
3.0	500	**0**	**4**
	750	**0**	**12**
	1000	**0**	**23**
	1250	**0**	**37**
	1500	**0**	**59**

[8] Pentium M, 1.6 GHz 760 MB RAM