FULLY NONLINEAR STOCHASTIC PDE

Theory & Applications

(joint work with P.-L. Lions)

OUTLINE OF LECTURE

- problem
- applications
- mathematical difficulties
- stochastic integration
- "correct" equation
- key observations
- theory
- phase transitions
- work in progress & open problems

$$(E)\quad \begin{cases} du = F(D^2u, Du, x, t, \omega)\,dt + H(Du, x, t, \omega)\cdot dW_t & \text{in } \mathbb{R}^N\times(0,\infty) \\ u = u_0 \text{ on } \mathbb{R}^N\times\{0\} \end{cases}$$

$u_0 \in BUC(\mathbb{R}^N)$

F degenerate elliptic, i.e. $F(A,\dots) \geqq F(B,\dots)$ if $A \geqq B$

$H = (H_1,\dots,H_m)$, F continuous

$dW_t = (dW_t^1, \dots, dW_t^m)$ m-dim. WHITE NOISE

<u>Why study (E)?</u> important class of equs
many applications
"nothing" known nonlinear setting

<u>History</u>

linear uniformly elliptic operators & some uniformly elliptic quasi linear using MARTINGALE THEORY

Watanabe, Krylov, Kushner, Pardoux, ...

stochastic analysis, filtering and stochastic control with partial information

H linear in Du

F linear in D^2u Du & uniformly elliptic

example $\begin{cases} du = Du \, dW_t + \frac{1}{2}\Delta u \, dt \\ u_{|t=0} \; u_0 \end{cases}$

Itô's $\Longrightarrow$ $u(x,t = u_0(x+W_t$

Watanabe, Pardoux Zakai, Kunita, Krylov, Rozovsky,

- asymptotics of equations (processes) with rapidly oscillating (mixing) coefficients in time

eg $u^\varepsilon_t = \lambda u^\varepsilon_{xx} + \frac{1}{\varepsilon}\zeta(\frac{t}{\varepsilon^2})u^\varepsilon_x$, "$\frac{1}{\varepsilon}\zeta(\frac{t}{\varepsilon^2})dt \underset{\varepsilon\to 0}{\to} dW$"

$u^\varepsilon \to u$

$du \; (\lambda + \frac{1}{2} \; u_{xx} + u_x dW$

Watanabe

Kushner & Huang

nonlinear ?

· Pathwise stochastic control theory

(financial models with interest rates

typical problem:

$$(SDE)\begin{cases} dX_s = b(X_s,\alpha_s,\omega)\,ds + \Sigma(X_s,\alpha_s,\omega)\,d\tilde{W} + \sigma(X_s)\cdot dW(\omega) \\ X_0 = x \in \mathbb{R}^n \end{cases}$$

minimize over all α payoff $J(x,t,\alpha,\omega) = \tilde{\mathbb{E}}\,\mathbb{E}\,g(X(t))$

value function: $u(x,t) = \inf_\alpha J(x,t,\alpha)$

$$u_t = \inf_\alpha \left\{ b(x,\alpha)\cdot Du + \tfrac{1}{2}\operatorname{tr}(\Sigma\Sigma^T + \sigma\sigma^T) D^2u \right\}$$

Bellman equ

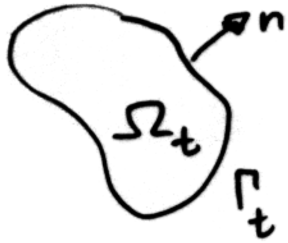
allow random dependence on u

$$U(x,t,\omega) = \inf_\alpha \tilde{\mathbb{E}}\, g(X(t,\omega))$$

$$dU = \inf_\alpha \left[b(x,\alpha,\omega)\cdot Du + \tfrac{1}{2}\operatorname{tr}(\Sigma\Sigma^T)(\alpha,x,\omega) D^2u \right] + \sigma Du\cdot dW$$

stochastic Bellman equ

• Front propagation with random normal velocities



$\dot{x} = Vn$, "$V = \frac{dW}{dt}$"

(nucleations)

<u>recall</u> level sets $\Gamma_t = \{u(t)=0\}$

$\Omega_t = \{u(\cdot,t) > 0\}$

u solves level set pde

$V = c$, $u_t = c|Du|$; $V = -\kappa$, $u_t = \Delta u - \frac{(D^2u\, Du, Du)}{|Du|^2}$

natural to expect

$V = \frac{dW}{dt}$, $du = |Du|\,dW$

$V = \alpha \frac{dW}{dt} - \beta\kappa$, κ = curvature

$$du = \alpha |Du|\,dW + \beta\left(\Delta u - \frac{(D^2u\, Du, Du)}{|Du|^2}\right)$$

some background on STOCHASTIC INTEGRATION
Stratonovich - Ito's Calculus

$$\int_a^b f(s)\,ds \approx \sum f(\tilde{s}_i)(s_{i+1}-s_i) \qquad \tilde{s}_i \in (s_i, s_{i+1})$$

$$\int_a^b f(s)\,dW(s) \approx \sum f(\tilde{s}_i)\left(W(s_{i+1})-W(s_i)\right)$$

different answer depending on the choise of $\tilde{s}_i$

Itô $\quad \tilde{s}_i = s_i$

Stratonovich: $\quad \tilde{s}_i = \frac{s_i+s_{i+1}}{2}$

$$\int_a^b f(s)\circ dW(s) = \int_a^b f(s)\,dW(s) + \frac{1}{2}\int_a^b f'(s)\,ds$$

X_t stochastic process

$d\phi(X_t) = \phi'(X_t)\,dX_t + \frac{1}{2}\phi''(X_t)\langle dX_t, dX_t\rangle$ - Ito's formula

$d\phi(X_t) = \phi'(X_t)\circ dX_t$

Mathematical Difficulties

- $du = |Du| dW$

level set theory needs $\beta(u)$ to be a solution $(\beta' > 0)$

$$d\beta(u) = \beta'(u)\, du + \frac{1}{2} \beta''(u) |Du|^2 dt$$

$$= D\beta(u) | du + \boxed{\frac{1}{2} \beta''(u) |Du|^2 dt}$$

not the same equ

- $du = u_x \, dW + \lambda u_{xx} \, dt \quad ; \quad \lambda \geqq 0$

$v(x,t) = u(x - W_t, t)$ solves

$$dv = du - u_x \, dW + \frac{1}{2} u_{xx} \, dt - u_{xx} \, dt$$

$$= (\lambda - \tfrac{1}{2}) u_{xx} \, dt = \boxed{(\lambda - \tfrac{1}{2}) v_{xx}} dt$$

not well posed for $0 < \lambda < 1/2$

REPLACE ITÔ'S INTEGRAL BY STRAT NOV CH'S

- $du = |Du| \circ dW$

  $d\beta(u) = \beta'(u)|\quad| \circ dW = \quad (u)| \circ dW$

- $du = u_x \circ dW + \lambda u_{xx} dt \qquad , \lambda \geqq 0$

  $v = u(x - W_t, t)$ solves

  $$dv = \lambda v_{xx} dt$$

$$du = F(D^2u, Du, x)\,dt + H(Du,x)\circ dW$$

$$= \left[F + \tfrac{1}{2}\left(D^2u\,\frac{\partial H}{\partial p}, \frac{\partial H}{\partial p}\right) + \tfrac{1}{2}\,\frac{\partial H}{\partial p}\cdot\frac{\partial H}{\partial x}\right]dt + H(Du,x)\cdot dW$$

History

$\left.\begin{array}{l}\text{linear (uniformly elliptic)}\\ \text{special quasilinear (unif. elliptic)}\end{array}\right\}$ martingale theory

Nothing nonlinear

SAMPLE OF MAIN DIFFICULTIES

- NO SMOOTH SOLNS IN GENERAL

  NOT ACCESSIBLE TO MARTINGALE THEORY WHEN NONLINEAR

  NO POINTWISE FORMULATION ("$dW = \pm\infty$ a.s.")

$$\begin{cases} u^\varepsilon_t = H(Du^\varepsilon)\dot{\zeta}^\varepsilon & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\varepsilon = u_0 & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

$\dot{\zeta}^\varepsilon$ changes sign "infinitely" often as $\varepsilon \to 0$

formally

$$u^\varepsilon(x,t) = U(x, \zeta^\varepsilon(t))$$

since ζ^ε may change signs

$$\begin{cases} U_\tau = H(DU) & \text{in } \mathbb{R}^n \times (-\infty,\infty) \\ U = u_0 & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

u^ε cannot be the "correct" solution, unless U is smooth.

<u>Example</u>: $\begin{cases} u^\varepsilon_t + |Du^\varepsilon|\dot{\zeta}^\varepsilon = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\varepsilon = |x| & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$

$$u^\varepsilon(x,t) = \max\left[(|x| - \zeta^\varepsilon(t))_+, \max_{0 \le s \le t}(-\zeta^\varepsilon_s)_+\right] = \left(|x| - (\zeta^\varepsilon(t) - \min_{0 \le s \le t} \zeta^\varepsilon(s))\right)_+ - \min_{0 \le s \le t} \zeta^\varepsilon(s)$$

$$\Big\downarrow \varepsilon \to 0$$

$$u(x,t) = \max\left[(|x| - W(t))_+, \max_{0 \le s \le t}(-W_s)_+\right] =$$

$$= \left(|x| - (W(t) - \min_{0 \le s \le t} W(s))\right)_+ - \min_{0 \le s \le t} W(s)$$

TRUE FOR ANY $\zeta^\varepsilon \to \zeta$ – not only W

IMPORTANT ISSUE

understand the oscillations - cancellations in time

notation

$$\begin{cases} v_t = H(Dv) \\ v_{|t=0} = v_0 . \end{cases} \qquad v(t) = S_H(t) v_0$$

Thm

$$\begin{cases} u_t = H(Du)\dot{\zeta} \\ u_{|t=0} = u_0 \end{cases} \qquad H = H_1 - H_2 \; ; \quad H_1, H_2 \text{ convex}$$

Then:

$$\leq u(t) \leq S_{H_1}\big(\max_{0\leq s\leq t} \zeta(s)_+\big) S_{H_2}\big(\max_{0\leq s\leq t} \zeta(s)_-\big)$$

contrast with "typical" estimate

$$u(t) \leq S_{|H|}\Big(\underbrace{\int_0^t |\dot{\zeta}(s)| ds}_{(= +\infty)} \Big)$$

$$\begin{cases} u^\varepsilon_t = H(Du^\varepsilon)\,J^\varepsilon & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\varepsilon = |x|^2 & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

<u>Question</u>. Assume $J^\varepsilon \to 0$ locally uniformly as $\varepsilon \to 0$.

When is it true that $u^\varepsilon(x,t) \xrightarrow[\varepsilon \to 0]{} u_0(x)$?

<u>Answer</u> iff H is the difference of two convex fns

$J \in C^1 \longleftrightarrow H$ continuous

$J \in C^{1/2}$ ("Bm") $\longleftrightarrow H$ Lipschitz

$J \in C \longleftrightarrow H$ difference of convex fns

MAIN RESULTS

- notion of stochastic viscosity solution
- existence and uniqueness

$$\begin{cases} du = F(D^2u, Du)\,dt + H(Du)\cdot dW \\ u|_{t=0} = u_0 \end{cases}$$

$H \in C^3$, F continuous & degenerate elliptic

<u>Thm</u> a.s. there exists a unique stochastic v.s.

EXISTENCE

Approximations to Brownian motion

$$\zeta^\varepsilon(t \xrightarrow[\varepsilon\to 0]{} W(t \quad \text{uniformly in } (0,T \quad \text{a.s.}$$

Examples: (i) $\zeta^\varepsilon\ W * \rho_\varepsilon \quad \rho_\varepsilon$ smooth

g $\quad \zeta^\varepsilon(t) = \dfrac{W(t+\varepsilon) - W(t)}{\varepsilon}$

(ii) $\zeta^\varepsilon(t = \frac{1}{\varepsilon}\zeta\left(\frac{t}{\varepsilon^2}\right.$ – mixing conditions

$M^t \quad \sigma\ \zeta(u,\omega) \quad s \le u \le t \qquad \beta(t) = \sup_{s \ge 0}\left\{ P(A \cap B) - P(A)P(B)\right|.\ A \in M_0^s, B \in M^\infty_{s+t}$

$$\int_0^\infty \beta^{\frac{1}{2}}(t)\,dt < \infty$$

Approximate problem

$$(IVP)_{\varepsilon, \zeta^\varepsilon, u_0^\varepsilon}\begin{cases} u_t^\varepsilon = F(D^2u^\varepsilon Du^\varepsilon) + H(Du^\varepsilon)\,\zeta^\varepsilon, & \mathbb{R}^N \times (0,T \\ u^\varepsilon\ u_0^\varepsilon \quad \text{on } \mathbb{R}^N\ \{0\} \end{cases}$$

Question $\qquad u^\varepsilon \to u$ as $\varepsilon \to 0$?

eqn for u ?

Assumptions: (i) H Lipschitz continuous & difference of convex fn

(ii) $F \in C(S^N \times \mathbb{R}^N)$ degenerate elliptic

(iii)* $\exists\, G \in C(S^{2N} \times \mathbb{R}^{2N})$ st

$$G\left(\begin{matrix} \lambda A & -\lambda A \\ -\lambda A & \lambda A \end{matrix}, P\right) = 0 \quad \& \quad F(X,p) - F(-Y,q) \leq G(Z,P)$$

if

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq Z \quad \& \quad P = (p,q)$$

THEOREM: · $(\zeta^\varepsilon)_{\varepsilon>0}, (\xi^\eta)_{\eta>0}$ approximations to Bm

· $(u^\varepsilon)_{\varepsilon>0}, (v^\eta)_{\eta>0}$ solutions of $(IVP)_{\varepsilon,\zeta^\varepsilon,u_0^\varepsilon}, (IVP)_{\eta,\xi^\eta,v_0^\eta}$

· $\|u_0^\varepsilon - v_0^\eta\|_{C(\mathbb{R}^N)} \to 0$, as $\varepsilon,\eta \to 0$

Then: $\|u^\varepsilon - v^\eta\|_{C(\mathbb{R}^N \times [0,T])} \to 0$ a.s. , as $\varepsilon,\eta \to 0$

In particular,

$(u_\varepsilon)_{\varepsilon>0}$ is Cauchy in $C(\mathbb{R}^N \times [0,T])$ a.s.

and

all approximations converge to the same limit.

* not ~~really~~ needed - it is used only to explain the ideas.

<u>(Sketch of) proof:</u> • possible strategy

1. define $\quad z(x,y,t) = u^\varepsilon(x,t) - v^\eta(y,t)$, then

$$z_t = F(D^2u^\varepsilon, Du^\varepsilon) - F(D^2v^\eta, Dv^\eta) + H(Du^\varepsilon)\dot{\zeta}^\varepsilon - H(Dv^\eta)\dot{\xi}^\eta$$
$$= F(D_x^2 z, D_x z) - F(-D_y^2 z, -D_y z) + H(D_x z)\dot{\zeta}^\varepsilon - H(-D_y z)\dot{\xi}^\eta$$

assumption on $F \implies$

$$(*) \quad z_t \leqq G(D^2 z, Dz) + H(D_x z)\dot{\zeta}^\varepsilon - H(-D_y z)\dot{\xi}^\eta$$

2. Fix $\lambda > 0$, define $\quad \bar{z}(x,y,t) = \lambda|x-y|^2 + C_{\varepsilon,\eta} t$ & compute

$$\bar{z}_t - \underbrace{G(D^2\bar{z}, D\bar{z})}_{\substack{=0 \\ \text{by assumption}}} - H(D_x\bar{z})\dot{\zeta}^\varepsilon + H(-D_y\bar{z})\dot{\xi}^\eta = C_{\varepsilon\eta} - H(\lambda(x-y))(\dot{\zeta}^\varepsilon - \dot{\xi}^\eta)$$

If $H(\lambda(x-y))$ bdd & $\underbrace{\dot{\zeta}^\varepsilon - \dot{\xi}^\eta \to 0}_{???}$, then $\exists\ C_{\varepsilon\eta} \to 0$ st

$\bar{z}$ supersolution of $(*)$

3. Then

$$z(x,y,t) \leq \lambda|x-y|^2 + C_{\varepsilon,\eta} t + \underbrace{\sup_{x,y}\left[u_0^\varepsilon(x) - v_0^\eta(y) - \lambda|x-y|^2\right]}_{\to 0 \text{ as } \varepsilon,\eta \to 0 \text{ \& } \lambda \to +\infty}$$

hence

$$\overline{\lim_{\varepsilon,\eta\to 0}}\ (u^\varepsilon(x,t) - v^\eta(x,t)) \leqq 0 \quad \ldots$$

STOCHASTIC HAMILTON-JACOBI EQNS

small time smooth solutions

assume $H \in C^3(\mathbb{R}^n)$, $\phi \in C^3(\mathbb{R}^n)$

$$(SHJ)_t \begin{cases} dv = H(Dv) \circ dW & \text{in } \mathbb{R}^n \times (t, \infty) \\ v = \phi & \text{on } \mathbb{R}^n \times \{t\} \end{cases}$$

Claim: method of characteristics gives a small time a.s. smooth solution $S(s,t)\phi$ of $(SHJ)_t$

cl teristics $\begin{cases} dX = -D_pH(P)\,dW \\ dP = 0 \end{cases}$ $(s > t)$; $\begin{matrix} X_t = x \\ P_t = p \end{matrix}$

$$X(s; x, t = x - D_pH(p)(W(s) - W(t) \;,\; P(s) = p \qquad (s > t$$

a.s. $\exists\, T(\phi, \omega) > 0$ s.t. $X^{-1}(s; x, t)$ exists in $[t, t + T(\phi, \omega))$

$$S(s,t)\phi(x) = \phi(X^{-1}(s,x,t)) + (W(s) - W(t)) \left[H(D\phi(X^{-1}(s,x,t))) - D_pH(D\phi(X^{-1}(\cdots))) D\phi(X^{-1}(\cdots \right]$$

smooth solution of $(SHJ)_t$

Back to the (sketch of the) proof

- $S^{\varepsilon,\eta}(s,t)\phi(x,y)$ smooth solution in $[t,t+T^{\varepsilon,\eta}(\phi,\omega))$ of
$$\begin{cases} U_s = H(D_xU)\dot{\jmath}^{\varepsilon} - H(-D_yU)\dot{\xi}^{\eta} & \text{in } \mathbb{R}^n \times [t,t+T^{\varepsilon,\eta}(\phi,\omega)) \\ U(t,x,y) = \phi(x,y) \end{cases}$$

- $S(s,t)\phi(x,y)$ smooth soln in $[t,t+T(\phi,\omega))$ of
$$\begin{cases} dU = (H(D_xU) - H(-D_yU))\circ dW & \text{in } [t,t+T(\phi,\omega)) \\ U(t,x,y) = \phi(x,y) \end{cases}$$

- $S^{\varepsilon,\eta}(\cdot,t)\phi \xrightarrow[\varepsilon,\eta\to 0]{} S(\cdot,t)\phi$ uniformly & a.s. in $\mathbb{R}^n \times [t,t+T(\phi,\omega)$ (stochastic calculus)

- $\phi(x,y) = \lambda|x-y|^2$; $S(s,t)(\lambda|x-y|^2) = \lambda|x-y|^2$ (explicit calc.
$$S^{\varepsilon,\eta}(s,t)(\lambda|x-y|^2) \xrightarrow[\varepsilon,\eta\to 0]{} \lambda|x-y|^2$$

- $Z^{\varepsilon,\eta}(x,y,s,t) = S^{\varepsilon,\eta}(s,t)(\lambda|x-y|^2)$ smooth (local time) supersolution of $Z_t = G(D^2Z,DZ) + H(D_xZ)\dot{\jmath}^{\varepsilon} - H(-D_yZ)\dot{\xi}$

- comparison $\Longrightarrow$
$$\underbrace{u^{\varepsilon}(x,t+s) - v^{\eta}(y,t+s)}_{-\lambda|x-y|^2} \leq \underbrace{S^{\varepsilon,\eta}(s,t)(\lambda|x-y|^2)}_{-\lambda|x-y|^2} + \sup_{x,y}\left[u^{\varepsilon}(x,t) - v^{\eta}(y,t) - \lambda|x-y|^2\right]$$

- $$\lim_{\lambda\to 0} \sup_{\substack{(x,y)\in\mathbb{R}^{2N} \\ t\in[0,T]}} \lim_{\varepsilon,\eta\to 0} \left(u^{\varepsilon}(x,t) - v^{\eta}(y,t) - \lambda|x-y|^2\right) = 0$$

INTRINSIC CHARACTERIZATION - NOTION OF WEAK SLN

$$v_t = F(D^2 v, Dv) + H(Dv)\mathring{\zeta}$$

subsolution

ϕ smooth , at max of $v - \phi$

$$\phi_t \leq F(D^2\phi, D\phi) + H(D\phi)\dot{\zeta}$$

problem if $|\dot{\zeta}| = +\infty$

- equivalent definition

ϕ smooth
g smooth ; $S(t+h,t)\phi$ smooth local time soln of $\begin{cases} v_s = H(Dv) \\ v(\cdot,t) = \phi \end{cases}$

at a max (x_0, h_0) of $v(x,t+h)$ $\left(S(t+h,t)\phi(x) + g(h)\right)$

$$g'(h_0) \leq F\left(D^2 S(t+h_0)\phi(x_0), DS(t+h_0)\phi(x_0)\right)$$

- ϕ smooth $\quad - F \equiv 0$

$$\frac{d}{dt} \max_x (v(x,t) - \phi(x)) \leq H(D\phi(\bar{x}(t)))\dot{\zeta}(t)$$

why? $\quad \frac{d}{dt}\left[v(\bar{x}(t),t) - \phi(\bar{x}(t))\right] = v_t + \underset{=0}{(v_x - \phi_x} \; (\bar{x}(t), t)$

Definition: $u : \mathbb{R}^N \times [0,T] \times \Omega \to \mathbb{R}$ subsolution iff

(i) $u(\cdot,\cdot,\omega) \in BUC(\mathbb{R}^N \times [0,T]$ a.s

(ii) $(t,\omega) \longrightarrow u(\ ,t,\omega) \in BUC(\mathbb{R}^N$ is $\mathcal{F}_t$-measurable

&

(iii) for every ϕ smooth, every g smooth, every t if

$u(\cdot, t+\cdot, \omega)$ $S(t+\cdot,t)\phi(\cdot) - g(t+$ attains a max at (x_0, h_0) then

$S(t+h,t)\phi$ is the local time smooth soln of $dv = H(Dv)\circ d\zeta$, $v(t) = \phi$

$$g'(t+h_0) \leq F(D^2 S(t+h_0,t)\phi(x_0)\ DS(t+h_0,t)\phi(x_0)$$

equivalent to

$$\max_x [u(\cdot, t+h)\quad S(t+h,t)\phi(x)] \leq \max_x [u(\cdot,t\quad \phi(x)]$$
$$+ \int_t^{t+h} F(D^2 S(t+s,t)\phi(x(s)), DS(t+s,t)\phi(x(s)\ )\,ds,$$

$x(s)$ max of $u(\cdot, t+s$ $S(t+s,t)\phi($

$\square$

Facts about weak solns

1. consistency smooth ($C^{2,\alpha}$) weak $\Leftrightarrow$ classical

2. stability limits of solns are solns

3. existence

4. short time behavior u smooth, H smooth

$$u(\cdot,t) = u_0 + t\,F(D^2u_0, Du_0) + W(t)\,H(Du_0)$$
$$+ \tfrac{1}{2}(W(t))^2 \left(D^2u_0\, DH(Du_0), DH(Du_0)\right)$$
$$+ O\left(t^2 + t \max_{0\le s\le t} |W(s)| + |W(t)|^3\right)$$

5. UNIQUENESS

UNIQUENESS

Thm $\exists!$ weak solution

$$du = F(D^2u, Du)\,dt + H(Du)\circ dW$$
$F \in C(S^n \times \mathbb{R}^n)$ degenerate elliptic
H Lipschitz & difference of two convex fns

(MAIN STEP) PROOF 1. u, v two solutions

$$U(x,y,t) = u(x,t) - v(y,t) \quad \text{solves}$$

$$dU \leq G(D^2_{x,y}U, DU)\,dt + [H(D_xU) - H(-D_yU)]\circ dW$$

2. key ingredients : ϕ smooth, $|t-t_0| \ll 1$
$S(t,t_0)\phi$ smooth sol^n of
$d\psi = H(D\psi)\circ dW$

- define $\hat{u}(\xi,t) = \sup_x [u(x,t) - S(t,t_0)\phi(\xi - x)]$

then (i) $\hat{u}_t \leq C$; $D^2_\xi \hat{u} \geq -C$ in a nbgd of t_0

(ii) $\hat{u}_t \leq F(D^2\hat{u}, D\hat{u})$

- NO NEED TO DOUBLE TIME

REMARK: Any path, not only Brownian paths.

BACK TO INTERFACES

$$V = v_1(Du, n, x, t)\,dt + v_2(n, x, t)\,dW$$

level set pde

$$du = F(D^2u, Du, x, t)\,dt + H(Du, x, t)\circ dW$$

$\Gamma_t = \{u(\cdot, t) = 0\}$ well defined stable

distance fn

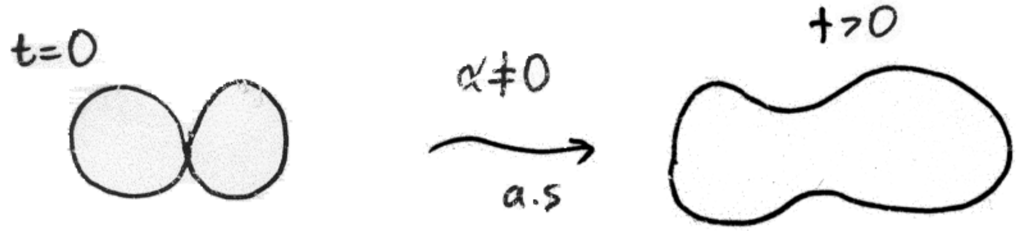
$\rho(x,t)$ - signed distance to front

$$d\rho \text{ `` '' } v_1(D^2\rho, D\rho, x-\rho D\rho, t)\,dt + v_2(D\rho, x-\rho D\rho, t)\,dW$$

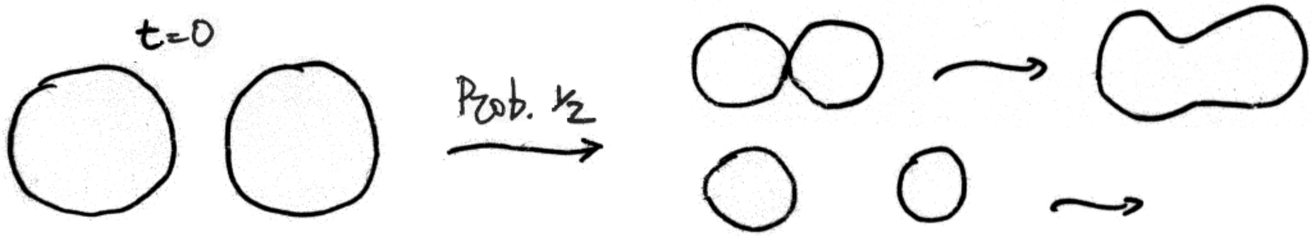
fattening

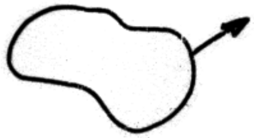
Yip time step min

ex $V = -mc + \alpha\, dW$



NO FATTENING
(Yip &. S)





understand evolution

$$V \quad v(Dn, n, x, t)\,dt + v_2(n, x, t)\,dW$$

moving interfaces (between two coexisting phases) arise as

SCALING LIMITS OF VARIOUS TYPES OF MODELS

describing the dynamics of phase boundaries generated in the limit

macroscopic modes Allen-Cahn-type equations

"Ginzburg-Landau" —"—

microscopic models spin or particle system

croscopic ⇒ macroscopic hydrodynamic space-time scaling

intermediate (mesoscopic) reaction-diffusion equations with additive noise

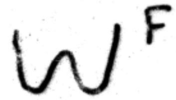
effect of randomness is not averaged out in the hydrodynamic limit

fluctuations yield in the limit an additive noise as a correction term

Spohn, Presutti et all Kawasaki, Ohta, ...

Example #1 (Funaki)

$$-\Delta u + f(u) + \varepsilon^{\gamma+\frac{1}{2}} a(\varepsilon^{\frac{1}{2}} x)\, \mathring{W} = 0$$

"cubic" f 

, $\ddot{q} = f(q)$ standing wave

$p(x,y,t)$ fund. soln of $\partial_t - \Delta + f'(q)$

goal: understand long time behavior & effect of noise

(i) $v(x,t) = u(\varepsilon^{-\frac{1}{2}}x, \varepsilon^{-1}t)$

$$\Delta v + \frac{1}{\varepsilon} f(v) + \varepsilon^{\gamma} a(x)\, \mathring{W} = 0 \qquad (a^{\frac{1}{2}}\mathring{W}(at) \cong \mathring{W}(t))$$

Thm 1. As $\varepsilon \to 0$, $v(x,t) \approx q\left(\varepsilon^{-\frac{1}{2}} d(x, \Gamma_t)\right)$

Γ_t moves by mean curvature

NO EFFECT OF NOISE

(ii) $z(x,t) = v(x, \varepsilon^{-2\gamma-\frac{1}{2}}t) = u(\varepsilon^{-\frac{1}{2}}x, \varepsilon^{-2\gamma-3/2}t)$ (LONGER TIME SCALES)

$$z_t - \frac{1}{\varepsilon^{2\gamma+\frac{1}{2}}}\left[\Delta z + \frac{1}{\varepsilon} f(z)\right] + \frac{1}{\varepsilon^{1/4}} a(x)\, \mathring{W} = 0$$

main contribution of noise: $O(\varepsilon^{1/4})$

width of interface: $O(\varepsilon^{1/2})$

"Thm" 2 $z(x,t) \approx q(\varepsilon^{-\frac{1}{2}}(x - \xi_t))$; $\dot{\xi}_t = \alpha_3 a^2(\xi_t)$

$$\alpha_3 = \lim_{t\to\infty} \frac{1}{t} \int_0^t ds \int f''(q(y))\, q'(y) \left[\int_0^s \int p(s-\lambda, y, z)\, dz\, dW_\lambda\right]^2 dy$$

(iii) what if α_3 0 (eg $f(\cdot)$ $f(-u)$)

$$\chi(x,t) \quad z \times \varepsilon^{-\frac{1}{2}} t = v \; x, \varepsilon^{-2\gamma-1} t \qquad u \; \varepsilon^{\frac{1}{2}} x \quad \varepsilon^{2\gamma-2} t$$

$$\chi_t \quad \varepsilon^{2\gamma+1} \left[\Delta \chi + \tfrac{1}{\varepsilon} f(\chi) \right] + \tfrac{1}{\varepsilon^{\frac{1}{2}}} a(x) \overset{\circ}{W} \quad 0$$

noise contr. $O(\varepsilon^{1/2})$ width of interface $O(\varepsilon^{\frac{1}{2}}$

*Thm 3 $\quad \chi(x,t \approx q(\varepsilon^{\frac{1}{2}}(x-\xi_t)$

$$d\xi_t \quad \alpha_1 a(\xi_t) dW + (\tfrac{1}{2}\alpha_1 + \alpha_2 \quad a(\xi_t) a'(\xi_t) dt$$

$$\alpha_1 = \|q'\|_{L^2}^{-2} (q(+\infty) - q(-\infty))$$

$$\tfrac{1}{2}\alpha_1 + \alpha_2 = -\|q'\|_{L^2}^{2} \int_0^\infty dt \int x \, p(t,x,z) p(t,y,z) f''(q(z)) q'(z) \, dx\,dy\,dz$$

$$\alpha_2 \neq 0$$

α_3 & α_2 are new terms due to noise

Thm 2, 3 $\quad$ Funaki (in law
Lions-S (pathwise

Example 2 Stochastic Allen Cahn (Lions & S)

$$du^\varepsilon \ \Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon) + \frac{1}{\varepsilon^{1/2}} a(x)\, dW^{(\varepsilon)} \ 0 \qquad (x,t) \in \mathbb{R}^N \times (0,\infty)$$

(dW^ε mild" smooth approximation to dW)
f cubic"

Thm As $\varepsilon \to 0$, interface moving with normal velocity

$$V = \text{mean curvature} + \alpha_1 a(x)\, dW + {}_2\alpha_1 a(x) Da(x) \cdot n$$

if $dW^\varepsilon \to \xi$ ξ "smooth" then

$$V = mc + \alpha_1 a(x) \xi(t$$

- $a(x) \equiv a$, N 2 , small time dW^ε Funaki
- $a(x) \equiv a$ $dW^\varepsilon = dW$ - formal result Kawasaki Ohta

WORK IN PROGRESS & OPEN PROBLEMS

- (x,t) - dependent case (Brownian paths)
- boundary conditions
- regularity of solutions [<u>simple</u> <u>problem</u> : are solus to $du = H(u_x) \circ dW + \lambda u_{xx} dt$ regular ?]
- regularizing effects / formulae / numerics
- asymptotic behavior as $t \to \infty$ & INVARIANT MEASURES
- front propagation
- turbulent reaction-diffusion, combustion
- competition of scales in motions in random environments
- $du = F(D^2u, Du, u, x, t)\,dt + G(u, x, t) \circ dW$

  $u(x,t) = \Psi(v(x,t), x, t)$
- stochastic conservation laws / kinetic formulation / regularizing effects

  $du = [\operatorname{div} F(u)] \circ dW$

  $du = f(u)_x dW + \frac{1}{2}[(f'(v))^2 v_x]_x dt$