Geometric Evolutions with Very Singular Diffusivity

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Recently, equations with very strong diffusivity attracts considerable interests in various fields. For example the gradient flow of total variations is used for removing noise from images [13]. It is also used to describe multi-grain phenomena in material sciences [12]. A typical form is

$$u_t - \operatorname{div}(\frac{\nabla u}{|\nabla u|}) = 0, \tag{1}$$

which has a divergence structure. However, there are several interesting problems which has no divergence structure. A typical example is the crystalline flow equations proposed by [2] and [14]. A simplest example is

$$V = -\operatorname{div}\xi(\mathsf{n}) \text{ on } \Gamma_t \tag{2}$$

where Γ_t is a closed simple curve in R^2 and V is the normal velocity in the direction of outward normal n of $\Gamma_t; \xi$ is the gradient of $\gamma : \mathsf{R}^2 \to [0, \infty)$ and γ is convex, piecewise linear and homogeneous of degree one.

If the equation is a gradient system, subdifferential approach [10], [11], [7] or nonlinear semigroup approach [1] do apply to provide reasonable notion of solutions. However, such an approach does not apply to provide notion of solutions (consistent with smooth problems) when the equation is not a gradient system.

To overcome these difficulties the author adjusted the theory of viscosity solutions for general curvature flow equation including (2) as a special example when the diffusivity is strong so that its effect is nonlocal. Based on [3], [4] we in particular established the level set method in [6]. To show significance of our results is this talk we show several applications of the theory [5].

Several other phenomena can be regarded as a result of singular diffusivity. Recently, a new notion of viscosity solution describing shock phenomena has been introduced by the author under the name of proper solutions [8]. The shock can be interpreted as a result of strong vertical diffusion [9]. Advantages of the theory over conventional theory of conservation law is that it applies to the equation of the form

$$u_{\mathrm{t}} - a(u)|\nabla u| = 0, \tag{3}$$

which does not have divergence structure. Here a is a given positive function and $r \mapsto a(r)$ is increasing. This equation (3) is considered as a crude model of bunchigs in the theory of crystal growth. In this talk we briefly review the theory developed in [8], [9].

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