Tutorial on Level Set Motion by Mean Curvature

L.C. Evans

I. Viscosity solutions of nonlinear PDE: "And now a brief message from our sponsor..."

Notation $Du = (\cdots, u_{x_i}, \cdots)$

$D^2u = ((\cdots u_{x_i x_j} \cdots))$

$$(\mathrm{P}) \qquad \boxed{F(D^2u, Du, u, x) = 0}$$

$$F : S^n \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

symmetric, $n \times n$ matrices

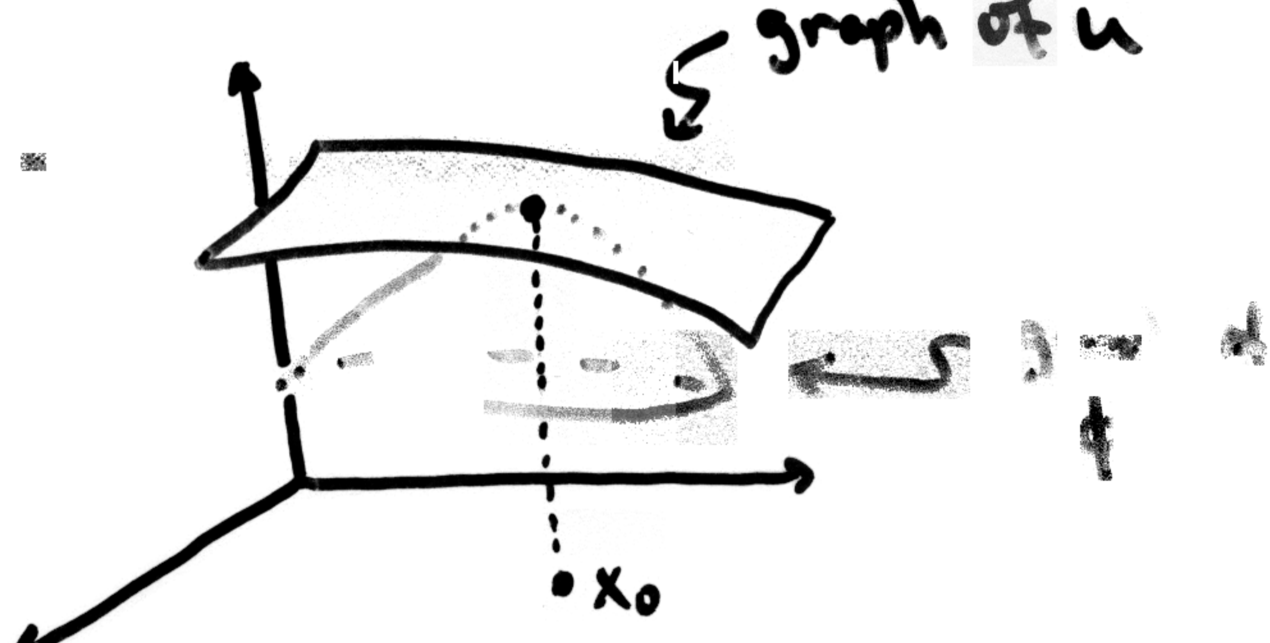
region where we solve the PDE

Def The PDE is elliptic if

$$\boxed{\underbrace{S \geq R}_{\text{partial ordering of sym. matrices}} \implies F(S,p,z,x) \underbrace{\leq}_{\text{ordering of real numbers}} F(R,p,z,x)}$$

- In general, $\nexists$ smooth solution of (P) (+ boundary conditions)
- But, temporarily, assume u is smooth

Let $\phi \in C^\infty$



At x_0, $Du = D\phi$, $D^2 u \geq D^2 \phi$

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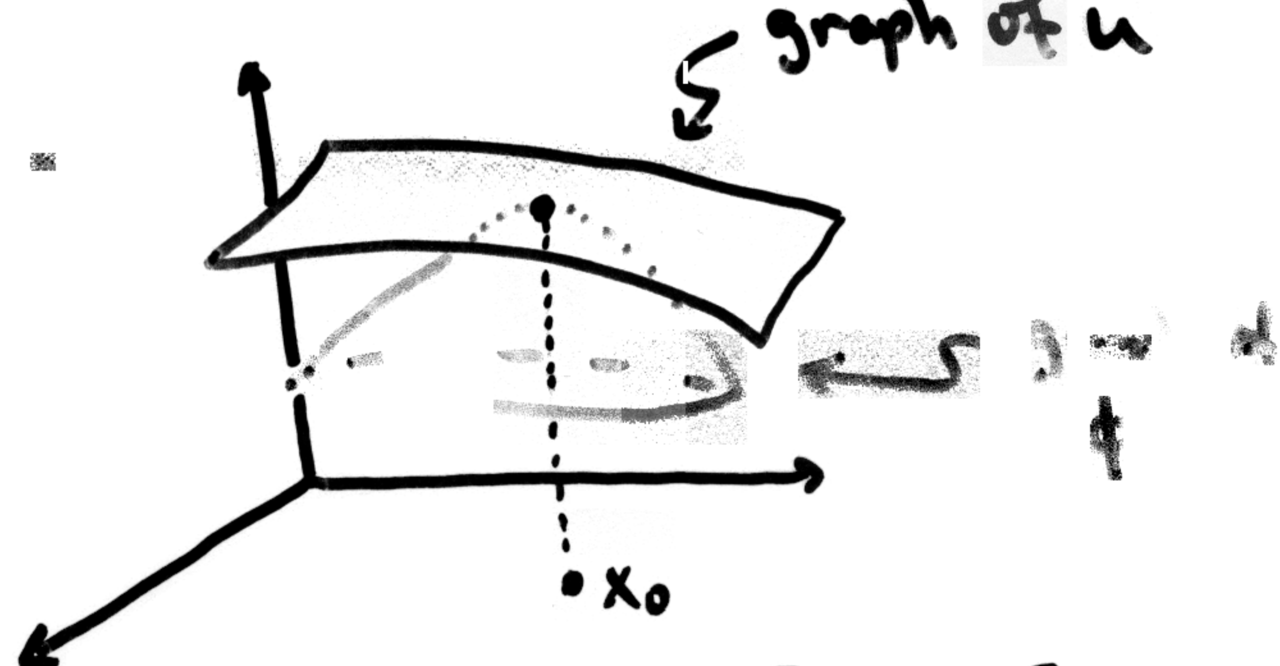


At x_0, $Du = D\phi$, $D^2 u \geq D^2 \phi$

So

$$0 = F(D^2u(x_0), Du(x_0), u(x_0), x_0)$$
$$\leq F(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0)$$

<u>Def</u> (i) We say $u \in C$ is a <u>viscosity supersol'n</u> if $\forall \phi \in C^\infty$ we have

• $$\boxed{0 \leq F(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0)}$$

at each point x_0 when the graph of ϕ touches the graph of u from <u>below</u>

(ii) u is a <u>viscosity</u> <u>subsolution</u> if $\forall \phi \in C^\infty$

•• $$\boxed{0 \geq F(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0)}$$

. . . . from <u>above</u>

(iii) u is a <u>viscosity</u> <u>solution</u> if it's both a super- and subsolution.

References Crandall-Lions, Trans AMS (1983)
Crandall-E-Lions, Trans AMS (1984)

Properties of Viscosity Solutions

- Consistency - A smooth sol'n is a viscosity sol'n. If a viscosity solution is differentiable at a point, it solves the PDE there

- Stability - If u_k $(k=1,2,\ldots)$ are viscosity sol'ns of (P) and $u_k \to u$ uniformly, then u is a sol'n

- Existence - Under reasonable assumptions, $\exists$ a viscosity sol'n of (P) (+ boundary conditions)

Existence:

Method 1 (Perron, Ishii)

Let $u := \sup\{v \mid v \text{ is a sub-solution}\}$

Method 2 (Other approximations)

Solve

smooth

(P_ε) $F_\varepsilon(D^2u^\varepsilon, Du^\varepsilon, x) = 0$

and show, e.g., $\sup_{0<\varepsilon\leq 1} \|Du^\varepsilon\|_{L^\infty} \leq C$

Let $u^\varepsilon \longrightarrow u$ uniformly.

Claim: $\boxed{u \text{ is solution of } (P)}$

Proof Let $\phi \in C^\infty$. Suppose $u - \phi$ has a strict min at x_0. Then $u^\varepsilon - \phi$ has a min at x^ε, with $x^\varepsilon \to x_0$

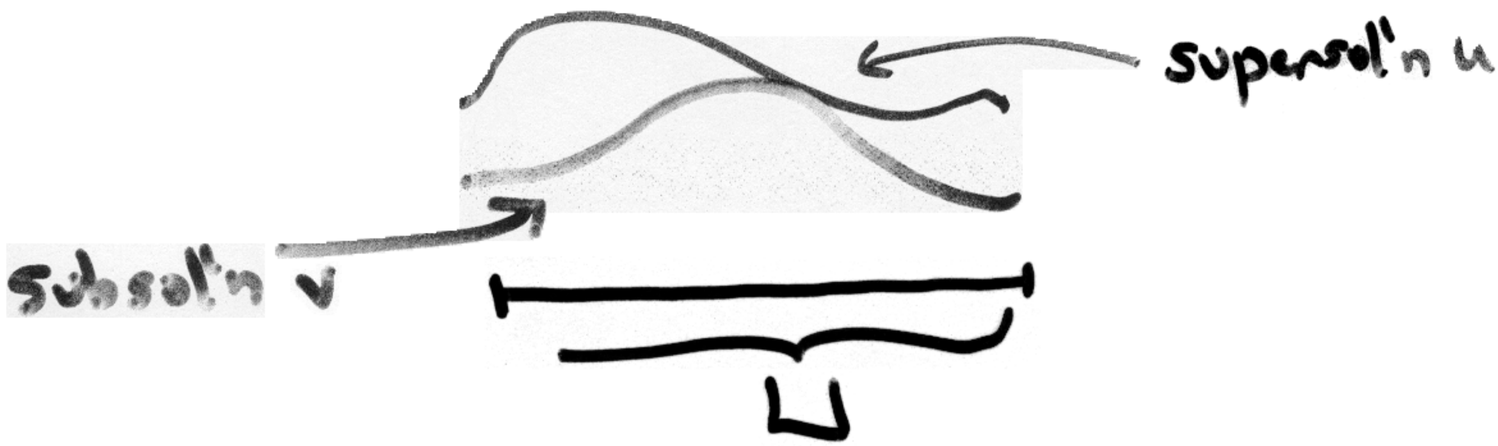
At x^ε, $Du^\varepsilon = D\phi$, $D^2u^\varepsilon \geq D^2\phi$

Thus

$$0 = F_\varepsilon(D^2u^\varepsilon, Du^\varepsilon, x^\varepsilon) \leq F_\varepsilon(D^2\phi, D\phi, x^\varepsilon).$$

Let $\varepsilon \to 0$ $\square$

- **Comparison** – If u is a supersol'n, v is a subsol'n, and $u \geq v$ on ∂U, then $u \geq v$ inside U



Reference Crandall article in Lecture Notes in Math #1660

Applications: We can apply these methods to highly nonlinear / degenerate PDE, provided smooth sol'ns have a comparison principle or maximum principle

Applications:

Dynamic programming for deterministic stochastic control, 2-person (zero-sum) differential games, H_∞ control

2. Large deviations theory - finding rate functions

Nonlinear homogenization

. Front dynamics in reaction-diffusion PDE (See Souganidis' lectures)

5. Geometric motion problems

Nonapplications:

Most systems of PDE (fluid mech, etc), nonlinear wave equations, conservation laws*, WKB for Schrödinger's equation*, dispersive equations, ...

II Motion of hypersurfaces by mean curvature

PDE: Heat equation

$$u_t = \Delta u$$

Geometry: Mean curvature motion in $\mathbb{R}^n$

$\Gamma(t)$ = surface at time t

Law of motion:

$$\boxed{V = H}$$

normal velocity

mean curvature $= \kappa_1 + \cdots + \kappa_{n-1}$

Analogy of heat flow and mean curvature flow: dissipation rates

<u>PDE</u> $u_t = \Delta u$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} \frac{|Du|^2}{2}\,dx\right) = \int_{\mathbb{R}^n} Du \cdot Du_t \,dx$$

$$= -\int_{\mathbb{R}^n} \Delta u \, u_t \,dx = -\int_{\mathbb{R}^n} (\Delta u)^2 \,dx \leq 0$$

<u>Geometry</u> $V = H$

$$\frac{d}{dt}\big(\underbrace{\mathcal{H}^{n-1}(\Gamma(t))}_{\text{area}}\big) = \int_{\Gamma(t)} \underline{v} \cdot \underline{H} \, d\mathcal{H}^{n-1}$$

$$= -\int_{\Gamma(t)} H^2 \, d\mathcal{H}^{n-1} \leq 0$$

<u>Notation</u> $\underline{\nu}$ = unit normal to surface

mean curv. vector $\longrightarrow$ $\underline{H} = -H\underline{\nu}$

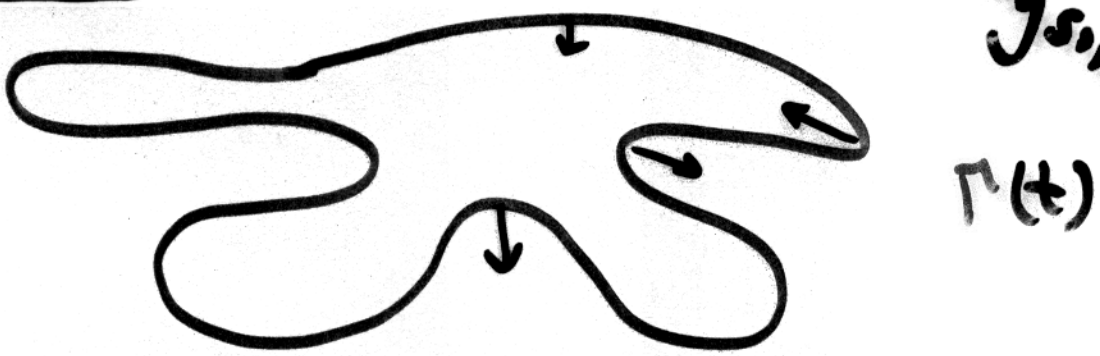
velocity vector $\longrightarrow$ $\underline{v} = V\underline{\nu}$

$\mathcal{H}^{n-1}$ = $(n-1)$ dimensional Hausdorff measure

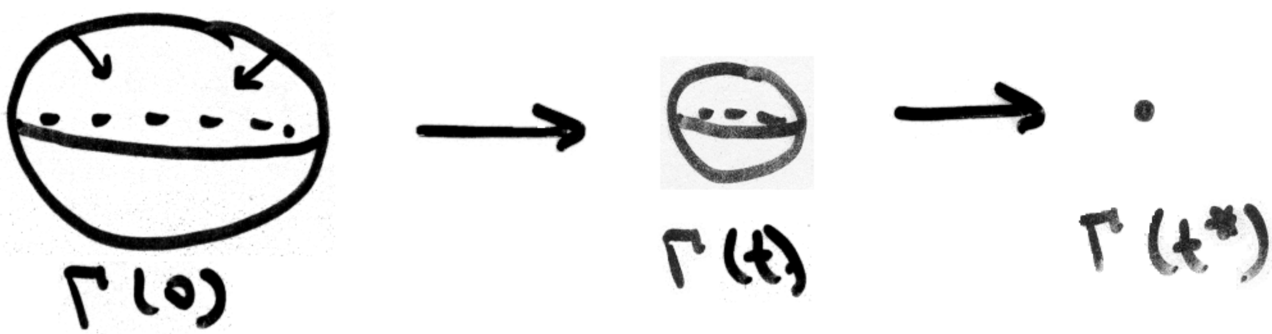
Examples:

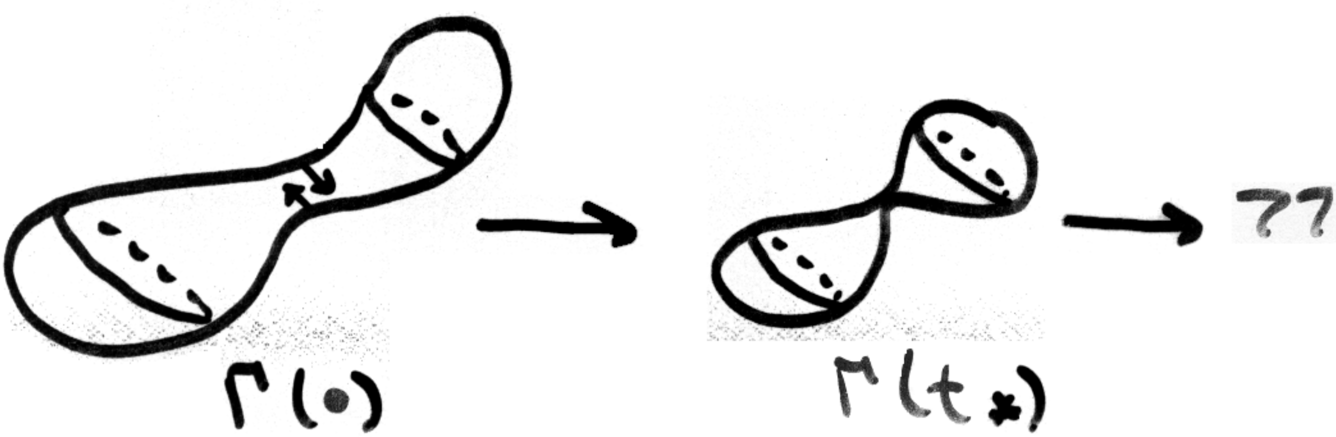
$\mathbb{R}^2$

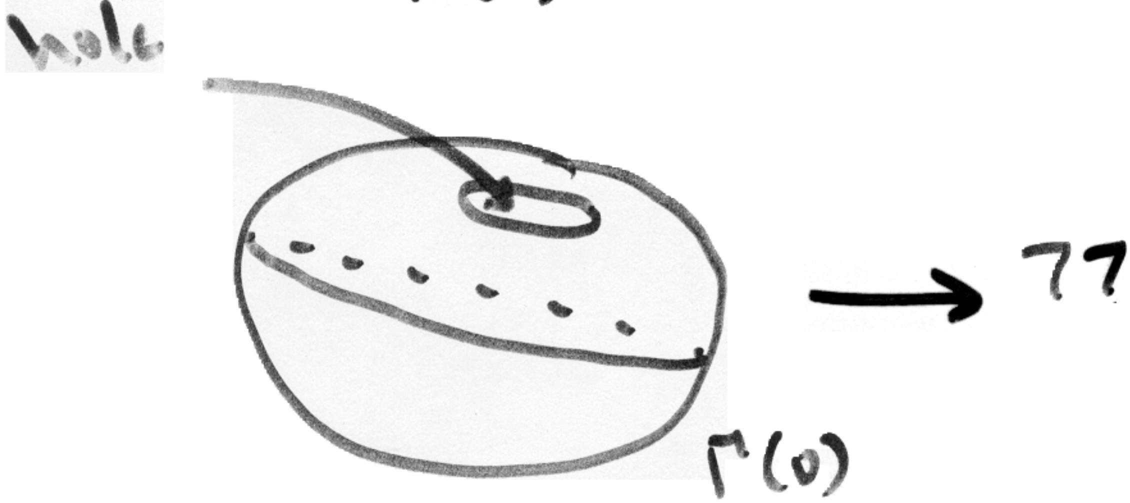
M. Grayson



$\mathbb{R}^3$

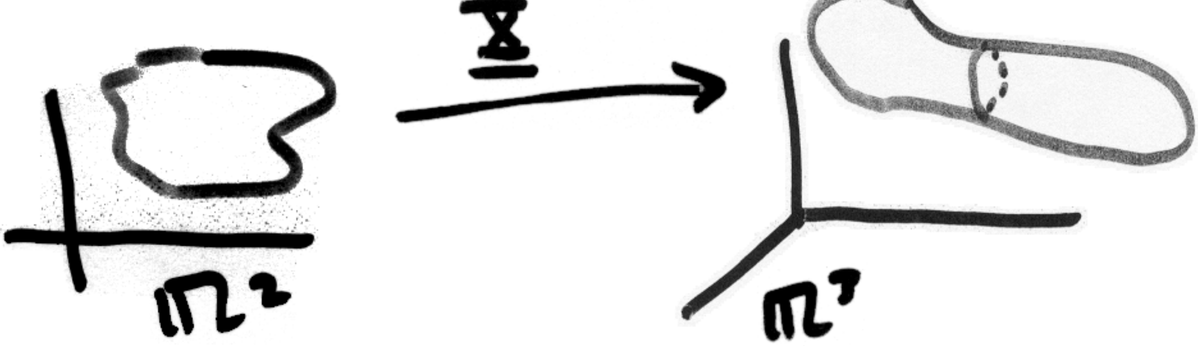






How to build the motion by mean curvature?

- Parametrize the surfaces $\Gamma(t)$



"Lagrangian coordinates"

The PDE system for $\underline{X} = (X^1, X^2, X^3)$ develops singularities in finite time

- Geometric measure theory – moving varifolds

__Ref__ K. Brakke, "The Motion of a surface by its Mean Curvature", Princeton U Press, 1978 – B. White

"$$\overline{D}\|V_t\|(\phi) \leq -\int |\underline{h}(V_t, x)|^2 \phi(x)\, d\|V_t\| x + \int S^{\perp}(D\phi(x)) \cdot \underline{h}(V_t, S)\, dV_t(x,S)$$"

★ Level set method

Write down a PDE for $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$, which says that

$$\boxed{\text{each level set of } u \text{ is evolving by mean curvature flow}}$$

Study this PDE to deduce properties of $\Gamma(0) \longmapsto \Gamma(t)$ for $t>0$

References → Osher–Sethian, J. Comp. Phys. (1988)
numerics

theory → Chen–Giga–Goto, J. Diff. Geom. (1991)
→ E–Spruck, J. Diff. Geom (1991)

• Distance function method

Study the PDE satisfied by the (signed) distance function to $\Gamma(t)$

Ref M Soner, J Diff Eq (1993)

• Barrier method

Define a generalized flow as one that does not cross smooth barriers

Ref De Giorgi, see articles by Ambrosio and by Bellettini - Novaga in "Calculus of Variations and PDE", Springer, 1999 - Ilmanen

III Level set method

Look for $u = u(x,t)$ $(x \in \mathbb{R}^n, t \geq 0)$

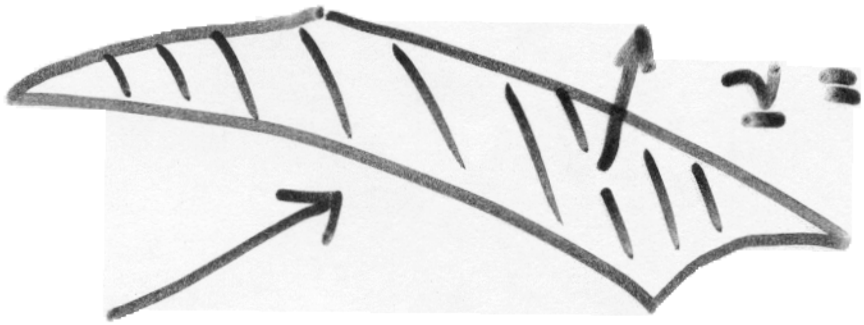
"order parameter" or "Eulerian marker"

so that

$$\Gamma(t) := \{x \in \mathbb{R}^n \mid u(x,t) = 0\}$$

evolves by mean curvature motion.

In fact, we want all level sets of u to evolve by m.c. motion



$\underline{\nu}$ = unit normal

a piece of $\Gamma(t) = \{u(\cdot,t) = 0\}$

<u>Geometry/calculus facts</u>:

$$\underline{\nu} = \frac{Du}{|Du|} \qquad Du = (u_{x_1}, \dots, u_{x_n}) = \text{gradient in } x$$

mean curvature $\rightarrow$ $H = \operatorname{div}(\underline{\nu}) = \operatorname{div}\left(\frac{Du}{|Du|}\right)$

normal velocity of level set $\rightarrow$ $V = \dfrac{u_t}{|Du|}$

Since $V = H$, we get the PDE

$$\text{(M)} \quad \boxed{\begin{aligned} u_t &= |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) \\ u_t &= \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} \end{aligned}}$$

Invariance under relabelling

Let $\boxed{v = \Phi(u)}$

Calculate

$$\left(\delta_{ij} \quad \frac{v_{x_i}v_{x_j}}{|Dv|^2}\right) v_{x_ix_j}$$

$$\left(\delta_{ij} - \frac{\cancel{(\Phi')^2}\, u_{x_i}u_{x_j}}{\cancel{(\Phi')^2}\,|Du|^2}\right)\left(\Phi' u_{x_ix_j} + \Phi'' u_{x_i}u_{x_j}\right)$$

$$= \Phi'\left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right)u_{x_ix_j}$$

$$\cancel{+ \Phi''\left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right)u_{x_i}u_{x_j}}$$

$$\Phi' u_t \qquad v_t$$

So $\underline{v = \Phi(u) \text{ also solves (M)}}$

Plan: Study (M) and use the solution u to define and investigate

$$\Gamma(t) \qquad \{u(\cdot,t) = 0\}$$

Interlude: Using the level set method to study other evolutions

Osher et al, Chen-Giga-Goto ← basic theory

Consider the PDE

$$u_t = F(D^2u, Du)$$

where $F: \mathbb{S}^n \times \mathbb{R}^n \to \mathbb{R}$. Let $v = \Phi(u)$

$$v_t \overset{?}{=} F(D^2v, Dv)$$

$$\Phi' u_t \overset{?}{=} F(\Phi' D^2u + \Phi'' Du \otimes Du, \Phi' Du)$$

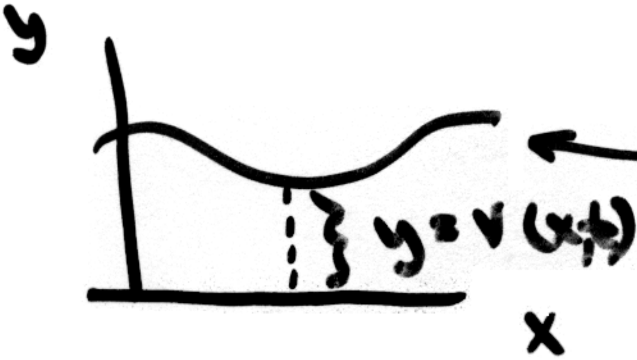
<u>Def</u> F is geometric if

$$\boxed{F(\lambda R + \mu\, p \otimes p, \lambda p) = \lambda F(R, p)}$$

If F is geometric, then u a solution

$\Longrightarrow v = \Phi'(u)$ is a solution

Example Level set method for the heat equation



$v_t = v_{xx}$

$u = C$

Suppose the graph of v is a level set of $u = u(x,y,t)$

$$u(x, v(x,t), t) = C$$

Differentiate

$$\begin{cases} u_y v_t + u_t = 0 \\ u_x + u_y v_x = 0 \\ u_{xx} + 2u_{xy}\underbrace{v_x}_{-\frac{u_x}{u_y}} + u_{yy}\overbrace{(v_x)^2}^{\frac{u_x^2}{u_y^2}} + \underbrace{u_y v_{xx}}_{= u_y v_t = -u_t} = 0 \end{cases}$$

Thus

$$\text{(H)} \quad \boxed{u_t = u_{xx} - 2u_{xy}\frac{u_x}{u_y} + u_{yy}\frac{u_x^2}{u_y^2}}$$

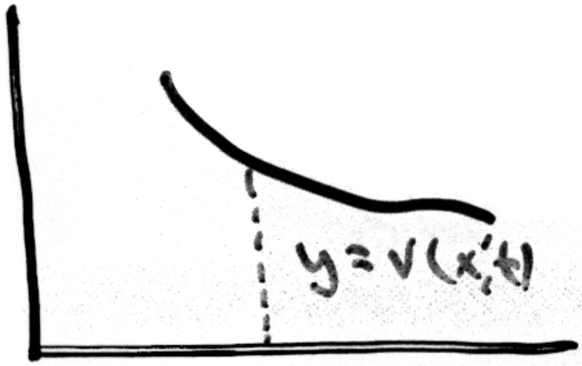
The PDE (H) says "each level set of u satisfies the heat equation"

Thus (H) is geometric, but it highly nonisotropic —

<u>Remark</u>: The heat equation is <u>not</u> invariant under a rotation of independent and dependent variables

$$v_t = v_{xx} \iff w_t = \underbrace{\frac{w_{yy}}{w_y^2}}_{\substack{\text{This PDE} \\ \text{integrable} \\ \text{(Rotate by } 90^\circ)}}$$

Flow of a graph by mean curvature:



This PDE is rotation invariant

$$v_t = (1 + |D'v|^2)^{1/2} \, \underbrace{\mathrm{div}' \left(\frac{D'v}{(1+|D'v|^2)^{1/2}} \right)}_{\substack{D'v = (v_{x_1}, \dots, v_{x_{n-1}}) \\ \mathrm{div}' = \left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_{n-1}}\right)}}$$

<u>Remark</u> Let $x = (x', x_n) \in \mathbb{R}^n$,

$$u(x,t) = -x_n + v(x', t)$$

Then

$$\text{(M)} \qquad u_t = |Du| \, \mathrm{div}\left(\frac{Du}{|Du|} \right)$$

Solving the initial value problem for (M)

$$\text{(IVP)}\quad \begin{cases} u_t - \left(\delta_{ij} - \dfrac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

$\uparrow$ given

Def (i) $u \in C$ is a <u>supersol'n</u> if $\forall \phi \in C^\infty$ if

$u - \phi$ has a max at (x_0, t_0),

then

(a) $\phi_t - \left(\delta_{ij} - \dfrac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} \geq 0$ at (x_0,t_0)

provided <u>$D\phi(x_0,t_0) \neq 0$</u>;

(b) $\phi_t - (\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j} \geq 0$

for some $|\eta| \leq 1$, provided <u>$D\phi(x_0,t_0) = 0$</u>

(ii) $u \in C$ is a <u>subsol'n</u>

if

$-\phi$ has a min at (x_0,t_0)

implies

(a) $$\phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} \leq 0$$ at (x_0,t_0)

if $D\phi \neq 0$;

(b) $$\phi_t - (\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j} \leq 0$$

for some $|\eta| \leq 1$, if $D\phi = 0$.

(iii) u is a <u>solution</u> if it's both a super- and subsolution.

<u>Remark</u> In the uniqueness proof, <u>case (b) comes up only when</u> <u>$D\phi(x_0,t_0) = 0$</u>, and so η really plays no role.

This is the main new idea for uniqueness

A. <u>Existence</u> – Let $\varepsilon > 0$.

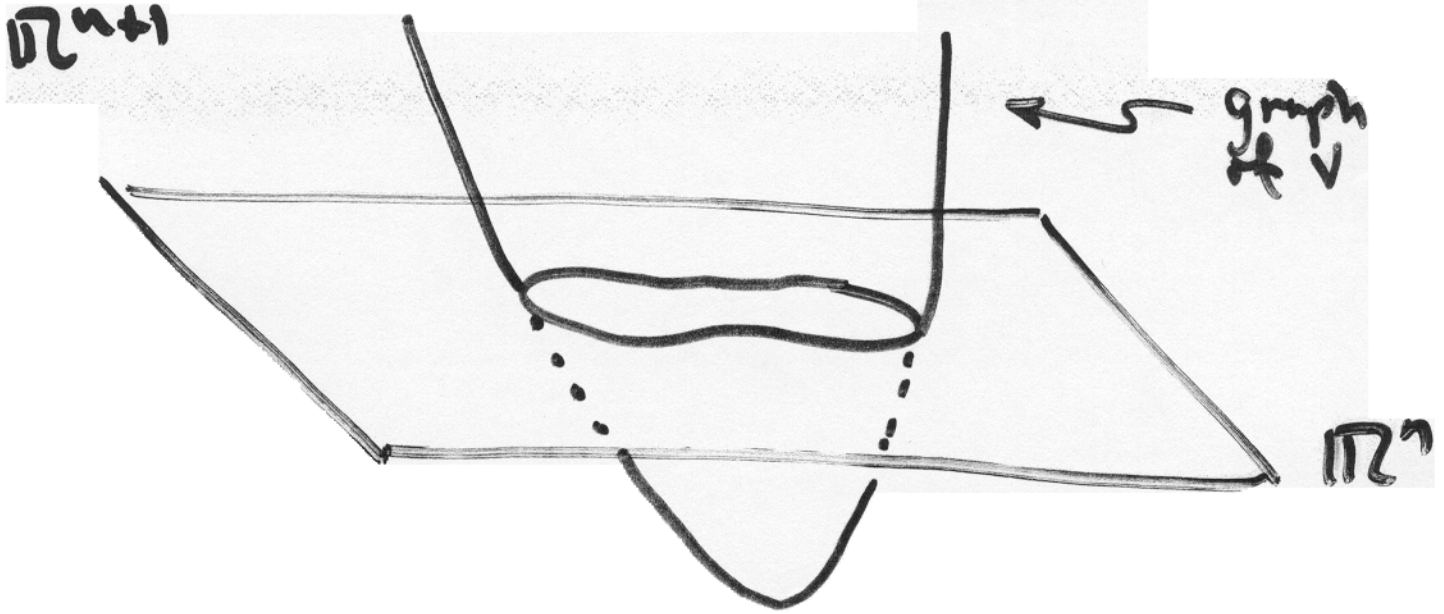
$$(\mathrm{IVP}_\varepsilon)\begin{cases} u^\varepsilon_t - \left(\delta_{ij} - \dfrac{u^\varepsilon_{x_i} u^\varepsilon_{x_j}}{(\varepsilon^2 + |Du^\varepsilon|^2)^{1/2}}\right) u^\varepsilon_{x_i x_j} \\ u^\varepsilon = g \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

$\boxed{\text{Geometric interpretation}}$ $\tilde{x} = (x, x_{n+1})$

Let $v(\tilde{x}, t) = -\varepsilon x_{n+1} + u(x,t)$. Then

$$v_t = \left(\delta_{ij} - \frac{v_{x_i} v_{x_j}}{|Dv|^2}\right) v_{x_i x_j} \quad \text{in } \mathbb{R}^{n+1} \times (0,\infty)$$

The level sets of v are graphs:



Estimates

$$\sup_{0<\varepsilon\leq 1} \| D_x u^\varepsilon, u^\varepsilon_t \|_{L^\infty} \leq C < \infty$$

Thus $u^{\varepsilon_j} \to u$ locally uniformly, and $\underbrace{u \text{ is a (viscosity) sol'n of (IVP)}}_{\text{similar to earlier proof}}$

B. Uniqueness, Comparison

Theorem (i) $\exists$ at most one solution of (IVP)

(ii) If u is a sol'n corresponding to initial data g, and if $\hat{u}$ corresponds to $\hat{g}$, then

$$\boxed{g \geq \hat{g} \text{ in } \mathbb{R}^n \implies u \geq \hat{u} \text{ in } \mathbb{R}^n \times [0,\infty)}$$

Comparison principle

Idea of proof: Look at points (x_0, y_0, t_0, s_0) where

$$\Phi(x,y,t,s) = \hat{u}(x,t) - u(y,s) - \frac{1}{\varepsilon^2}\left(|x-y|^4 + (t-s)^2\right)$$

has a max.

We have a "4", and not a "2", here to avoid problems with case (b) in the definition of super- and subsolutions.

We also use Jensen's regularizations:

Let $u \in C$ be a (viscosity) sol'n of $F(D^2u, Du) = 0$.

Set

$$u^\delta(x) = \sup_y \left\{ u(y) - \frac{|x-y|^2}{\delta} \right\}$$

Then $u^\delta \geq u$ and u^δ is a sub sol'n.

u^δ is smoother than u

C Invariance under relabelling of level sets

Theorem Let u solve (IVP) and suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous.

Then $$\boxed{v = \Phi(u)}$$
solves (IVP), with initial data $\underbrace{\Phi(g)}_{h =}$

$$\boxed{\text{How to build the generalized flow } \{\Gamma(t)\}_{t \geq 0}}$$

Step 1: Given the compact set $\Gamma(0)$, take any $g \in C(\mathbb{R}^n)$ such that
$$\Gamma(0) = \{x \mid g(x) = 0\}$$

Step 2: Solve (IVP) with initial data g

Step 3: Define for $t > 0$:
$$\Gamma(t) := \{x \mid u(x,t) = 0\}$$

Is this well-defined?

Theorem Suppose

$$\Gamma(0) = \{g = 0\} = \{\hat{g} = 0\}.$$

Then

$$\Gamma(t) = \{u(\cdot,t) = 0\} = \{\hat{u}(\cdot,t) = 0\}$$

<u>Proof</u> Design a continuous function Φ such that $\Phi(0) = 0$ and

$$h = \Phi(\hat{g}) \geq |g|$$

Let v solve (IVP) with initial data h. Then

$$\underbrace{v = \Phi(\hat{u})}_{\text{invariance under relabelling}} \underbrace{\geq |u|}_{\text{Comparison}}$$

$$\therefore \quad x \in \hat{\Gamma}(t) \implies \Phi(\hat{u}(x,t)) = 0 \geq |u(x,t)|$$

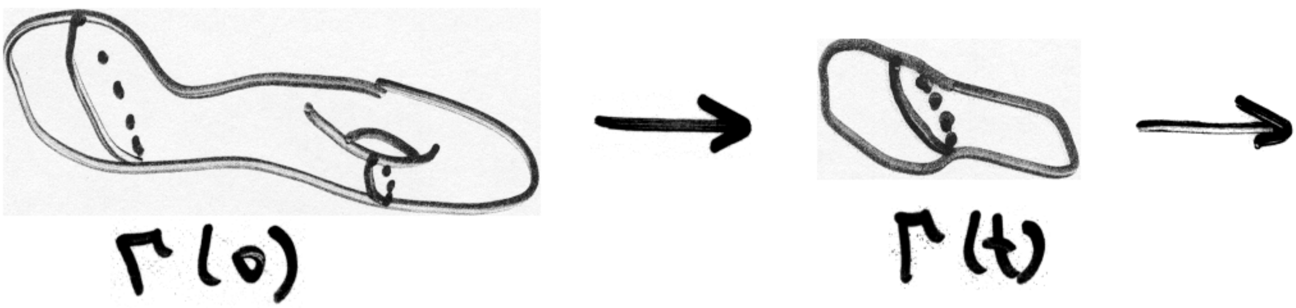
$$\implies x \in \Gamma(t)$$

Thus

$$\hat{\Gamma}(t) \subseteq \Gamma(t)$$

$\square$

IV Properties of Mean Curvature flow $\{\Gamma(t)\}_{t\geq 0}$



Consistency with classical flow:

Theorem Let $\{\tilde{\Gamma}(t)\}_{0\leq t<t_*}$ be a smooth flow starting from $\tilde{\Gamma}(0) = \Gamma(0)$. Then

$$\underbrace{\Gamma(t)}_{\text{generalized flow}} = \underbrace{\tilde{\Gamma}(t)}_{\text{smooth flow}} \quad \text{for } 0 \leq t < t_*$$

t_*: time of first singularity

Idea of proof – Let

$d(x,t)$ = (signed) distance of x to $\tilde{\Gamma}(t)$

Write

$$\underline{d} := \alpha e^{-\lambda t} d$$

appropriate constants

Then $$\underline{d}_t - \left(\delta_{ij} - \frac{\underline{d}_{x_i} \underline{d}_{x_j}}{|D\underline{d}|^2}\right) \underline{d}_{x_i x_j} \le 0$$

near $\hat{\Gamma}(t)$

Compare $\underline{d}$ and u, etc. $\square$

$$\boxed{\text{\underline{Theorem} If } \Gamma(0) \subseteq \hat{\Gamma}(0), \text{ then } \Gamma(t) \subseteq \hat{\Gamma}(t) \quad \forall t \ge 0}$$

Comparison principles for generalized flow by mean curvature

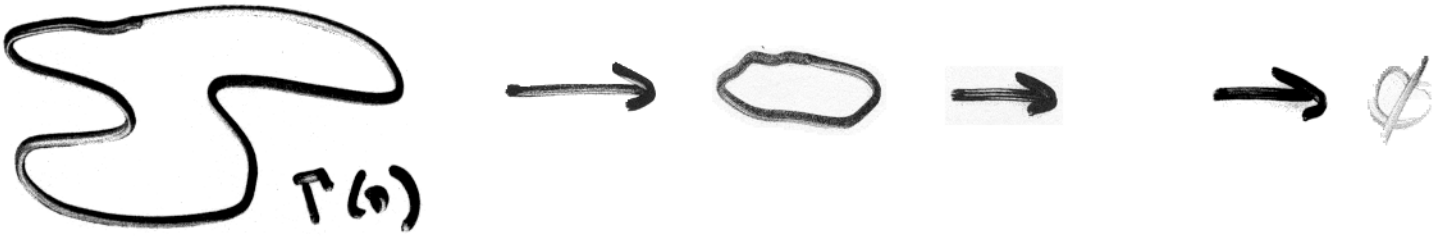
Theorem

$$\operatorname{dist}(\Gamma(0), \hat{\Gamma}(0)) \le \operatorname{dist}(\Gamma(t), \hat{\Gamma}(t))$$

$\forall t \ge 0$

Examples ($n = 2$)

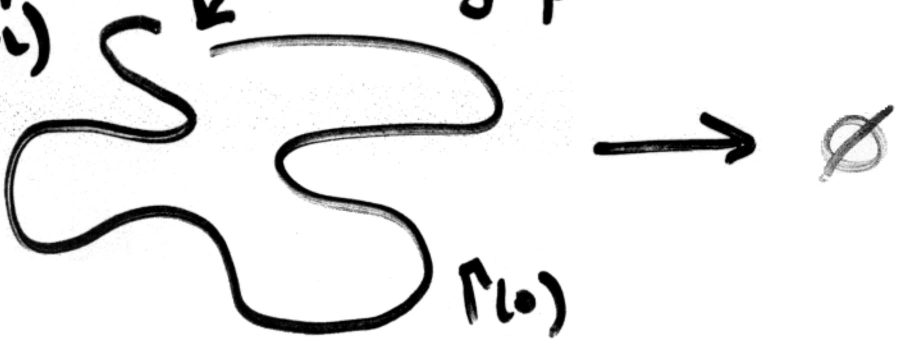
(i)



$\Gamma(0)$

(ii)

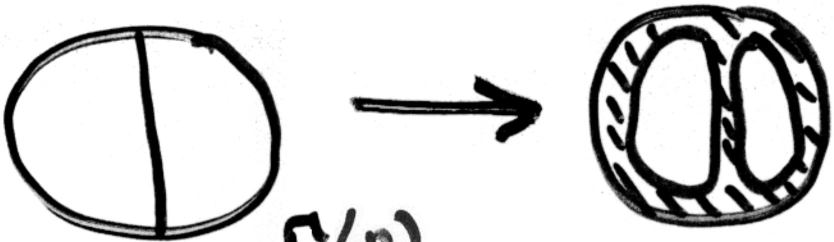
gap



$\Gamma(0)$

$\Gamma(t)$ vanishes for $t > 0$

(iii)

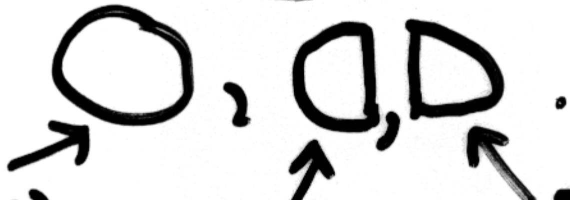


$\Gamma(0)$

$\Gamma(t)$ develops interior "fattening"

Proof of (iii) The set  $\leftarrow \Gamma(0)$

contains the subsets

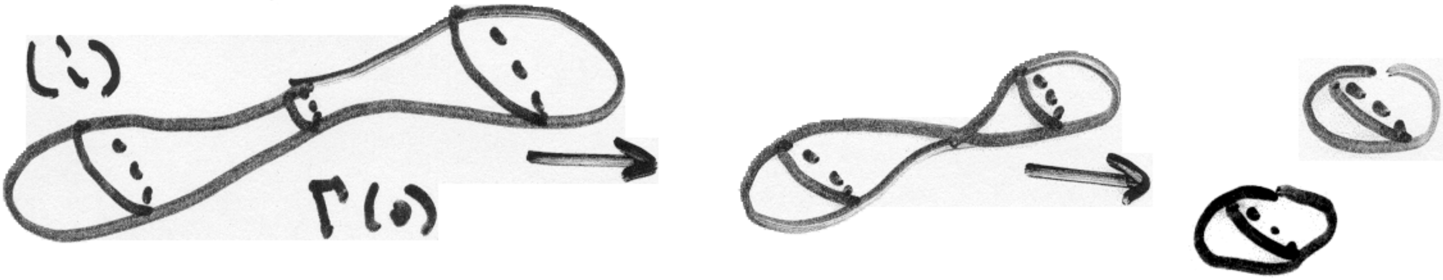


$\Gamma_1(0)$ $\Gamma_2(0)$ $\Gamma_3(0)$

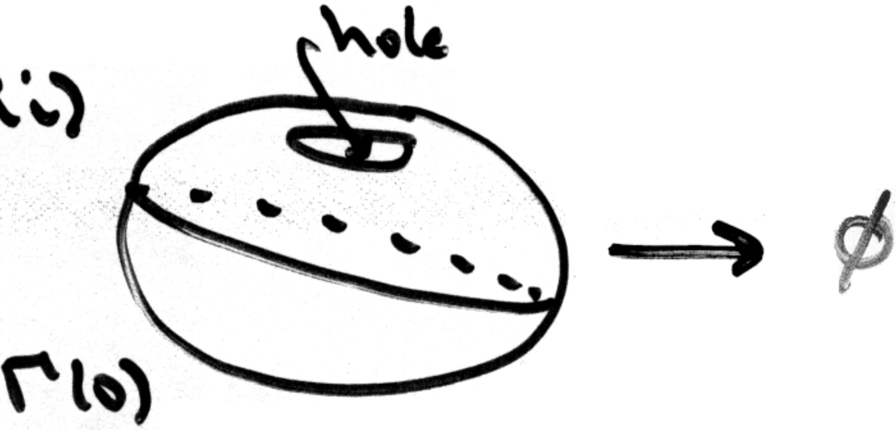
Thus $\Gamma(t)$ contains $\Gamma_i(t)$ for $i = 1, 2, 3$

Examples $(n=3)$

(i)



(ii)

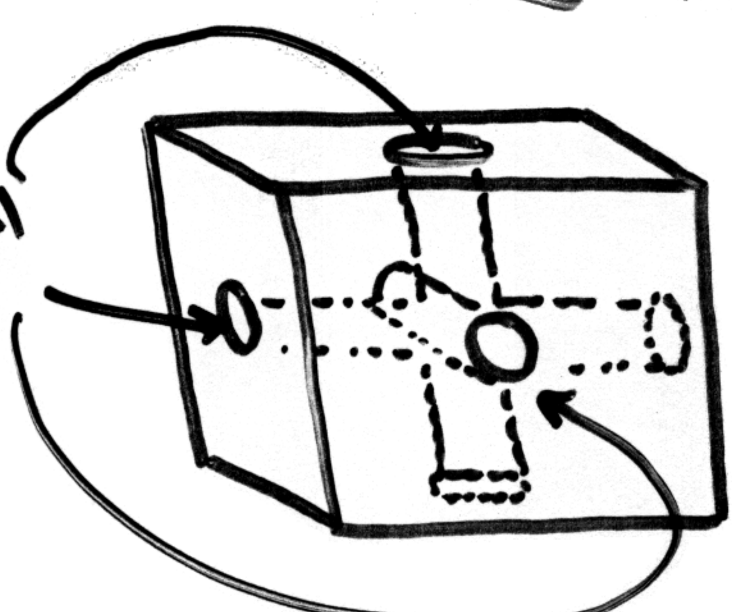


$\Gamma(t)$ vanishes for $t>0$

Self-similar flows

Sphere, torus (Angenent), higher genus (Chopp) ← numerics only

Cylindrical holes thru each face



If the holes are the right size, this evolves to a self-similar shape (Chopp)

Numerical challenges

Compute past changes of topology

Detect instantaneous vanishing

Detect fattening

(Ref Chopp, Exper. Math (1994)
Angenent, Ilmanen, Velazquez)

- Compute unstable motions

Also, K. Brakke's Surface Evolver

(available free on the web)

Better computations would be useful to help further development of the rigorous theory —

Interlude: Connections with calculus of variations (?)

Informal principle: The flow is (fairly) well-behaved, since $\mathbb{R}^n$ "filled" with the various level sets each moving by mean curvature.

Analogy (?) in calculus of variations

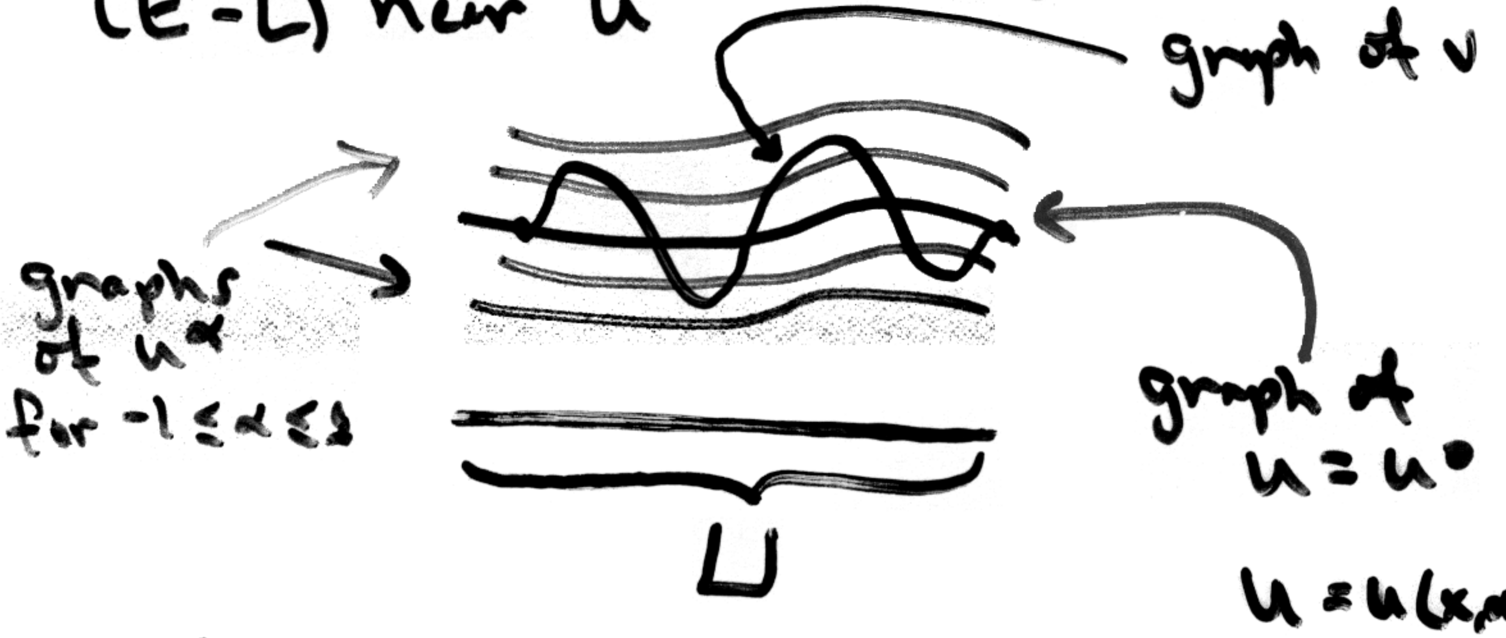
$$I[v] = \int_U F(Dv, v)\, dx$$

$F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $F = F(p, z)$

Assume $p \mapsto F(p,z)$ is convex

$$\text{(E-L)} \quad -\operatorname{div}(D_pF(Du,u)) + F_z(Du,u) = 0$$

- Any <u>minimizer</u> of $I[\cdot]$ (subject to given boundary conditions) solves the <u>Euler-Lagrange equations</u> (E-L)
- But, a solution of (E-L) need not be a minimizer
- Suppose, however, $\exists$ many sol'ns of (E-L) near u



$u = u(x, \alpha)$

Let

$$v(x) = u(x, \theta(x))$$

where $u^\alpha(x) = u(x,\alpha)$, $\theta : U \to (-1,1)$, $\theta = 0$ on ∂U.

Theorem $I[u^0] \le I[v]$

Hence, a sol'n of (E-L) is a local minimizer, & $\exists$ many other nearby sol'ns

__Proof__ (Ball-Murat) $Dv = Du + u_\alpha D\theta$

$$F(Dv, v, x) = F(Du + u_\alpha D\theta, v, x)$$

$$\ge F(Du, v, x) + D_p F(Du, v, x) D\theta\, u_\alpha$$

Convexity

$\therefore I[v] - I[u^0]$

$$(*) \quad \ge \int_U F(Du\, v\, x) + D_p F \cdot D\theta\, u_\alpha - F(Du^0, u^0, x)\, dx$$

Compute also

$$\frac{\partial}{\partial x_i}\left(\int_0^{\theta(x)} F_{p_i}(Du, u, x)\, u_\alpha\, d\alpha\right)$$

$$- F_{p_i}(Du(x, \theta(x)), \dots)\, u_\alpha(x, \theta(x))\, \theta_{x_i}$$

$$= \int_0^{\theta(x)} \frac{\partial}{\partial x_i}\left(F_{p_i}\right) u_\alpha + F_{p_i} u_{\alpha, x_i} \, d\alpha$$

by (E-L)

$$= \int_0^{\theta(x)} F_z u_\alpha + F_{p_i} u_{\alpha, x_i} \, d\alpha$$

$$= \int_0^{\theta(x)} \frac{d}{d\alpha} F(Du, u, x) \, d\alpha$$

$$= F(Du(x, \theta(x)), v, x) - F(Du^0, u^0, x)$$

Plug this calculation into (*)

$$I[v] - I[u^0] \geq \int_U \frac{\partial}{\partial x_i}\left(\int_0^{\theta(x)} \cdots \, d\alpha\right)$$

$$= 0, \text{ since } \theta = 0 \text{ on } \partial U$$

$\square$

<u>Question</u> Is there a "dynamic" analogue of this, relevant to level set motion?

<u>Ref</u> E, SIAM J Appl. Math (1996)

V Further properties

How can we steal calculations from diff. geometry and apply these to the level set flow?

Rate of change of area:

- Recall that if $\{\Gamma(t)\}$ is a smooth flow,

$$\frac{d}{dt}\left(\mathcal{H}^{n-1}(\Gamma(t))\right) = -\int_{\Gamma(t)} H^2 \, d\mathcal{H}^{n-1}$$

- Suppose now u is a smooth sol'n of

$$u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right) u_{x_ix_j}$$

Compute:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} |Du| \, dx\right) = \int_{\mathbb{R}^n} \frac{Du}{|Du|} \cdot Du_t \, dx$$

$$= -\int \operatorname{div}\left(\frac{Du}{|Du|}\right) u_t \, dx$$

$$= - \int \underbrace{\left(\operatorname{div}\left(\frac{Du}{|Du|}\right)\right)^2}_{= H^2} |Du| \, dx$$

Integrate

$$(*) \quad \int_{\mathbb{R}^n} |Du(T)| \, dx + \int_0^T \int_{\mathbb{R}^n} H^2 |Du| \, dx\, dt$$

$$\int_{\mathbb{R}^n} |Dg| \, dx$$

How can we extract information about a given level set?

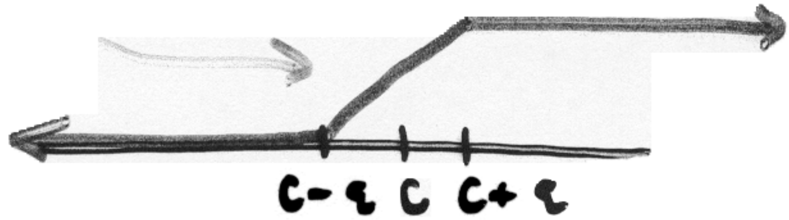
<u>Coarea formula</u> Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous, $f \in C_c$. Then

$$\boxed{\int_{\mathbb{R}^n} f |Du| \, dx = \int_{-\infty}^{\infty} \left(\int_{\{u = \lambda\}} f \, d\mathcal{H}^{n-1} \right) d\lambda}$$

This is a "curved" form of Fubini's Thm

Apply (*) with $v = \Phi(u)$ replacing u, where Φ is this

$\Phi' = \frac{1}{2\varepsilon}$



$$\int_{\mathbb{R}^n} \Phi' \, |Du| \, dx + \int_0^T \int_{\mathbb{R}^n} H^2 |Du| \, \Phi' \, dx \, dt$$

$$= \int_{\mathbb{R}^n} \Phi' \, |Dg| \, dx$$

Use Coarea formula

$$\frac{1}{2\varepsilon} \int_{c-\varepsilon}^{c+\varepsilon} \mathcal{H}^{n-1}(\{u = \lambda\}) \, d\lambda$$

$$+ \frac{1}{2\varepsilon} \int_0^T \int_{c-\varepsilon}^{c+\varepsilon} \left(\int_{\{u=\lambda\}} H^2 \, d\mathcal{H}^{n-1} \right) d\lambda$$

$$\frac{1}{2\varepsilon} \int_{c-\varepsilon}^{c+\varepsilon} \mathcal{H}^{n-1}(\{g = \lambda\}) \, d\lambda$$

Let $\varepsilon \to 0$

$$\mathcal{H}^{n-1}(\{u(\cdot,T) = c\}) + \int_0^T \int_{\{u=c\}} H^2 \, d\mathcal{H}^{n-1} \, dt$$

(**)

$$= \mathcal{H}^{n-1}\{g = c\}$$

A rigorous calculation –

(* *) implies

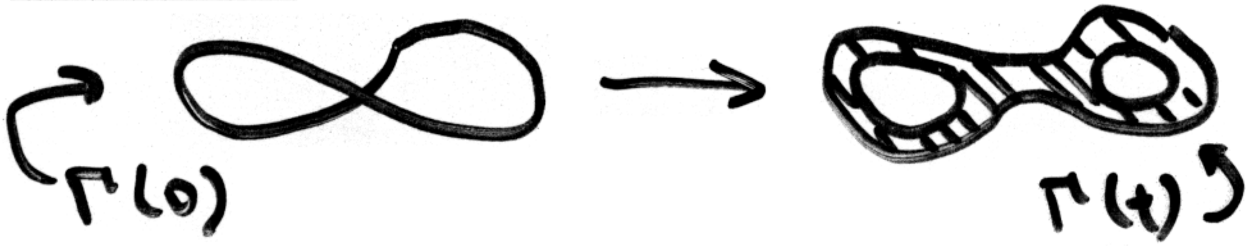
$$(***)\qquad \boxed{\mathcal{H}^{n-1}(\Gamma^c(t)) \le \mathcal{H}^{n-1}(\Gamma^c(0))}$$

for the level set

$$\Gamma^c(t) := \{u(\cdot,t) = c\}$$

But this formal deduction can be false:

Example $(n=2)$



Theorem For a.e. c, t, the inequality $(***)$ holds

Proof Recall the approximations

$$u^\varepsilon_t = (\varepsilon^2 + |Du^\varepsilon|^2)^{1/2} \operatorname{div}\left(\frac{Du^\varepsilon}{(\varepsilon^2+|Du^\varepsilon|^2)^{1/2}}\right),$$

the solution of which is smooth

Compute

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} (|Du^\varepsilon|^2+\varepsilon^2)^{1/2}-\varepsilon \; dx\right)$$

$$= \int \frac{Du^\varepsilon \cdot Du^\varepsilon_t}{(|Du^\varepsilon|^2+\varepsilon^2)^{1/2}}\, dx$$

$$= -\int \operatorname{div}\left(\frac{Du^\varepsilon}{(\quad)^{1/2}}\right) u^\varepsilon_t \; dx$$

$$= -\int \left(\operatorname{div}(\quad)\right)^2 (\varepsilon^2+|Du^\varepsilon|^2)^{1/2} dx \le 0.$$

Integrate, let $\varepsilon \to 0$, recall

$$\int |Du|\, dx \le \liminf_{\varepsilon\to 0} \int |Du^\varepsilon|\, dx :$$

$$\int_{\mathbb{R}^n} |Du(\cdot,t)|\, dx \le \int_{\mathbb{R}} |Dg|\, dx$$

Replace u by $\Phi(u)$ (as above), use Coarea formula

$$\frac{1}{2\varepsilon}\int_{c-\varepsilon}^{c+\varepsilon} \mathcal{H}^{n-1}(\Gamma^\lambda(t))\, d\lambda$$

$$\frac{1}{2\varepsilon}\int_{c-\varepsilon}^{c+\varepsilon} \mathcal{H}^{n-1}(\Gamma^\lambda(0))\, d$$

Now $\lambda \mapsto \mathcal{H}^{n-1}(\Gamma^\lambda(t))$ is summable, since $\int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\Gamma^\lambda(t))\,d\lambda = \int_{\mathbb{R}^n} |Du| < \infty$

Thus for almost every level c, we can let $\varepsilon \to 0$ in previous inequality □

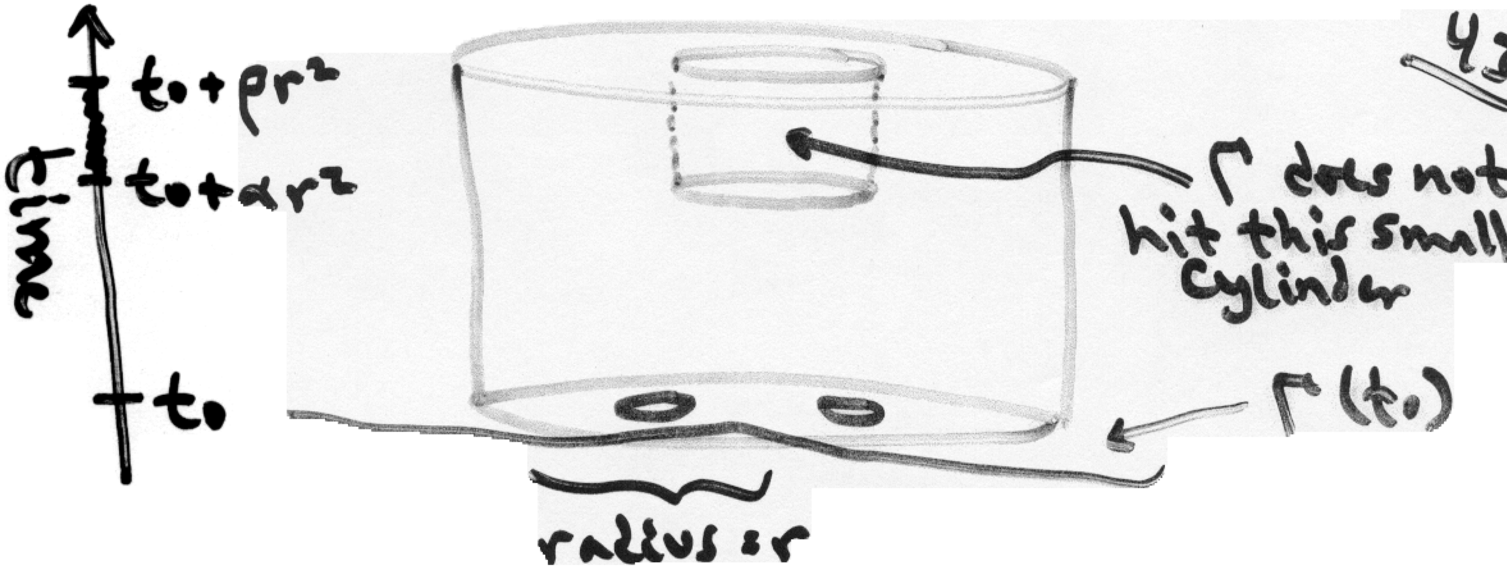
Remark This result suggests that for a.e. c, the flow $\{\Gamma^c(t)\}_{t \geq 0}$ is well behaved.

This is correct: see
E – Spruck, J Geom Analysis (1992)
— " —, J Geom Analysis (1995)

Clearing Out Lemma

(K. Brakke)

Idea: Show that if there is not much of $\Gamma(t_0)$ within a ball $B(x_0, r)$, then $\Gamma(t) \cap B(x_0, \frac{r}{2})$ for some later times



Theorem $\exists$ $\alpha, \beta, \eta > 0$ such that if

$$\mathcal{H}^{n-1}(\Gamma(t_0) \cap B(x_0, r)) \leq \eta r^{n-1},$$

then $\Gamma(t) \cap B(x_0, r/2) = \phi$ for

$$\alpha r^2 \leq t - t_0 \leq \beta r^2$$

Proof Assume $x_0 = t_0 = 0$, $r = 1$,

and so

$$\mathcal{H}^{n-1}(\Gamma(0) \cap B(0, 1)) \leq \eta$$

Assume $\{\Gamma(t)\}_{t \geq 0}$ is a smooth flow by mean curvature

1 Let $\Phi(t) \int_{\Gamma(t)} f(x,t)\, d\mathcal{H}^{n-}$

$$\Phi'(t) = \int_{\Gamma(t)} f_t - (S_{ij} - \nu_i \nu_j) f_{x_i x_j} - f H^2 \, d\mathcal{H}^{n-1}$$

standard calculation in diff. geometry (Hamilton, Huisken)

Let $f(x,t) = h(1 - |x|^2 - \mu t)$,

$$h(z) = \begin{cases} z^3 & z \geq 0 \\ 0 & z \leq 0 \end{cases}$$

Then

(1) $\Phi' \leq \int_{\Gamma(t)} (H^2 h + h)\, d\mathcal{H}^{n-1} < 0$

2 Also

$$\Phi \leq \left(\int_{\Gamma(t)} f^{\frac{n-1}{n-2}} \, d\mathcal{H}^{n-1} \right)^{\frac{n-2}{n+1}} \left(\int_{\Gamma(t)} f^{2/3} \, d\mathcal{H}^{n-1} \right)^{3/(n+1)}$$

$$\le C\left(\int_{\Gamma(t)} |Df| + |fH| \, d\mathcal{H}^{n-1}\right)^{\frac{n-1}{n+1}} \left(\int_{\Gamma(t)} f^{2/3} \, d\mathcal{H}^{n-1}\right)^{\frac{2}{n+1}}$$

Sobolev type
inequality on the surface $\Gamma(t)$

Now h' $3h^{2/3}$ and $f = h(|x|^2 - \mu t)$

$\therefore$

$$\Phi \le C\left(\int_{\Gamma(t)} h + H^2 h \, d\mathcal{H}^{n-1}\right)^{\frac{n-1}{n+1}} \left(\int_{\Gamma(t)} h \, d\mathcal{H}^{n-1}\right)^{\frac{2}{n+1}}$$

(2)

$$\le C\left(\Phi'\right)^{\frac{n+2}{n+1}}$$

3 Rewrite formula (2).

$$\Phi' \le -C\,\Phi^{\frac{n+1}{n+2}}$$

Since this power $s < 1$, the solution of the differential inequality goes to zero in finite time

4. Apply above formal proof to the approximations u^ε, which correspond to a smooth mean curvature flow in $\mathbb{R}^{n+1} \times (0,\infty)$

$\square$.

Application: Estimate of extinction time t^*

$$\boxed{t^* \leq C\, \mathcal{H}^{n-1}(\Gamma(0))^{\frac{2}{(n-1)}}}$$

Proof Pick $r > 0$ so large that

$$\mathcal{H}^{n-1}(\Gamma(0)) = \eta\, r^{n-1}.$$

Then $\mathcal{H}^{n-1}(\Gamma(0) \cap B(x_0, r)) \leq \eta\, r^{n-1}$

and so $\Gamma(t) \cap B(x_0, r/2) = \phi$

for $t = \alpha r^2$. This holds for all points x and so $t^* \leq \alpha r^2 \leq C\, \mathcal{H}^{n-1}(\Gamma(0))^{2/(n-1)}$

$\square$

Giga and Yamauchi showed

$$\boxed{t^* \geq 2\,\frac{|U(0)|^2}{\mathcal{H}^{n-1}(\Gamma(0))^2}}$$

provided $\Gamma(0)$ is the boundary of an open set $U(0)$, with Lebesgue measure $|U(0)|$

VI <u>Generalizations</u>, <u>applications</u>

A. <u>Higher codimension flows</u>

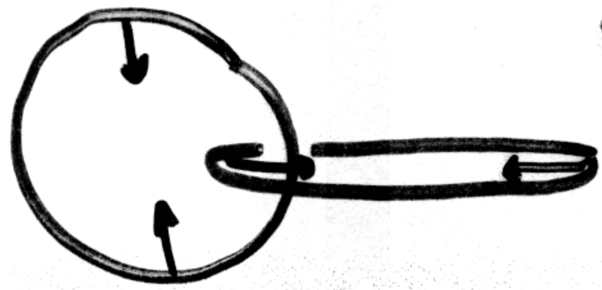
Law of evolution $\underline{v} = \underline{H}$

velocity vector

mean curvature vector

Ref Ambrosio - Soner, J Diff. Geom (1996)

Example



$\Gamma(0)$

What happens when the rings intersect?

B. Approximations

$\mathbb{K}$ = cpt. subsets of $\mathbb{R}^n$

General framework -

Def Define for each time $t \geq 0$, the mapping

$$\mathcal{M}(t) : \mathbb{K} \to \mathbb{K}$$

by

$$\mathcal{M}(t) K = \Gamma(t)$$

for $\Gamma(0) := K$

generalized m.c. flow (as above)

By uniqueness

$$\mathcal{M}(t)\,\mathcal{M}(s) = \mathcal{M}(t+s)$$

$\forall\, s,t \geq 0$

Call $\mathcal{M}(\cdot)$ the mean curvature flow semigroup

Approximation Assume a family of mappings

$$\mathcal{A}(t) : \mathbb{K} \to \mathbb{K} \qquad (t \geq 0)$$

is given such that

(i) $\mathcal{A}(0) = \text{identity}$

(ii) "$\mathcal{A}(t)$ approximates $\mathcal{M}(t)$" for small t

Question Do we have

$$\boxed{\lim_{n\to\infty} \mathcal{A}(t/n)^n K = \mathcal{M}(t) K}$$

$\forall\, t > 0,\ K \in \mathbb{K}$?

Cf Chernoff formula for semigroups

Example Bence-Merriman-Osher (BMO algorithm)

Given $t>0$, $K \subset \mathbb{R}^n$, solve the heat flow

$$
(H) \qquad \boxed{\begin{aligned} v_t &= \Delta v \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ v &= \chi_K \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}}
$$

Define the set $\mathcal{H}(t)K$ by saying

$$
x \in \mathcal{H}(t)K \iff v(x,t) \geq \frac{1}{2}
$$

"heat flow approximation"

Heuristic principle

$$
\text{"} \mathcal{H}(t/n)^n K \to \mathcal{M}(t)K \text{"}
$$

Ref Barles - Georgelin, SIAM J Numer. Analysis (1995)
E - Indiana U Math J (1993)
Mascarenhas, Tech Report UCLA (1992)