Caustic localization and Multi-valued Solution of the Eikonal equation

Jean-David Benamou (INRIA)
Rémi Sentis (CEA)
Ian Solliec (INRIA)
Olivier Lafitte (CEA)

Geometric optics Wave equation

$$u_{tt}(t,\mathbf{x}) - c(x)\Delta u(t,\mathbf{x}) = 0, \quad u(t,.)|_{\Gamma} = u_0(.)e^{-i\omega t}$$

Asymptotic frequency domain

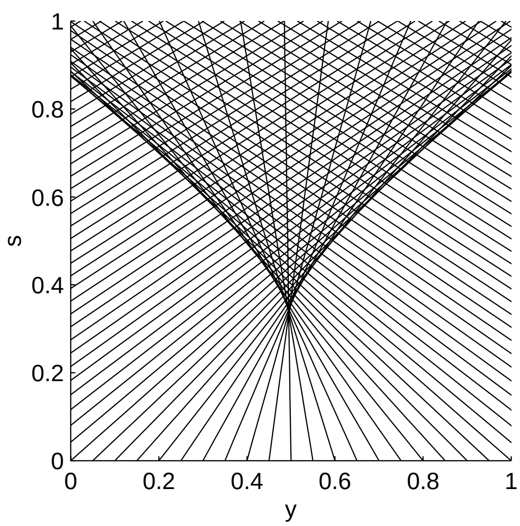
$$u(t,\mathbf{x}) \stackrel{t\to+\infty}{\longrightarrow} u_\omega(\mathbf{x})e^{-i\omega t}$$

For large ω, $u_\omega(\mathbf{x})$ is too expensive to compute. It is approximated by the GO ansatz.

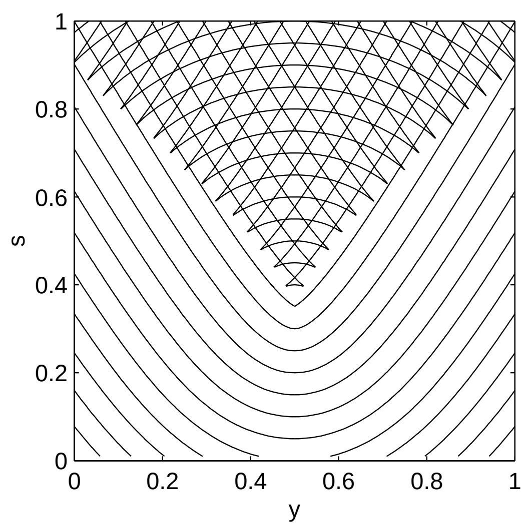
$$u_\omega(\mathbf{x}) \simeq a(\mathbf{x})\, e^{i\omega\phi(\mathbf{x})}$$

Lagrangian Method

$$u_\omega(\mathbf{x}) \simeq a(y(s))e^{i\omega\phi(y(s))} \quad \text{pour} \quad \mathbf{x} = y(s)$$



Rayons : y



Fronts : ϕ

Ray tracing - Multi-Valued Solution

$$\begin{cases} \dot{y}(s,y^0) = H_p(s, y(s,y^0), p(s,y^0)), & y(0,y^0) = y^0 \\ \dot{p}(s,y^0) = -H_y(s, y(s,y^0), p(s,y^0)), & p(0,y^0) = \phi^0_{y^0}(y^0) \\ \dot{\varphi}(s,y^0) = p \cdot H_p(s,y,p) - H(s,y,p), & \varphi(0,y^0) = \phi^0(y^0) \end{cases}$$

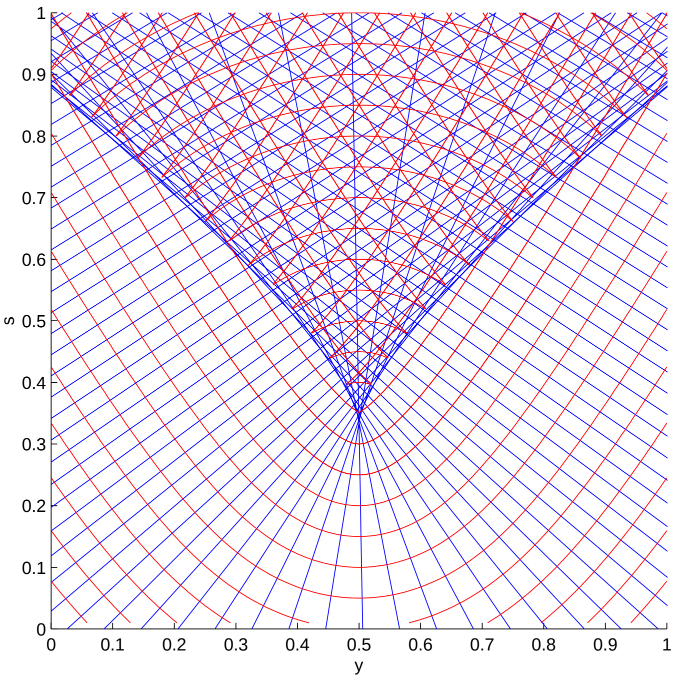
Hamiltonian function given by :

$$H(s,y,p) = -\sqrt{1-p^2}$$

Initial phase :

$$\phi^0(y^0) = \int_0^{y^0} \frac{p^0(x)}{\sqrt{1+p^0(x)^2}} dx$$

avec $p^0(x) = \frac{-3(x-0.5)}{\sqrt{1+3(x-0.5)^2}}$



Caustic points

$(s, y(s, y^0))$ is a caustic point $\Leftrightarrow det(\frac{\partial y}{\partial y^0}(s, y^0)) = 0$.

In practice :

$$\begin{pmatrix} \frac{\partial \dot{y}}{\partial y^0}(s, y^0) \\ \frac{\partial \dot{p}}{\partial y^0}(s, y^0) \end{pmatrix} = \begin{pmatrix} H_{py}(s, y, p) & H_{pp}(s, y, p) \\ -H_{yy}(s, y, p) & -H_{yp}(s, y, p) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial y}{\partial y^0}(s, y^0) \\ \frac{\partial p}{\partial y^0}(s, y^0) \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial y}{\partial y^0}(0, y^0) \\ \frac{\partial p}{\partial y^0}(0, y^0) \end{pmatrix} = \begin{pmatrix} Id \\ \frac{\partial^2 \phi^0}{\partial^2 y^0}(s, y^0) \end{pmatrix}$$

Some elementary geometry

Bicharacteristics : $\Lambda = \{(s, y(s, y^0), p(s, y^0)); (s, y^0) \in R_s^+ \times R_y\}$.

Λ is a 2-D (Lagrangian) sub-manifold of phase space :
$R_s^+ \times R_y \times R_p$.

Rays : $\Pi_y(\Lambda)$ with
$$\begin{array}{rl} \Pi_y \ : \ R_s^+ \times R_y \times R_p & \longrightarrow R_s^+ \times R_y \\ (s, y, p) & \longrightarrow (s, y). \end{array} .$$

In 2-D (s, y) (stable,local) generic caustics are

- The Fold : $\Lambda = \{(s,\ y = 3\,p^2, p); (s, p) \in [0, s_l] \times [-p_l, p_l]\}$
  et $\varphi = 2\,p^3$.
- The Cusp : $\Lambda = \{(s,\ y = 4\,p^3 + 2\,ps, p); (s, p) \in [0, s_l] \times [-p_l, p_l]\}$
  et $\varphi = 3\,p^4 + p^2 s$.

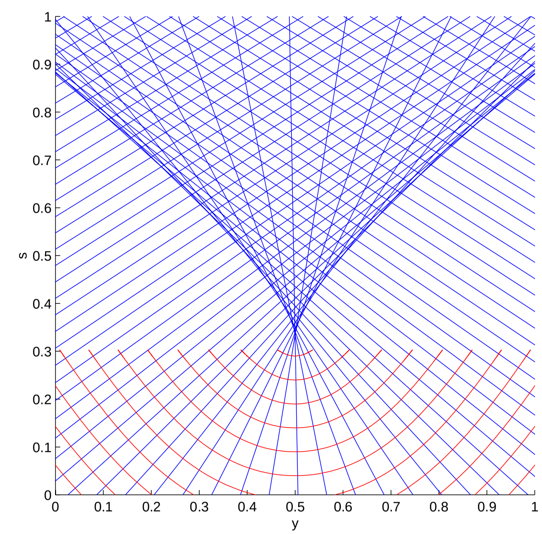
Switching to Eulerian coordinates - Classical Solution

As long as $det(\frac{\partial y}{\partial y^0}(s, y^0)) \neq 0$, one can invert $y^0 \rightarrow y(s, y^0)$ and define on $\Pi_y(\Lambda)$ a Eulerian phase $\phi(s, x)$:

$$\phi(s, y(s, y^0)) = \varphi(s, y^0).$$

Then $\phi_x(s, y(s, y^0)) = p(s, y^0)$ and ϕ is a classical solution of

$$\begin{cases} \phi_s(s, x) + H(s, x, \phi_x(s, x)) = 0 \\ \phi(0, x) = \phi^0(x). \end{cases}$$



Squetch of the Proof

$$\frac{d}{ds}\{\phi(s, y(s, y^0))\} = \dot{\varphi}(s, y^0)$$

$$\Leftrightarrow \quad \phi_s(s, y(s, y^0)) + \dot{y}(s, y^0) \cdot \phi_x(s, y(s, y^0)) = p \cdot H_p(s, y, p) - H(s, y, p)$$

$$\Leftrightarrow \quad \phi_s(s, y(s, y^0)) = -H(s, y(s, y^0), \phi_x(s, y(s, y^0)))$$

$$\Leftrightarrow \quad \phi_s(s, x) = -H(s, x, \phi_x(s, x))$$

Remark $V(s, x)$ s. t. $V(s, y(s, y^0)) = \begin{pmatrix} \frac{\partial y}{\partial y^0}(s, y^0) \\ \frac{\partial p}{\partial y^0}(s, y^0) \end{pmatrix}$ satisfies

$$V_s(s, x) + H_p(s, x, \phi_x(s, x)) \cdot V_x(s, x) =$$

$$\begin{pmatrix} H_{py}(s, x, \phi_x(s, x)) & H_{pp}(s, x, \phi_x(s, x)) \\ -H_{yy}(s, x, \phi_x(s, x)) & -H_{yp}(s, x, \phi_x(s, x)) \end{pmatrix} \cdot V(s, x)$$

and $\beta(s, y(s, y^0)) = det(\frac{\partial y}{\partial y^0}(s, y^0)) \to \beta(s, x) = det(V_1(s, x))$.

Remark (to be used) As long as $det(\frac{\partial p}{\partial y^0}(s,y^0)) \neq 0$, one can invert $y^0 \longrightarrow p(s,y^0)$ and define on $\Pi_p(\Lambda)$, $X(s,p)$, $\phi(s,p)$:

$$X(s,p(s,y^0)) = y(s,y^0).$$
$$\phi(s,p(s,y^0)) = \varphi(s,y^0)$$

X and ϕ are solution of

$$\begin{cases} X_s(s,p) - H_y(s,X,p)\cdot X_p(s,p) = H_p(s,X,p) \\ \phi_s(s,p) - H_y(s,X,p)\cdot \phi_p(s,p) = p\cdot H_p(s,X,p) - H(s,X,p) \\ X(0,p) = x, \quad \phi(0,p) = \phi^0(x) \end{cases}$$

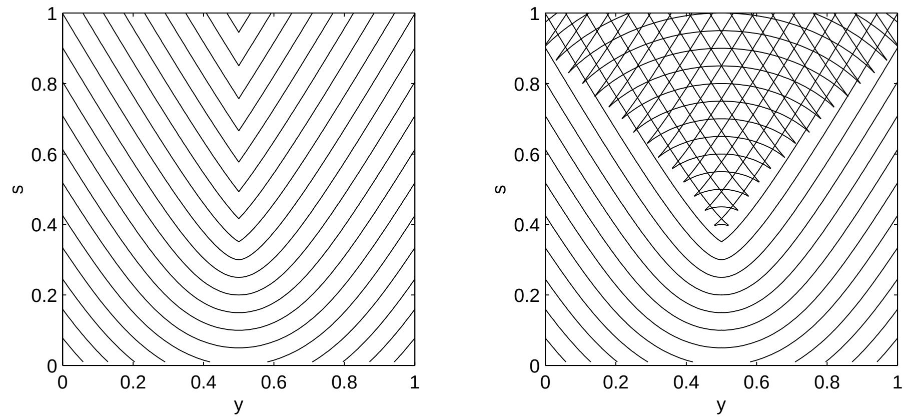
In the presence of Caustics

$$\phi^{Visc}(s,x) = \inf_{\left\{\begin{array}{c} y^0 \in R,\ y(.) \in W^{1,+\infty}(R) \\ t.q.\ y(0) = y^0,\ y(s) = x \end{array}\right\}} \int_0^s L(t, y(t), \dot{y}(t))dt + \phi^0(y^0),$$ where $L(s,x,v) = \sup_{p \in R_p}\{p \cdot v - H(s,x,p)\}$.

$\phi^{Visc}(s,x)$ is the (global, single-valued) viscosity solution of

$$\phi_s(s,x) + H(s,x,\phi_x(s,x)) = 0, \quad \phi(0,x) = \phi^0(x)$$

$$\phi^{Visc}(s,x) = \min_{\left\{\begin{array}{c} y^0 \in R_y\ t.q. \\ y(s,y^0) = x \end{array}\right\}} \varphi(s,y^0)$$



In the Eulerian framework

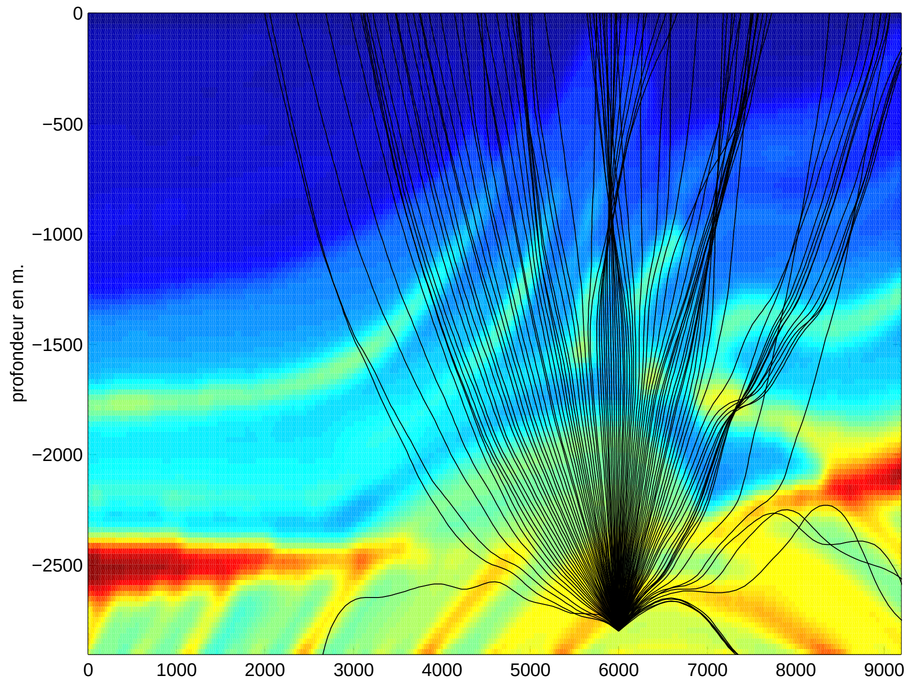
The viscosity solution theory (Crandall-Lions, Barles, Souganidis, Perthame...) offers

- a comprehensive framework.
- Stability and convergence results for numerical upwind schemes.

But, these schemes (for instance finite-difference schemes) (Shu-Osher, Abgrall, Tadmor, ...) cannot compute the Multi-Valued Solution.

Eulerian versus Lagrangian ?

Spatial resolution



Computing H.-F. wave ansatz (Keller-Ludwig)

$$u(x) \sim \frac{\omega^{\frac{1}{6}}}{2\sqrt{\pi}} e^{-i\frac{\pi}{4}} \left(a^{-}(x) e^{i\omega\phi^{-}(x)} + a^{+}(x) e^{i\omega\phi^{+}(x)} \right)$$

Who cares ?

- Engquist-Fatemi-Osher ('95), Ruuth-Merriman-Osher ('99), Steinhoff ('99...)
- Symes ('96), Symes-Quian ('99).
- Brenier-Corrias ('96), Engquist-Runborg ('96).
- Engquist-Runborg-Tornberg ('00).
- JDB ('96, '99), Abgrall-JDB ('99), Solliec-JDB ('00) , Golse-Lafitte-Sentis-Montarnal ('99), Lafitte-Sentis-Solliec-JDB
- ...

Eulerian Splitting of the Multi-Valued Solution (JDB '99)

We assume here that the exact location of the caustics is known and given by $x = C(s)$

$$\phi_s^{-C}(s,x)+H(s,x,\phi_x^{-C}(s,x))=0, \quad \phi^{-C}(0,x)=\phi^{0-C}(0,x), \;\; x<C(0)$$

$+$ in-coming B.C.

$$\phi_s^{+C}(s,x)+H(s,x,\phi_x^{+C}(s,x))=0, \quad \phi^{+C}(s,C(s))=\phi^{-C}(s,C(s))$$

$+$ out-going B.C.

Fold caustic localisation **x= C(s)**

$\{(s, C(s), p_c(s)),\ s > s_c\} \subset \Lambda$: the caustic trajectory in phase space $R_s^+ \times R_y \times R_p$.

In term of rays :

$$\begin{aligned} C(s) &= y(s, y^0_{C(s)}) \\ p_C(s) &= p(s, y^0_{C(s)}) \end{aligned}$$

, $y^0_{C(s)}$ unknown.

We find the equation

$$\begin{cases} \dot{C}(s) & = \dot{y}(s, y^0_{C(s)}) + \frac{\partial}{\partial s}(y^0_{C(s)}) \cdot \frac{\partial y}{\partial y^0}(s, y^0_{C(s)}) \\ & = \dot{y}(s, y^0_{C(s)}) \qquad\qquad (\frac{\partial y}{\partial y^0}(s, y^0_{C(s)}) = 0) \\ & = H_p(C(s), p_C(s)) \end{cases}$$

$p_C(s)$ **not available in** (s,x) **space.**

Recall : $\phi_x(s, y(s, y^0)) = p(s, y^0)$ then

$$\frac{\partial y}{\partial y^0}(s, y^0)\phi_{xx}(s, y(s, y^0)) = \frac{\partial p}{\partial y^0}(s, y^0) \text{ and } \phi_{xx}(s,x) \nearrow +\infty \text{ as } x \to C(s)$$

Finite difference approximations do not converge. Need to pass in (s,p) **coordinates.**

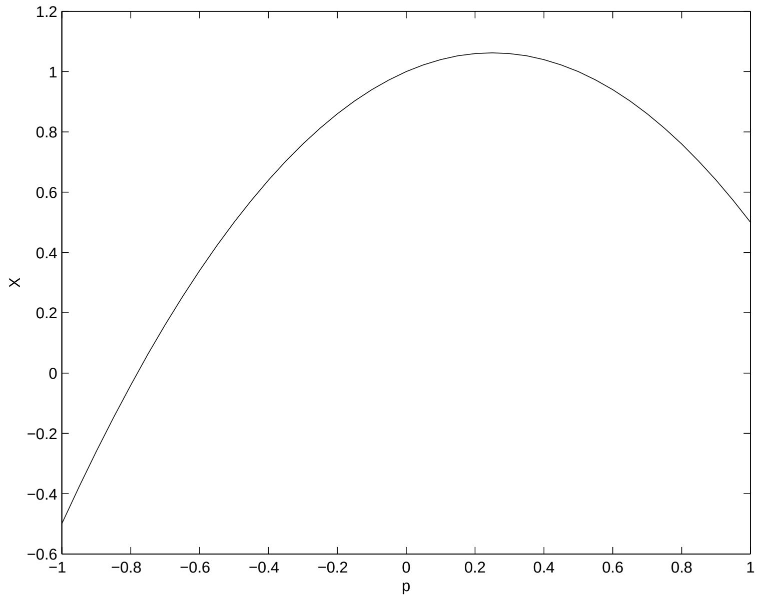
Recall : $X(s, p(s, y^0)) = y(s, y^0)$ and
$$C(s) = X(s, p_c(s))$$
$$X_p(s, p_c(s)) = 0$$

For a Fold, we can write (locally)

$$X(s,p) = C(s) + a\,(p - p_{C(s)})^2 + \text{l.o.t.}$$

Computation of a and $p_c(s)$

Let $p^- = \phi_x^{-C}(s, x^-)$, $\quad p^+ = \phi_x^{+C}(s, x^+)$



then $\begin{cases} x^- = C(s) + a\,(p^- - p_{C(s)})^2 \\ x^+ = C(s) + a\,(p^+ - p_{C(s)})^2 \end{cases}$

Remark : if $C(s) - x^- = C(s) - x^+$, we simply find
$p_c(s) = 0.5\,(\phi_x^{-C}(s, x^-) + \phi_x^{+C}(s, x^+))$

The final closed system

$$\phi_s^{-C}(s,x)+H(s,x,\phi_x^{-C}(s,x))=0, \quad \phi^{-C}(0,x)=\phi^{0-C}(0,x), \;\; x<C(0)$$

$$\phi_s^{+C}(s,x)+H(s,x,\phi_x^{+C}(s,x))=0, \quad \phi^{+C}(s,C(s))=\phi^{-C}(s,C(s))$$

$$\dot{C}(s)=H_p(C(s),p_C(s)), \quad C(0) \text{ given}$$

$$p_C(s))=NCF(\phi_x^{-C}(s,x^-),\phi_x^{+C}(s,x^+)$$

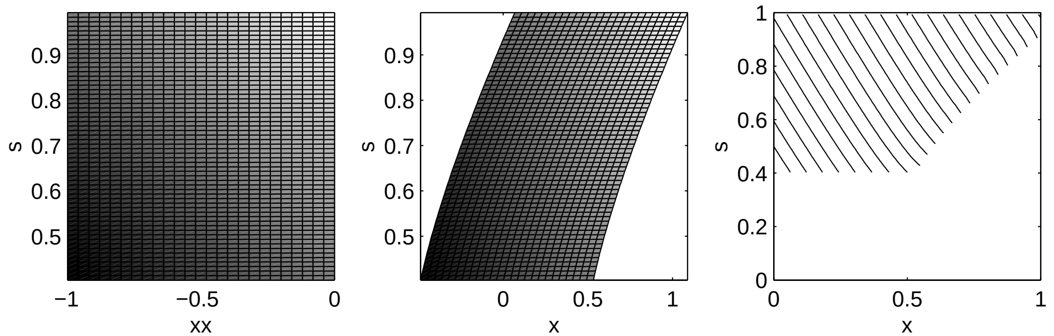
Straightening Caustics - Change of variable in space

Let $(s, \tilde{x}(s,x) + C(s)) \leftarrow (s,x)$, then

$\tilde{\phi}(s,\tilde{x}) = \phi^{-C}(s,x)$ satisfies

$$\begin{cases} \tilde{\phi}_s(s,\tilde{x}) + H(s, \tilde{x} + C(s), \tilde{\phi}_{\tilde{x}}(s,\tilde{x})) - H_p(s, C(s), p_c(s)) \cdot \tilde{\phi}_{\tilde{x}}(s,\tilde{x}) = 0, \\ \qquad for \quad \tilde{x} \in]-M, 0[. \end{cases}$$

We invert the change of variable $\phi(s, \tilde{x} + C(s)) = \tilde{\phi}^{-C}(s,\tilde{x})$



Two branches

Let $(s, a(s)\tilde{x} + b(s)) = (s, x)$ with $\begin{cases} a(s) = \dfrac{Cl(s) - Cr(s)}{2\ M} \\ b(s) = \dfrac{Cl(s) + Cr(s)}{2} \end{cases}$.

The equation for $\tilde{\phi}(s, \tilde{x}) = \phi^{+C}(s, x)$ is

$$\begin{cases} \tilde{\phi}_s(s, \tilde{x}) + H(s, a(s)\tilde{x} + b(s), \dfrac{\tilde{\phi}_{\tilde{x}}(s, \tilde{x})}{a(s)}) - (\dot{a}(s)\tilde{x} + \dot{b}(s)) \cdot \dfrac{\tilde{\phi}_{\tilde{x}}(s, \tilde{x})}{a(s)} = 0, \\ \qquad for \quad \tilde{x} \in] - M, M[. \end{cases}$$