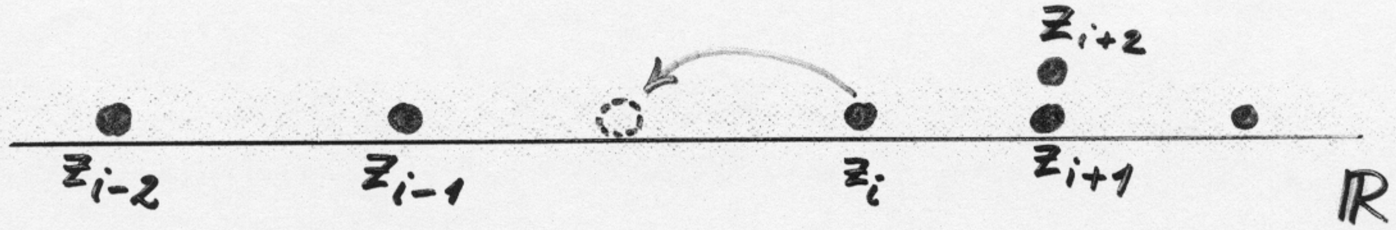
MACROSCOPIC LIMITS AND FLUCTUATIONS FOR THE ALDOUS-DIACONIS-HAMMERSLEY PROCESS



STATE: $z(t) = \left(z_i(t) : i \in \mathbb{Z} \right)$

$z_i(t) \in \mathbb{R}$ = LOCATION OF PARTICLE i

$z_{i-1}(t) \leq z_i(t)$ FOR ALL i, t

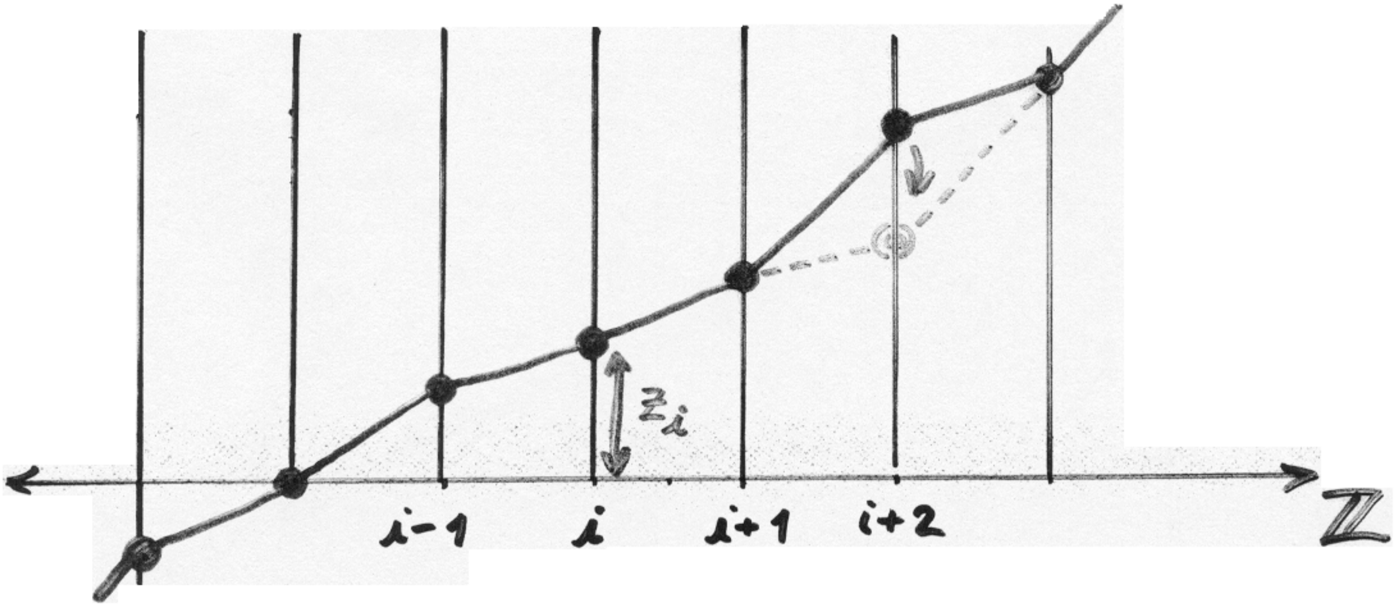
DYNAMICS: * PARTICLES JUMP LEFT

* z_i JUMPS AT RATE $z_i - z_{i-1}$

  MEANS: WAITING TIME OF JUMP HAS EXP. DISTR. W. MEAN $(z_i - z_{i-1})^{-1}$

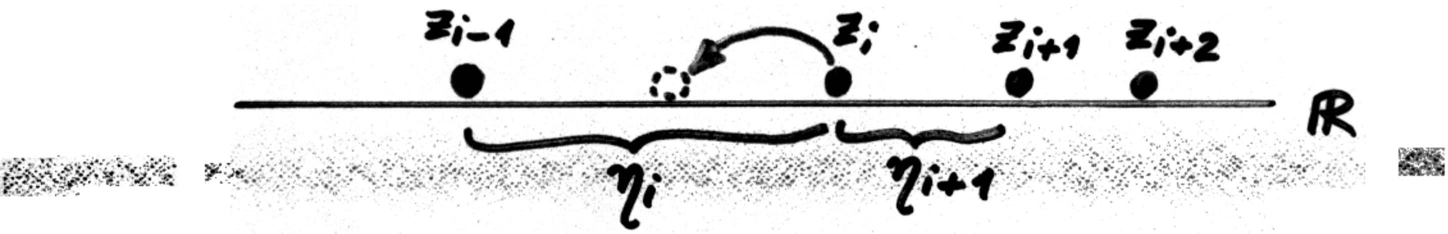
* NEW LOCATION $z_i' \sim \mathrm{UNIF}(z_{i-1}, z_i)$.

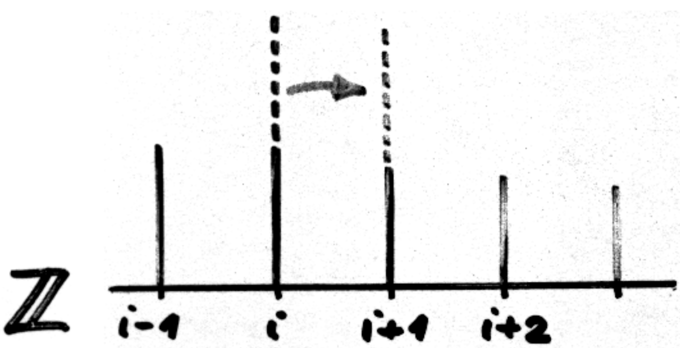
NTERFACE PICTURE



STICK MODEL PICTURE

$\eta_i = z_i - z_{i-1}$ = SLOPE IN INTERFACE PICTURE

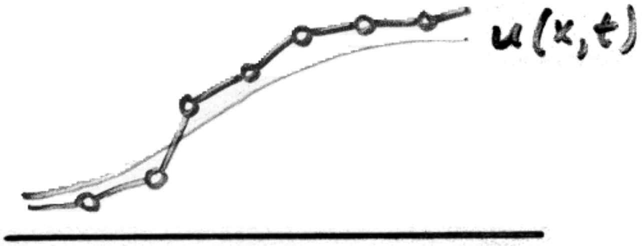


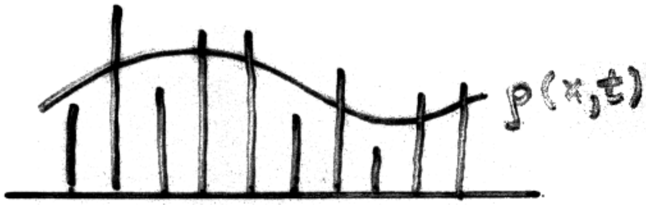


STICK PROCESS

$$\eta(t) = (\eta_i(t) : i \in \mathbb{Z})$$

THE QUESTIONS





1. LAWS OF LARGE NUMBERS

$$\frac{1}{n} Z_{[nx]}(nt) \xrightarrow[n\to\infty]{} u(x,t)$$

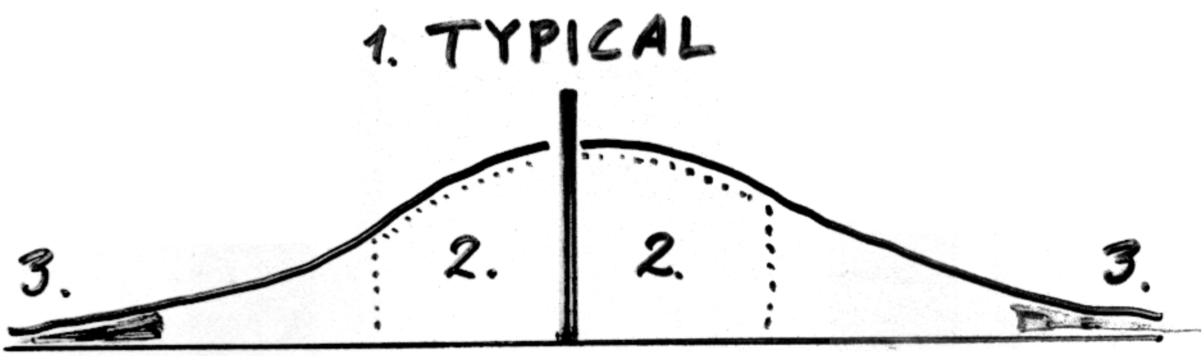
$$\frac{1}{n}\sum_{[na]\le i\le [nb]} \eta_i(nt) \xrightarrow[n\to\infty]{} \int_a^b \rho(x,t)\,dx$$

2. FLUCTUATIONS

$$\frac{Z_{[nx]}(nt)-nu(x,t)}{n^{\beta}} \xrightarrow{d} \zeta(x,t)$$

3. LARGE DEVIATIONS

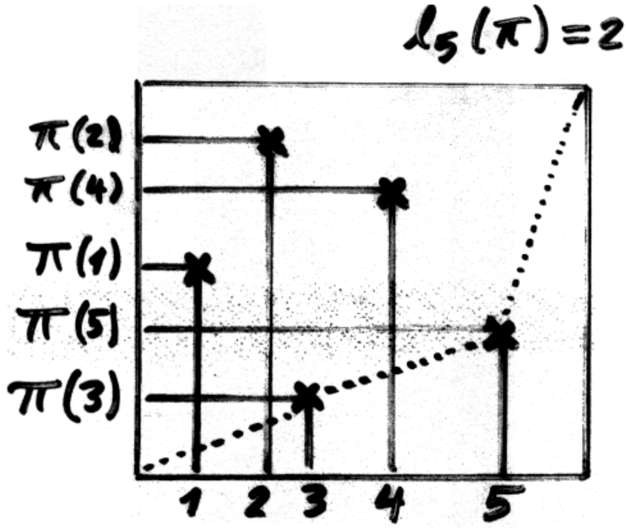
$$P\left\{Z_{[nx]}(nt)-nu(x,t) \geq nr\right\} = e^{-n^{\alpha} I(r)+o(n^{\alpha})}$$



HISTORY

HAMMERSLEY 1972:

$$\frac{\ell_N}{\sqrt{N}} \longrightarrow c$$



$\ell_N :=$ length of longest increasing subseq of a random $\pi \in S_N$

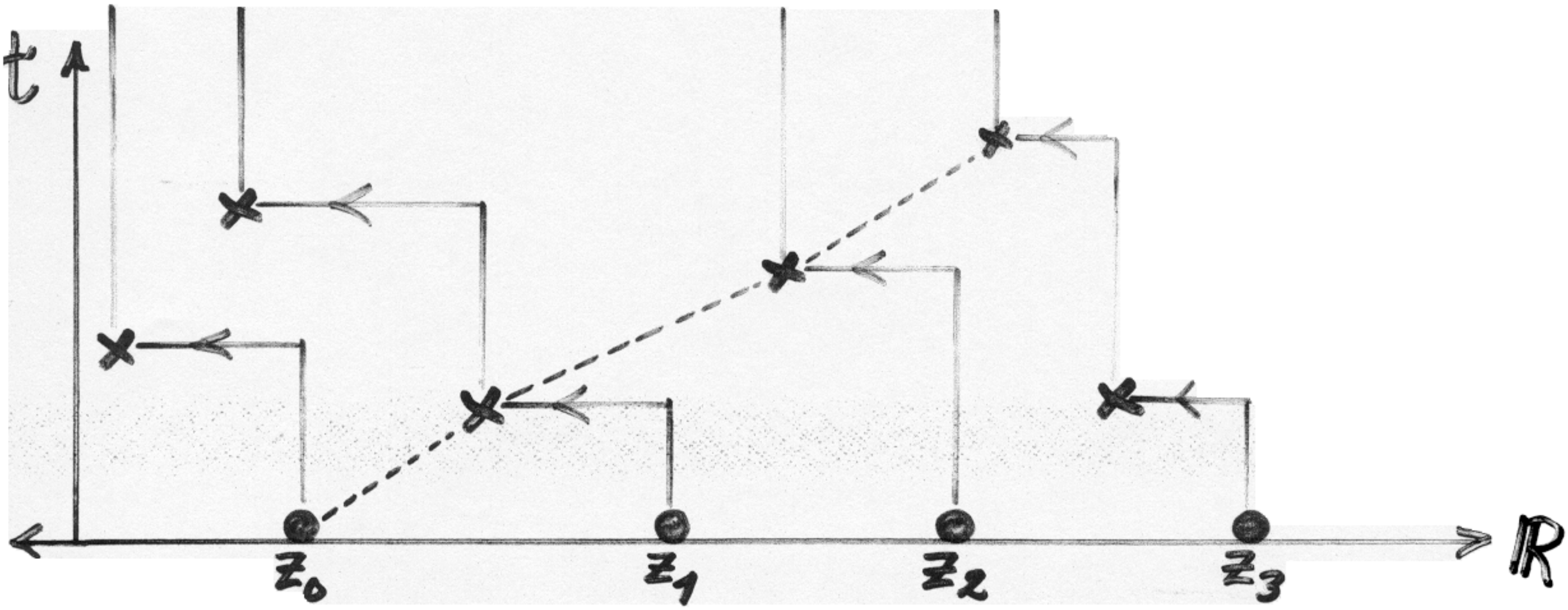
VERSHIK-KEROV 1977: $c=2$
LOGAN-SHEPP 1977: $c \geqslant 2$

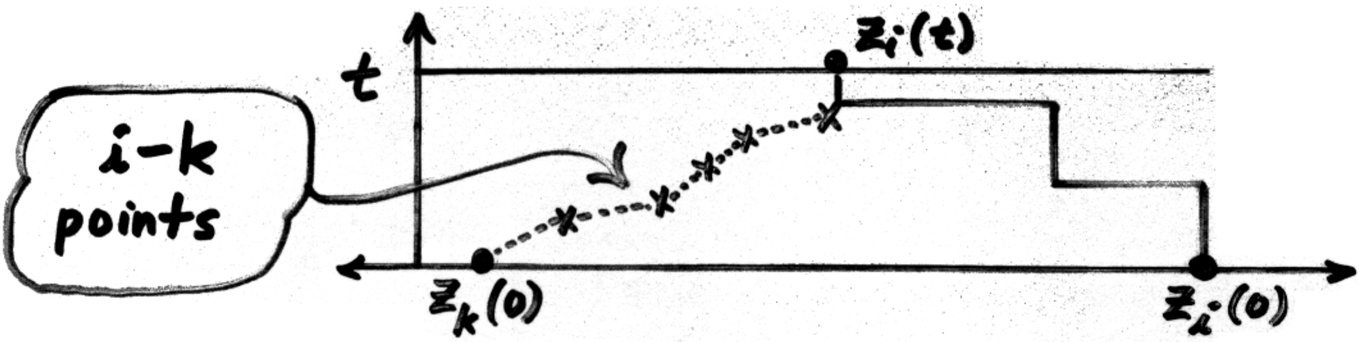
ALDOUS-DIACONIS 1995:

Expand Hammersley's idea to ∞ particle process, adapt (ROST 1981) to study process and prove $c=2$ w/o recourse to combinatorics of permutations (Young diagrams)

WORK CONTINUES: Most recent entirely original pf. of $c=2$ by GROENEBOOM 2000.

CONSTRUCTION W. SPACE TIME POISSON POINT PROCESS $\mathbf{x}$

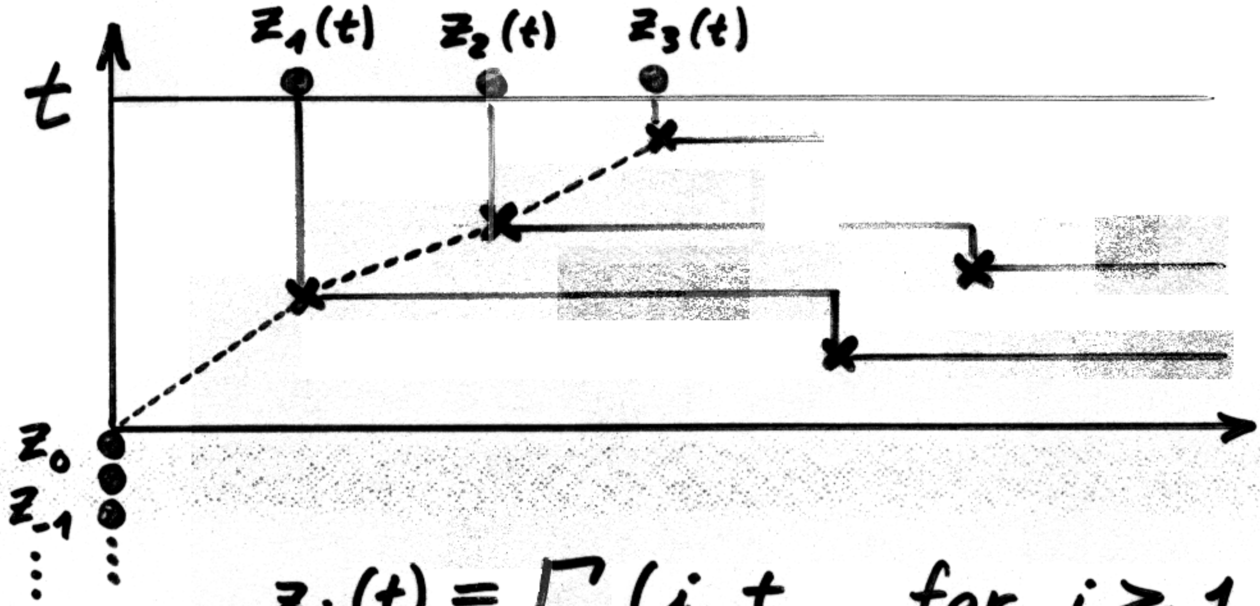




$$\Gamma_k(m,t) = \inf\Big\{ h>0 : (z_k, z_k+h] \times (0,t] \text{ contains } \nearrow \text{path of } m \text{ points} \Big\}$$

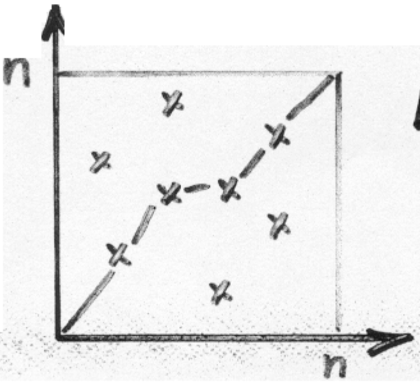
<u>LEMMA</u> $z_i(t) \quad \inf\limits_{k<i} \Big\{ z_k(0) + \Gamma_k(i-k, t) \Big\}$

SPECIAL CASE: $z_i(0) = \begin{cases} 0 & \text{for } i \le 0 \\ \infty & i \ge 1 \end{cases}$

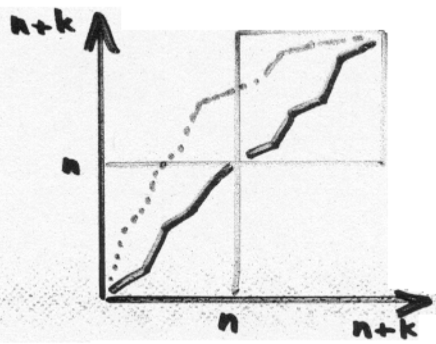


$z_i(t) = \Gamma_0(i, t$ for $i \ge 1$

$L(n,n) :=$ MAX. # POINTS ON $\nearrow$ PATH IN $(0,n] \times (0,n]$



$L(n,n) = 4$



Superadditivity: $\dfrac{L(n,n)}{n} \longrightarrow c$

Cor: $\dfrac{1}{n}\Gamma([nx], nt) \longrightarrow \dfrac{x^2}{c^2 t}$

STEADY STATES

$\nu^{\rho} :=$ PROBAB DISTR. ON $[0,\infty)^{\mathbb{Z}} \ni \{\eta_i\}_{i\in\mathbb{Z}}$

ARE I.I.D. EXP'S W. $E^{\nu^{\rho}}[\eta_i] = \rho$

<u>THEOREM</u>: $\{\nu^{\rho}\}_{0\le\rho<\infty}$ INVARIANT FOR $\eta(t)$.

"JUSTIFICATION", BY ANALOGY W. M/M/1 QUEUES

ASSUME STICKS INDEPENDENT EXP-R.V.'S.

$$\text{RATE}(\eta_i \to \eta_i + u) = E\left[\eta_{i-1}\cdot\frac{1}{\eta_{i-1}}\cdot \mathbb{1}\{\eta_{i-1}\geq u\}\right] = e^{-\frac{u}{\rho}}$$

$$\nu^{\rho}(\eta_i)\cdot\text{RATE}(\eta_i \to \eta_i+u) \stackrel{?}{=} \nu^{\rho}(\eta_i+u)\cdot\text{RATE}(\eta_i+u\to\eta_i)$$

$$\frac{e^{-\eta_i/\rho}}{\rho}\; e^{-u/\rho} \underset{\text{OK}}{=} \frac{e^{-\frac{\eta_i+u}{\rho}}}{\rho}\;\cancel{(\eta_i+u)}\frac{1}{\cancel{\eta_i+u}}$$

DETAILED BALANCE HOLDS FOR η_i

$\Rightarrow$ CAN FOLLOW QUEUEING ARGUMENT TO PROVE INVARIANCE.

HYDRODYNAMIC LIMIT

HYPOTHESES: A SEQUENCE $\{Z^n(\cdot)\}$ OF ADH-PROCESSES; A LEFT-CONT. NONDECR. FUNCTION u_0 ON $\mathbb{R}$ $\ni$

$$\forall y \in \mathbb{R} \quad \lim_{n\to\infty} \frac{1}{n} Z^n_{[ny]}(0) = u_0(y) \quad \text{IN PROB.}$$

+ TECHNICAL ASS.

THEOREM. $\forall (x,t) \in \mathbb{R} \times (0,\infty)$,

$$\lim_{n\to\infty} \frac{1}{n} Z^n_{[nx]}(nt) = u(x,t) \quad \text{in prob.}$$

The deterministic limit $u(x,t)$ is given by

$$u(x,t) = \inf_{y \le x} \left\{ u_0(y) + \frac{1}{4t}(x-y)^2 \right\} \quad (HL)$$

REMARK: $\frac{x^2}{4t}$ comes from $\lim \frac{1}{n} \Gamma([nx], nt)$

$$(HL) \Rightarrow u \text{ is a weak sol. of } \begin{cases} u_t + (u_x)^2 = 0 \\ u\big|_{t=0} = u_0 \end{cases}$$

Under some reg on u_0, u is the UNIQUE VISCOSITY SOL.

PF OF HYDRODYNAMIC LIMIT AND $c=2$

1. $$\frac{1}{n} Z^n_{[nx]}(nt) = \inf_{y \le x} \Big\{ \underbrace{\frac{1}{n} Z^n_{[ny]}(0)}_{\downarrow \; u_0(y)} + \underbrace{\frac{1}{n} \Gamma_{[ny]}([nx]-[ny], nt)}_{\downarrow \; \frac{(x-y)^2}{c^2 t}} \Big\}$$

$$\frac{1}{n} Z^n_{[nx]}(nt) \longrightarrow u(x,t) := \inf_{y \le x} \Big\{ u_0(y) + \frac{(x-y)^2}{c^2 t} \Big\}$$

2. To calculate c.

Equilibrium: $Z_0(0) = 0$, $\eta_i(0) \sim$ iid $Exp(\frac{1}{\rho})$

$$E Z_0(t) = -\rho^2 t$$

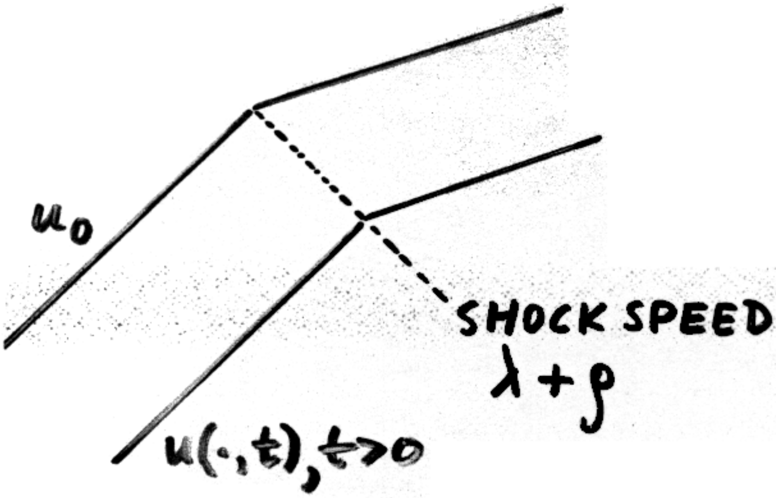
Step 1: $$\frac{1}{t} Z_0(t) \longrightarrow u(0,1) = \inf_{y \le 0} \Big\{ \rho y + \frac{y^2}{c^2 t} \Big\}$$

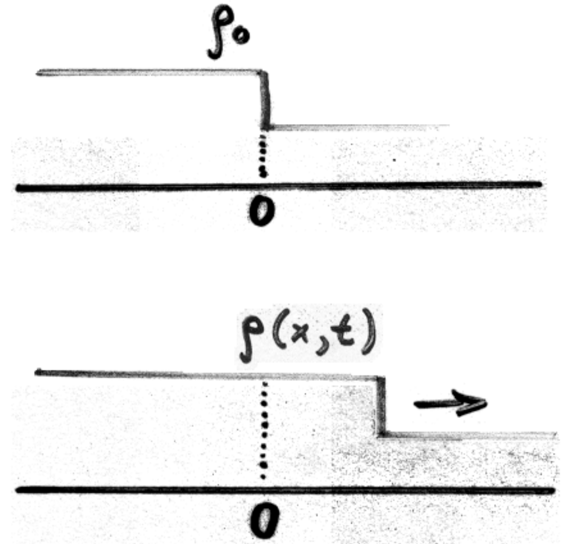
$$= -\frac{c^2}{4} \rho^2 t$$

$\Rightarrow c = 2$

EXAMPLES OF MACROSCOPIC EVOLUTIONS

SHOCK $\lambda > \rho$ $\quad u_0(y) = \begin{cases} \lambda y, & y < 0 \\ \rho y, & y \geq 0 \end{cases}$

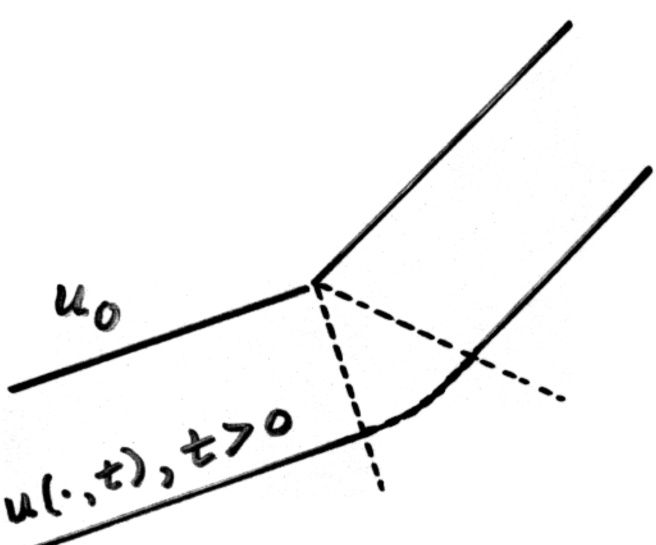


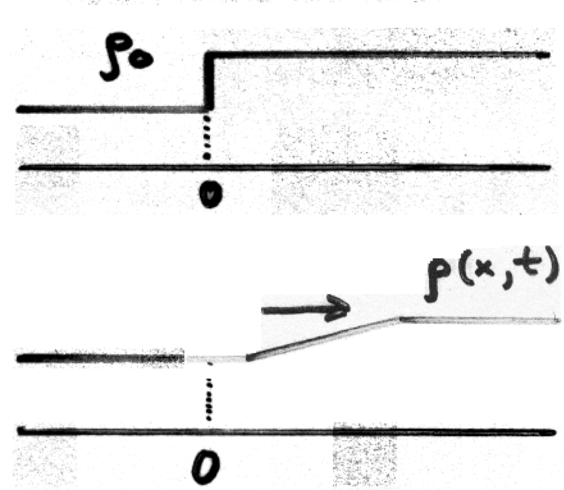


STICK PROFILE

$\rho(x,t) = u_x(x,t)$

RAREFACTION FAN $\lambda > \rho$ $\quad u_0(y) = \begin{cases} \rho y, & y < 0 \\ \lambda y, & y \geq 0 \end{cases}$





FLUCTUATIONS

1. INCREASING SEQUENCES (BAIK-DEIFT-JOHANSSON)

$$\lim_{n\to\infty} P\left\{ \frac{L(n,n)-2n}{n^{1/3}} \le t \right\} = F(t)$$

where the TRACY-WIDOM DISTRIBUTION def. by

$$F(t) = \exp\left(\int_t^\infty (x-t)u(x)^2\,dx \right),$$

u solves the Painlevé II eqn

$$u'' = 2u^3 + xu, \quad u \sim Ai(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2x^{1/4}\sqrt{\pi}} \quad x\to\infty.$$

TRACY & WIDOM 1994: F = limit distr. for (scaled) max eigenvalue of random $N\times N$ Hermitian matrix M

$$dM \sim \frac{1}{Z_N} e^{-tr(M^2} \prod_{i<j} dm_{ii}\, d(\mathrm{Re}\, m_{ij})\, d(\mathrm{Im}\, m_{ij}$$

(GUE)

$$P\left(\lambda_{max}^{(N)} \le 2\sqrt{N} + \frac{t}{N^{1/6}}\right) \xrightarrow[N\to\infty]{} F(t)$$

FLUCTUATIONS

DIFFUSIVE SCALE $\sqrt{n}$ IN HYDRODYNAMIC LIMIT

* INITIAL FLUCTUATIONS TRANSLATED ALONG CHARACTERISTICS OF P.D.E.

NO DYNAMICAL NOISE AT THIS SCALE

LET $$u(x,t) = \inf_{y \le x} \left\{ u_0(y) + \frac{1}{4t}(x-y)^2 \right\}$$

$$I(x,t) = \left\{ y : u(x,t) = u_0(y) + \frac{1}{4t}(x-y)^2 \right\}$$

$$\zeta_n(x,t) = \frac{z^n_{[nx]}(nt) \quad nu(x,t)}{\sqrt{n}}$$

<u>THM</u> SUPPOSE $\zeta_n(\cdot,0) \xrightarrow{d} \zeta_0(\cdot)$ IN THE TOP.

UNIFORM CONV. ON COMPACTS, AND $\zeta_0(\cdot)$

A.S. CONT. THEN

$$\lim_{n\to\infty} \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) = 0 \text{ in prob.}$$

$$\zeta_n(x,t) \xrightarrow{d} \zeta(x,t) := \inf_{y \in I(x,t)} \zeta_0(y)$$

"PF" $Z_{[nx]}(nt) - nu(x,t)$

$$= \inf_{i \le nx} \left\{ Z_i(0) + \Gamma_i([nx]-i, nt) - nu(x,t) \right\}$$

ESTIMATE

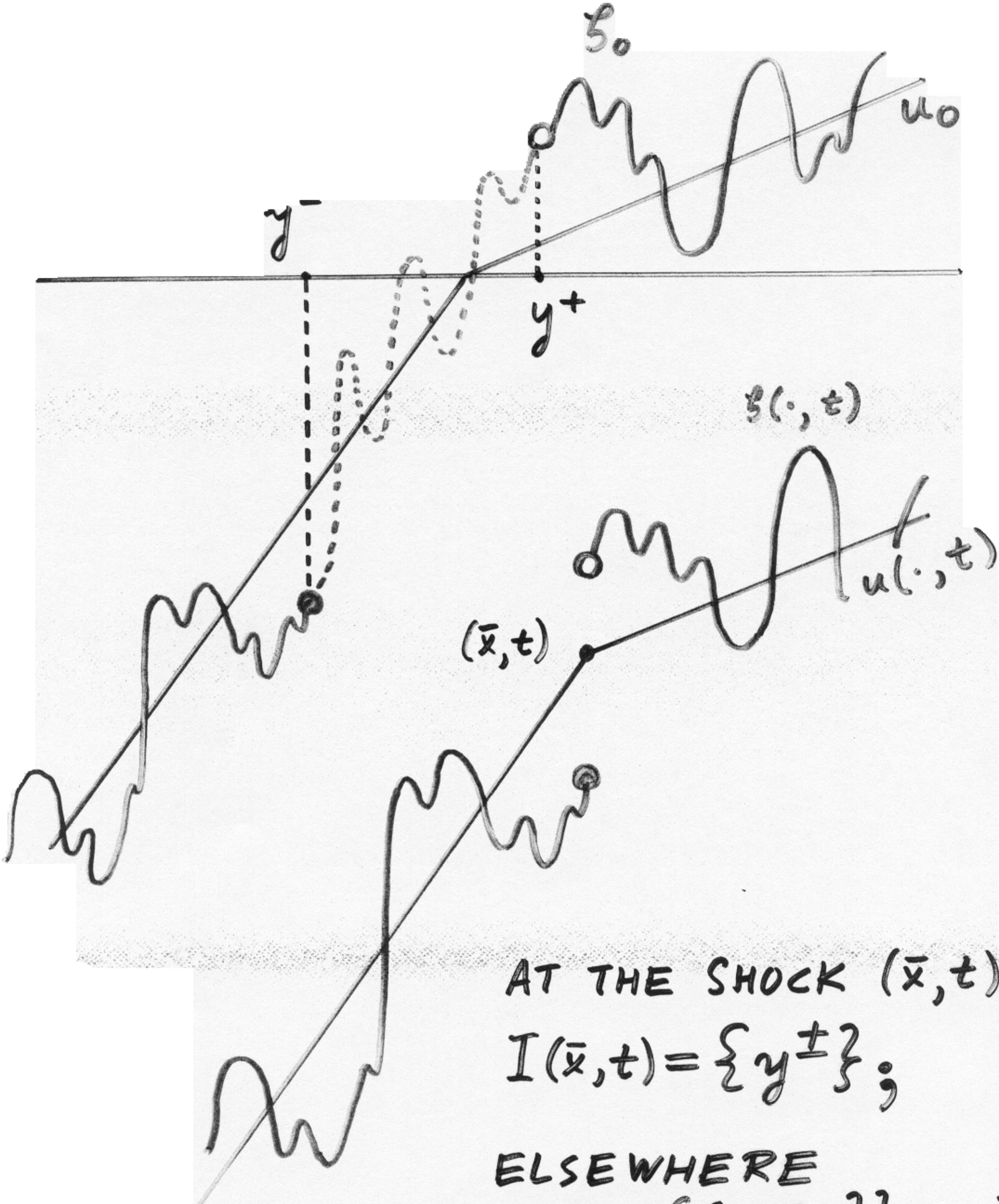
$$\approx \inf_{y \in I(x,t)} \left\{ Z_{[ny]}(0) + \Gamma_{[ny]}([nx]-[ny], nt) - nu(x,t) \right\}$$

$$u(x,t) = u_0(y) + \frac{1}{4t}(x-y)^2 \text{ for } y \in I(x,t)$$

$$\approx \inf_{y \in I(x,t)} \left\{ \underbrace{Z_{[ny]}(0) - nu_0(y)}_{O(\sqrt{n})} + \underbrace{\Gamma_{[ny]}([nx]-[ny], nt) - \frac{n(x-y)^2}{4t}}_{O(n^{1/3}) \text{ [BDJ]}} \right\}$$

$$\Rightarrow \zeta_n(x,t) = \inf_{y \in I(x,t)} \zeta_n(y,0) + O(n^{-\frac{1}{6}}).$$

SHOCK CASE



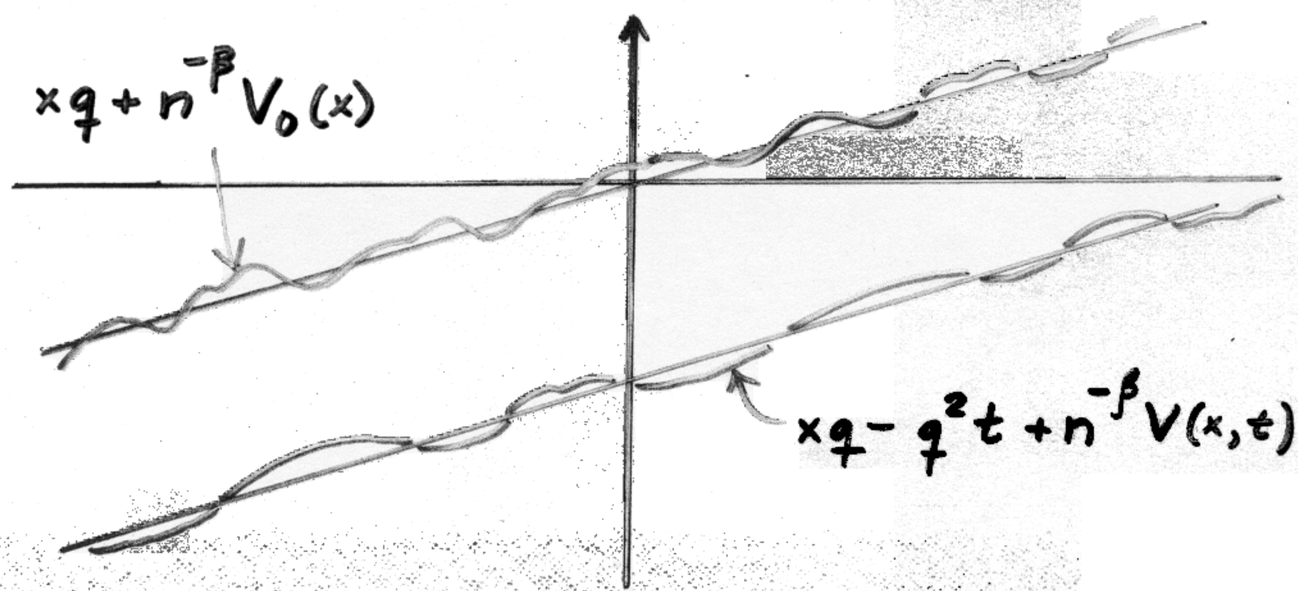
AT THE SHOCK $(\bar{x},t)$,

$$I(\bar{x},t)=\{y^\pm\};$$

ELSEWHERE

$$I(x,t)=\begin{cases}\{x-2t\lambda\}, & x<\bar{x}\\ \{x-2t\rho\}, & x>\bar{x}\end{cases}$$

PERTURBATION OF THE EQUILIBRIUM



$q > 0$ fixed reference slope.

$\beta \in (0, \frac{1}{2})$, V_0 Lipschitz function

Initially $z_k^n(0)$ sum of Exp r.v.'s w.

$$E\left[z_k^n(0)\right] = kq + n^{1-\beta} V_0\left(\tfrac{k}{n}\right)$$

<u>THM</u>. As $n \to \infty$,

$$z^n_{[nx] + [2n^{1+\beta} t q]}(n^{1+\beta} t) = n^{1+\beta} t q^2 + n x q + n^{1-\beta} V(x,t) + o(n^{1-\beta}) \quad \text{A.S}$$

where $V(x,t) = \inf_{y \in \mathbb{R}} \left\{ V_0(y) + \frac{(x-y)^2}{4t} \right\}$.