The Ergodic Theory of Traffic Jams

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Traffic mystery:

traffic jams with no visible means of support

Cars are basically interacting particles that try to repel each other. Yet sometimes they form large clumps (traffic jams) that:

-- Have a distinct spatial location

-- Are amazingly persistant in time

-- Are associated with no apparent external cause

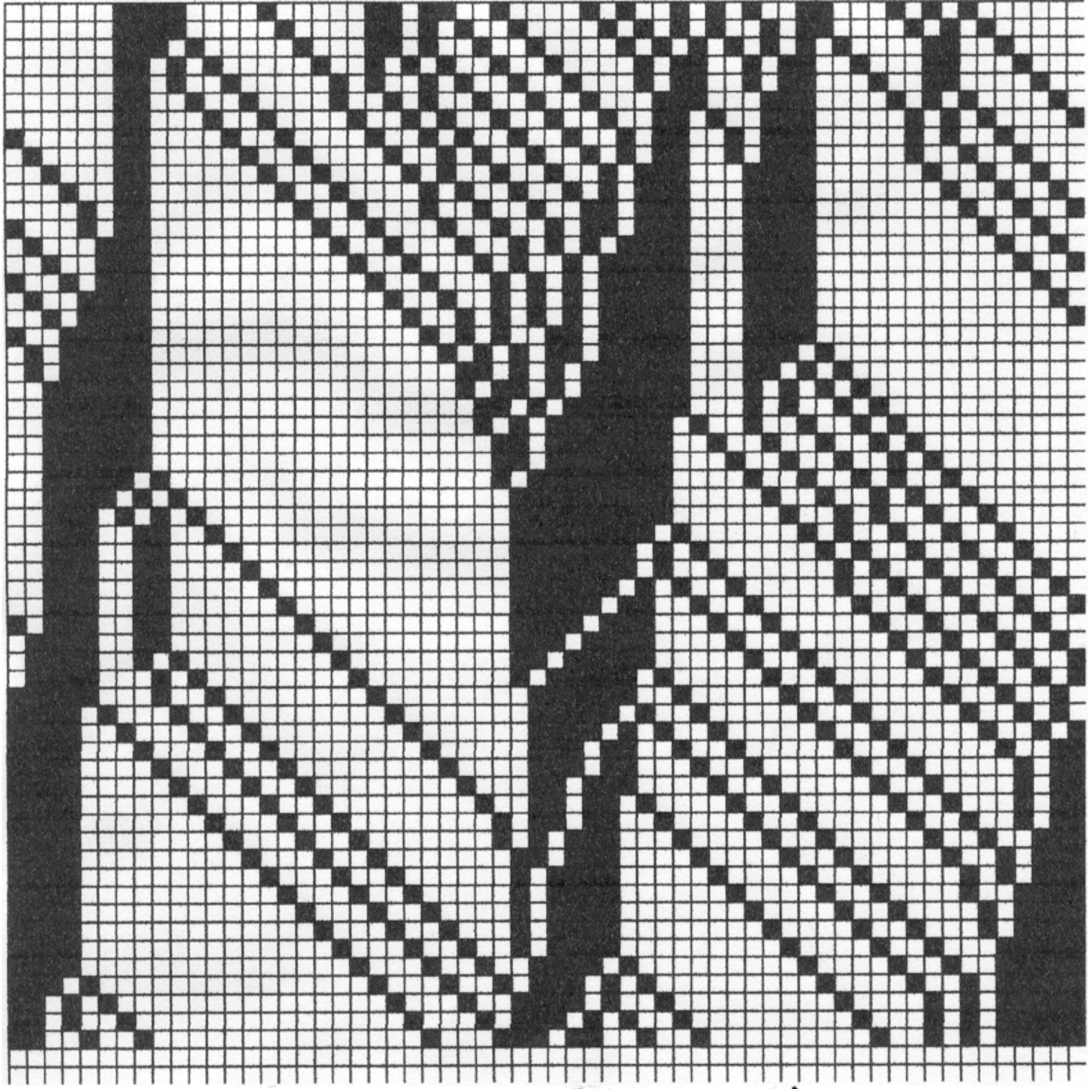
Problem: Find a particle-hopping model that captures this phenomenon.

-- Traffic jams should be precisely defined within context of model.

-- Model should be fun to play with and exhibit a variety of possible behaviors.

-- Model should be simple enough so that there is some hope of proving jams exist.

Nagel and Schreckenberg (1992) introduced a model satisfying first two criteria. Their model is slightly complicated, due to particles having velocities.



(time moves downward)

transition type	$(x-1)$	x	$(x+1)$	$(x+2)$	*jump probability*
accelerating	**1**	**1**	**0**	**0**	α
braking	**0**	**1**	**0**	**1**	β
congested	**1**	**1**	**0**	**1**	γ
driving	**0**	**1**	**0**	**0**	δ

Cars move one way in a single lane, with no on- or off-ramps, no passing, no collisions. Space and time are discrete, with synchronous updating. Jump probabilities correspond to velocities.

Initial conditions are determined by coin-flips with parameter ρ.

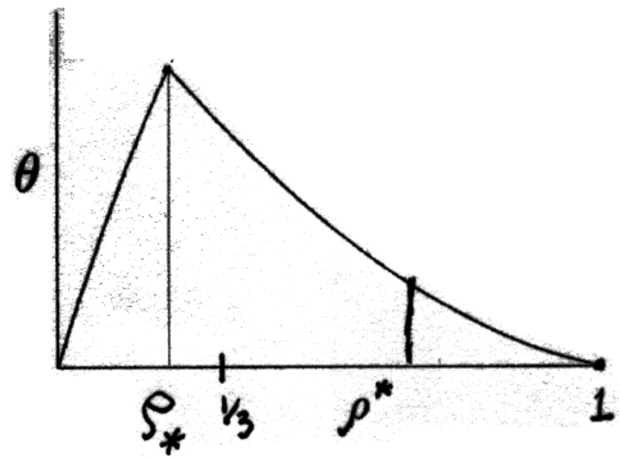
Basic notions and terminology

Slow-to-start: $\alpha < \delta$
Cruise-control limit: $\delta = 1$
Particle-hole symmetry: $\gamma = \delta$

Ansatz: For each value of ρ, system converges to an equilibrium probability distribution ν_ρ.

$$\theta(\rho) = \text{throughput in equilibrium } \nu_\rho$$

The "fundamental diagram" is the graph of $\theta(\rho)$



The following terminology applies to the cruise-control case ($\delta = 1$).

The "critical concentration" is denoted by

$$\rho_* = \sup \{\rho : \theta(\rho) = \rho\}$$

When $\rho \leq \rho_*$, the system enters "free-flow". For such ρ, ν_ρ is called a "free-flow state". These are the equilibria in which all cars are traveling at speed $\delta = 1$.

"Critical free-flow" is denoted by ν_*.

Ergodic (pure or unmixed) equilibria that are not free-flow states are called "jam states". Let

$$\rho^* = \inf \{\rho : \nu_\rho \text{ is a jam state}\}$$

The "critical jam" is denoted by ν^*

The phenomenon of interest occurs when ν_ρ is not ergodic, and instead, consists of a mixture of a free-flow state and a jam state. This will occur if $\rho_* < \rho < \rho^*$. Presumably in this case,

$$\nu_\rho \quad c\nu_* + 1 \quad c)\nu^*$$

where $c \quad (\rho^* \quad \rho \,/(\rho^* \quad \rho_*$

The **main problem** is to prove that $\rho_* < \rho^*$

What is known?

If $\alpha = \beta = \gamma = \delta = q$, we have the "Synchronous totally asymmetric exclusion process" (STASEP). This is not slow-to-start. For the STASEP, it is known that

$$\theta(\rho) = \frac{1-\sqrt{1-4q\rho(1-\rho)}}{2} \text{ for all } q \text{ and } \rho.$$

The remaining results apply to cruise-control.

If $\beta = 1$, we can prove that

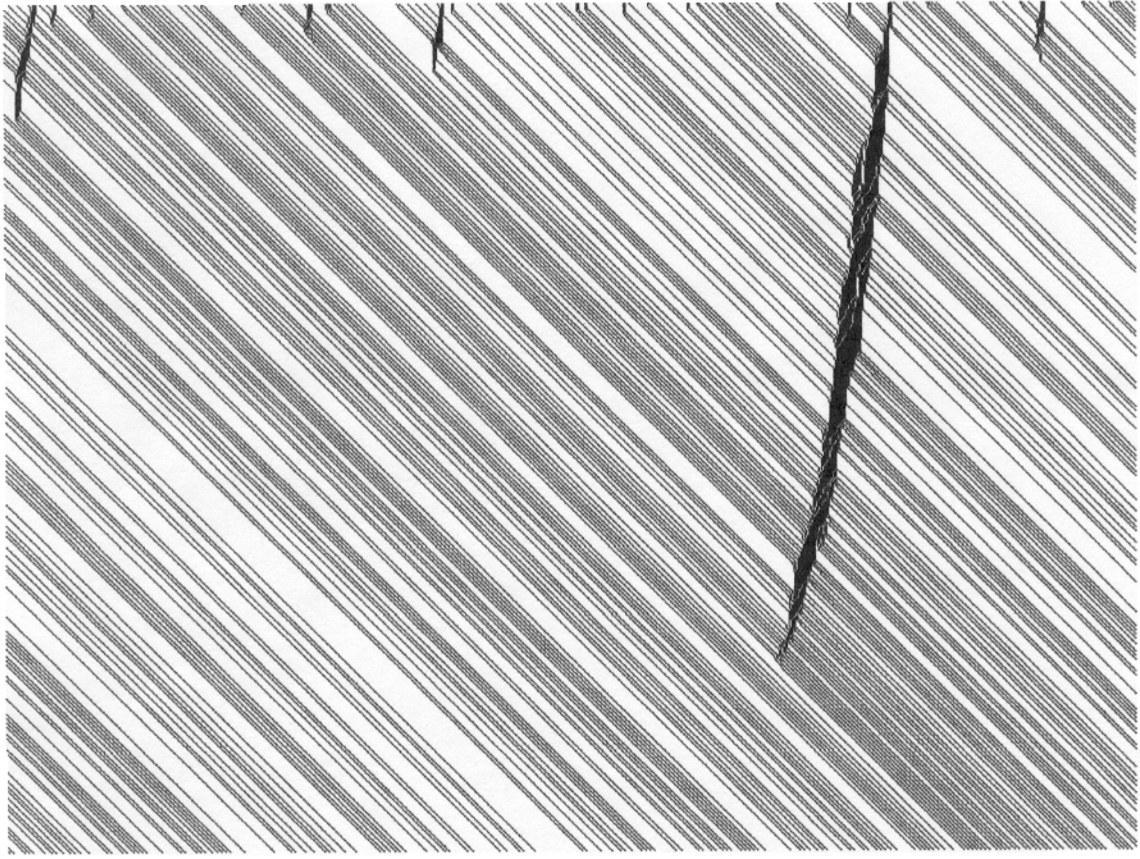
$$\rho_* = \frac{\alpha}{1+2\alpha-\gamma}$$

and

$$\theta(\rho) = \frac{(1-\rho)\alpha}{1+\alpha-\gamma} \text{ for } \rho > \rho_*.$$

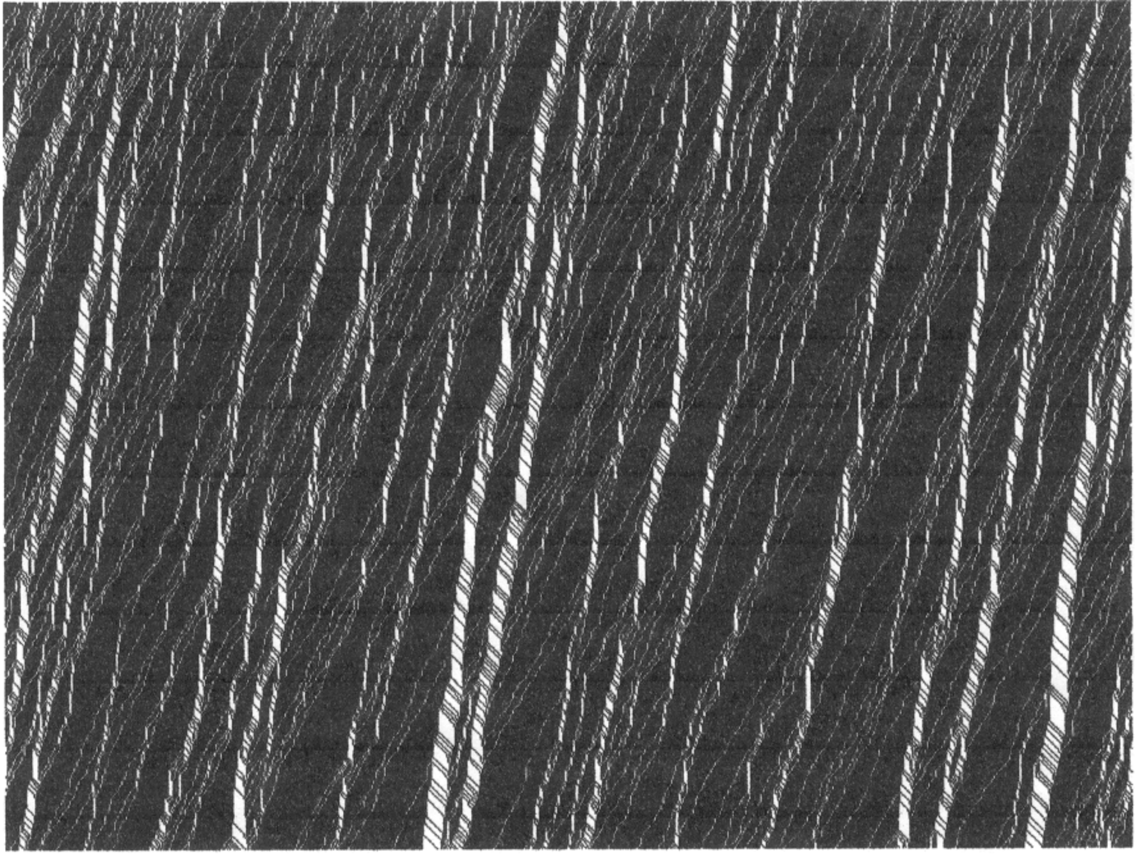
If $\beta = 0$, we can prove that $\rho_* = \frac{1}{3}$ and $\theta(\rho) = 1 - 2\rho$ for $\frac{1}{3} \leq \rho \leq \frac{1}{2}$.

If $\alpha = 0$, it is trivial to see that $\rho_* = \rho^* = 0$.
If $\alpha = 1$, we can prove that $\rho_* = \frac{1}{3}$.



$\alpha = .2$ (slow-to-start),
$\beta = \gamma = .5$ (moderate tailgaiting),
$\delta = 1$ (cruise-control),
18% cars (light traffic)

Traffic jams in initial state dissolve relatively quickly. This is slightly below the critical concentration.

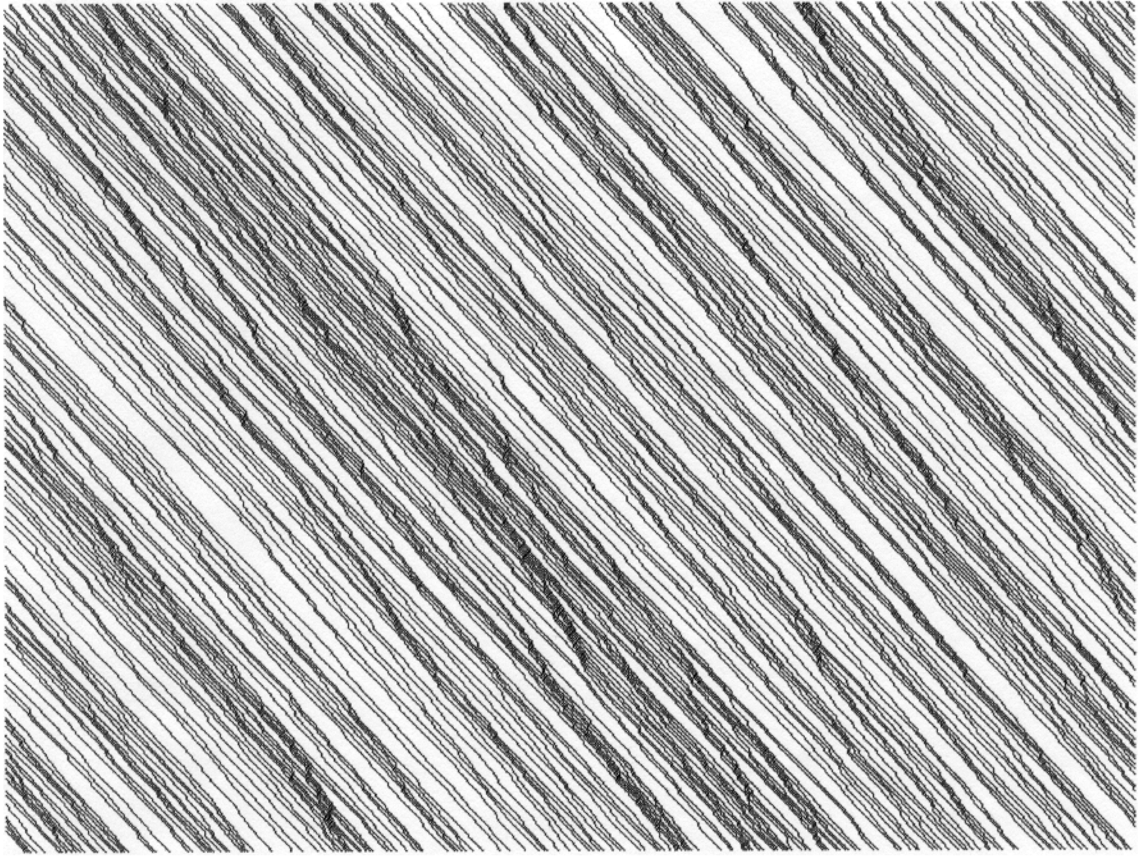


$\alpha = .2$ (slow-to-start),
$\beta = \gamma = .5$ (moderate tailgaiting),
$\delta = 1$ (cruise-control),
82% cars (heavy traffic)

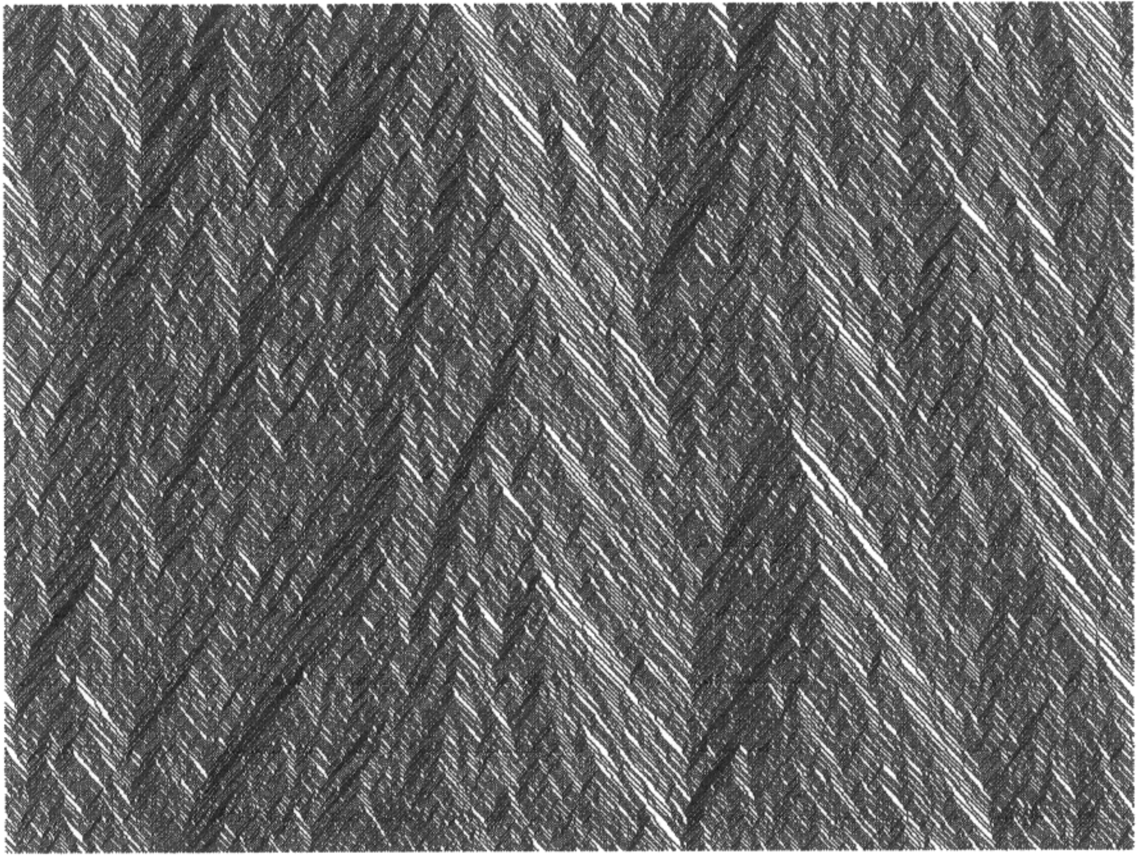
Traffic congeals into one large jam. Note transient short stretches of free-flow, typical of many jam states.

$\alpha = .2$ (slow-to-start),
$\beta = \gamma = .5$ (moderate tailgaiting),
$\delta = 1$ (cruise-control),
50% cars (moderate traffic)

Minimal traffic jam state covers a portion of space, surrounded by free-flow. Intervals of free-flow inside jam are typical of minimal jam state.



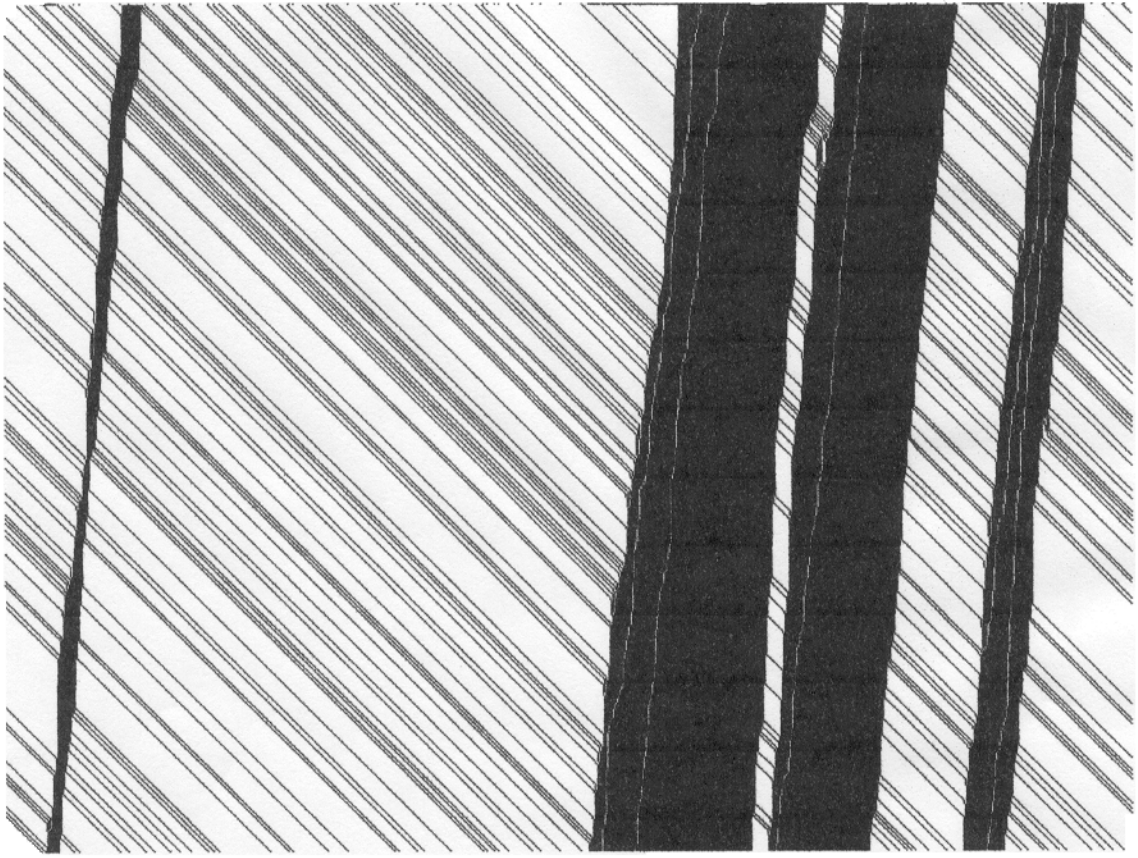
$\alpha = \beta = \gamma = \delta = .9$
(discrete-time version of asymmetric exclusion),
18% cars (light traffic)



$$\alpha = \beta = \gamma = \delta = .9$$

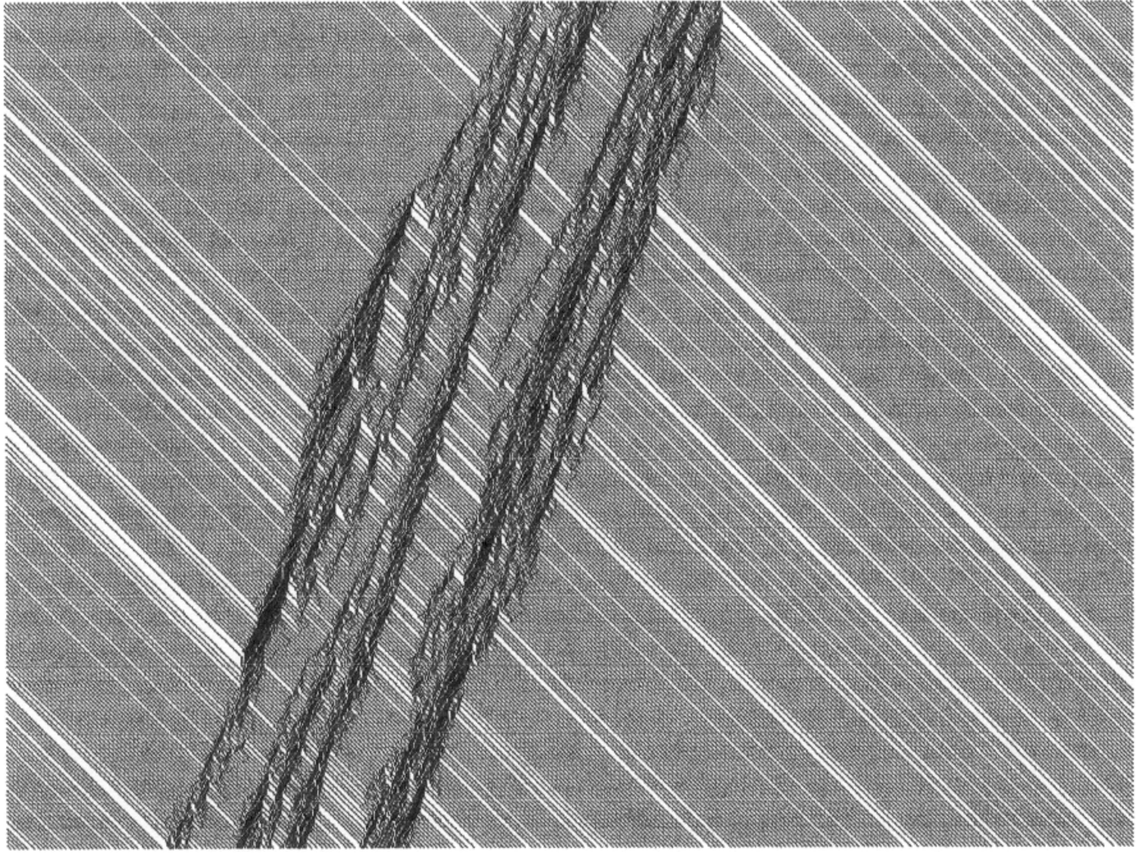
(discrete-time version of asymmetric exclusion), 50% cars (moderate traffic)

Note obvious lack of clustering. This model is monotone (cars repel each other) and hence does not have the slow-to-start feature.



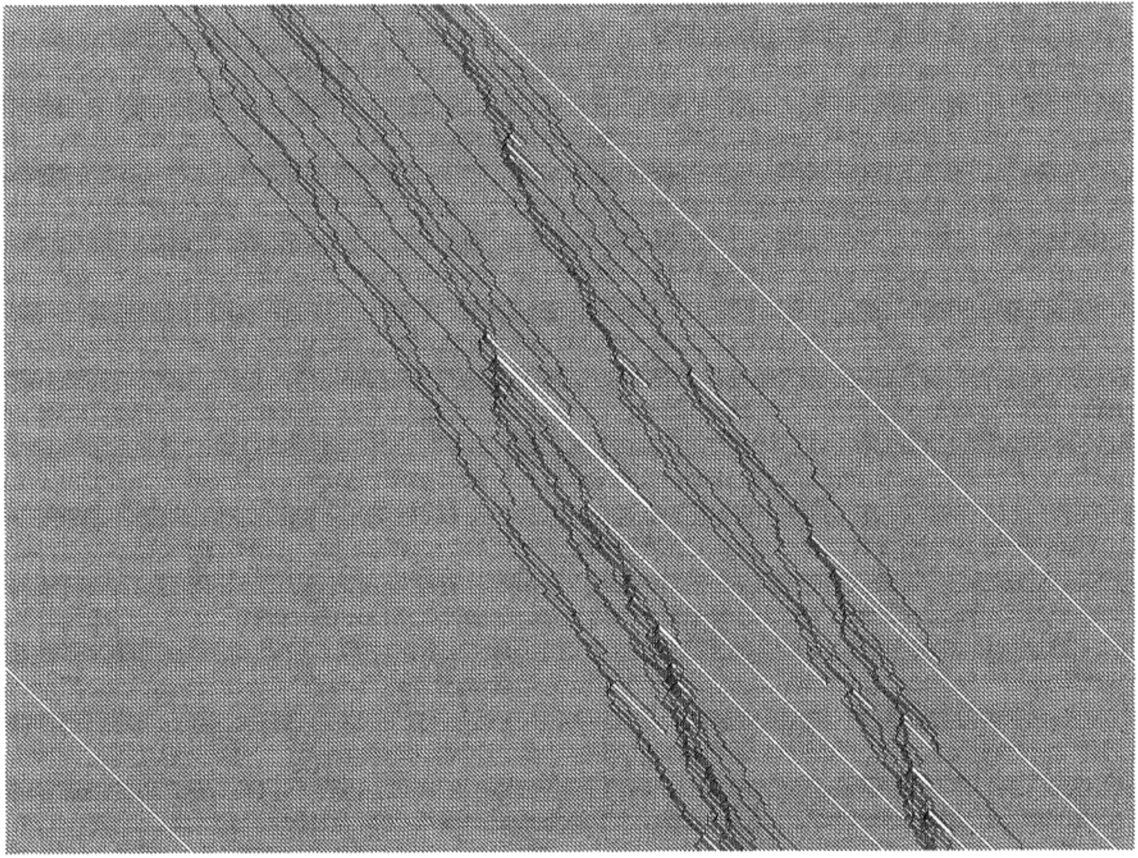
α γ 1 slow-to-start),
β 5 moderate **tailgaiting),**
δ 1 cruise-control).
34% cars (moderate traffic)

When α is less than β the **jams** have a tendency to be **fairly** solid.



$\alpha = \gamma = .6$ (moderate acceleration),
$\beta = .5$ (moderate tailgaiting),
$\delta = 1$ (cruise-control),
34% cars (moderate traffic)

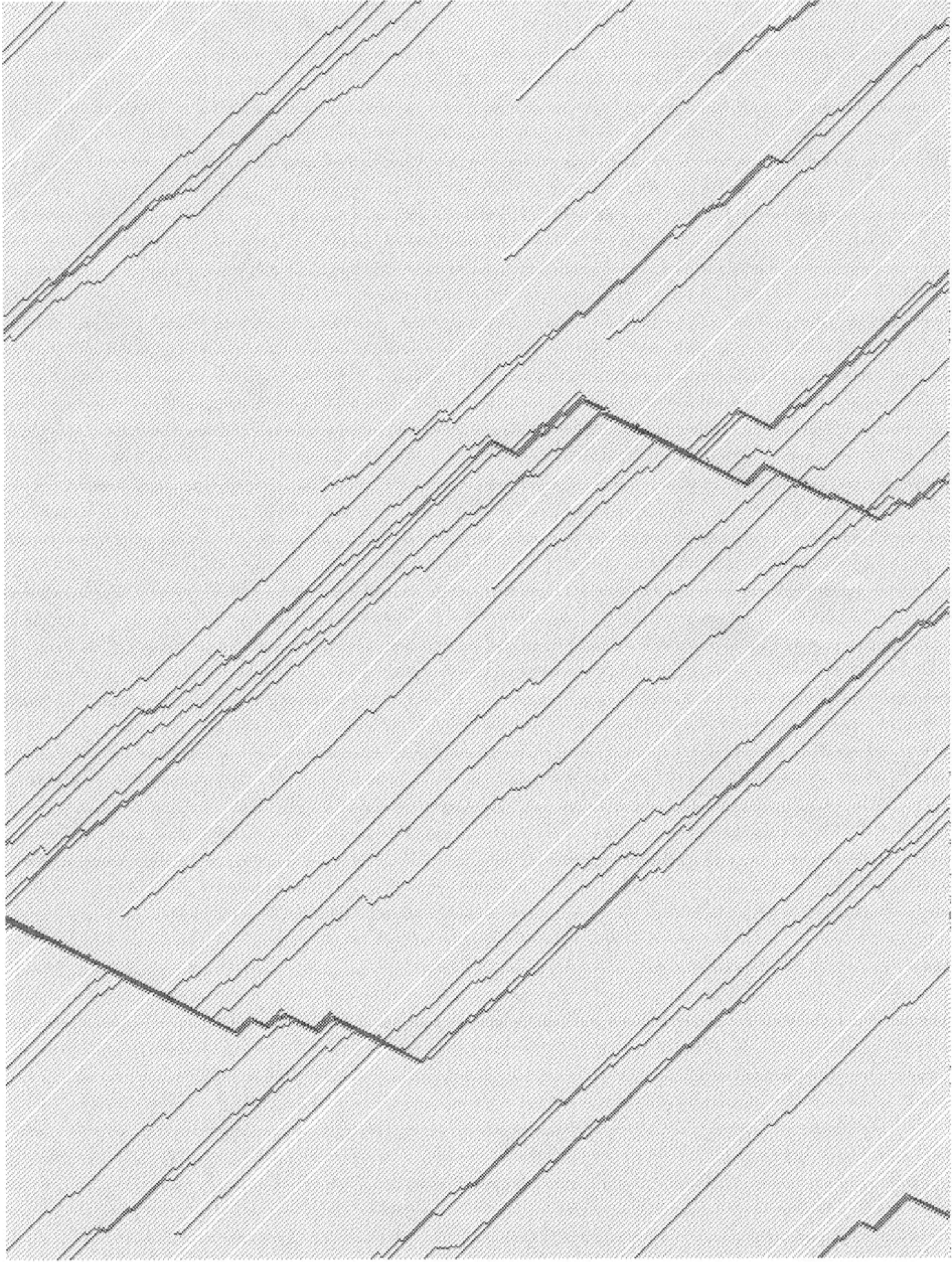
With $\alpha > \beta$, jam is not solid and contains relatively large intervals of free-flow.



$$\alpha = \beta = .9,\ \gamma = \delta = 1$$

(close to asymmetric exclusion),
34% cars (moderate traffic)

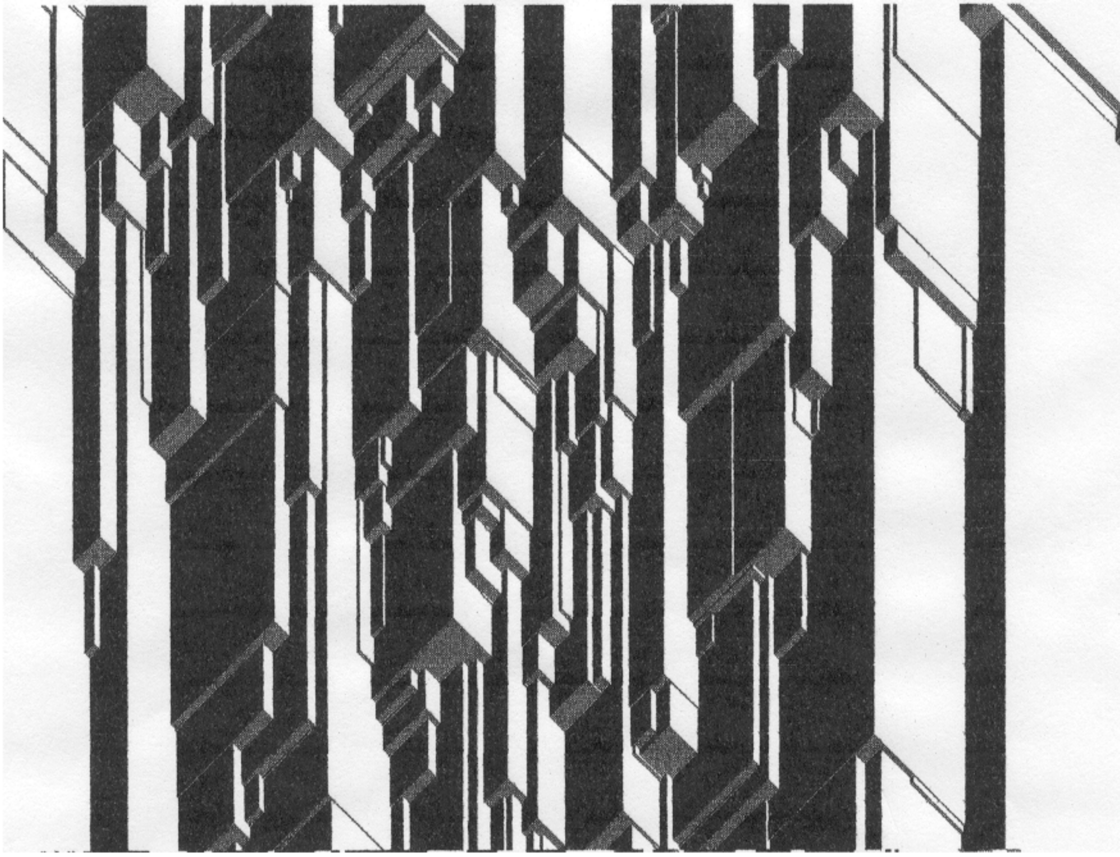
This model is just barely slow-to-start, yet clustering is still quite evident just above the critical concentration. Throughput inside jam state is higher than in free-flow, as evidenced by the forward-moving jam.



A strange, very + n ɹ state

α β 85 γ- δ

ρ 335



$\alpha =$ 01 very ow to tart

99 riou ta ga ting

$\gamma = \delta$ 1 (symmetric cru e contro)

50% cars (moderat to heavy traffic)

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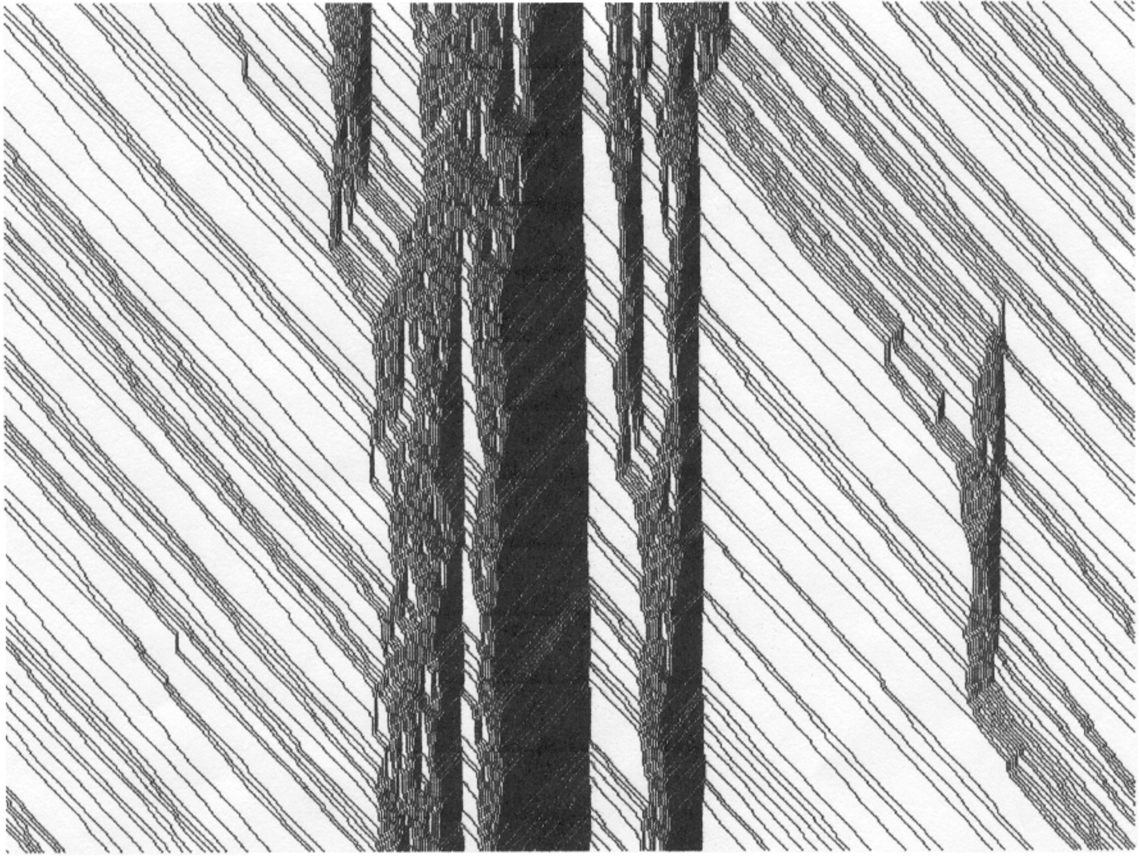
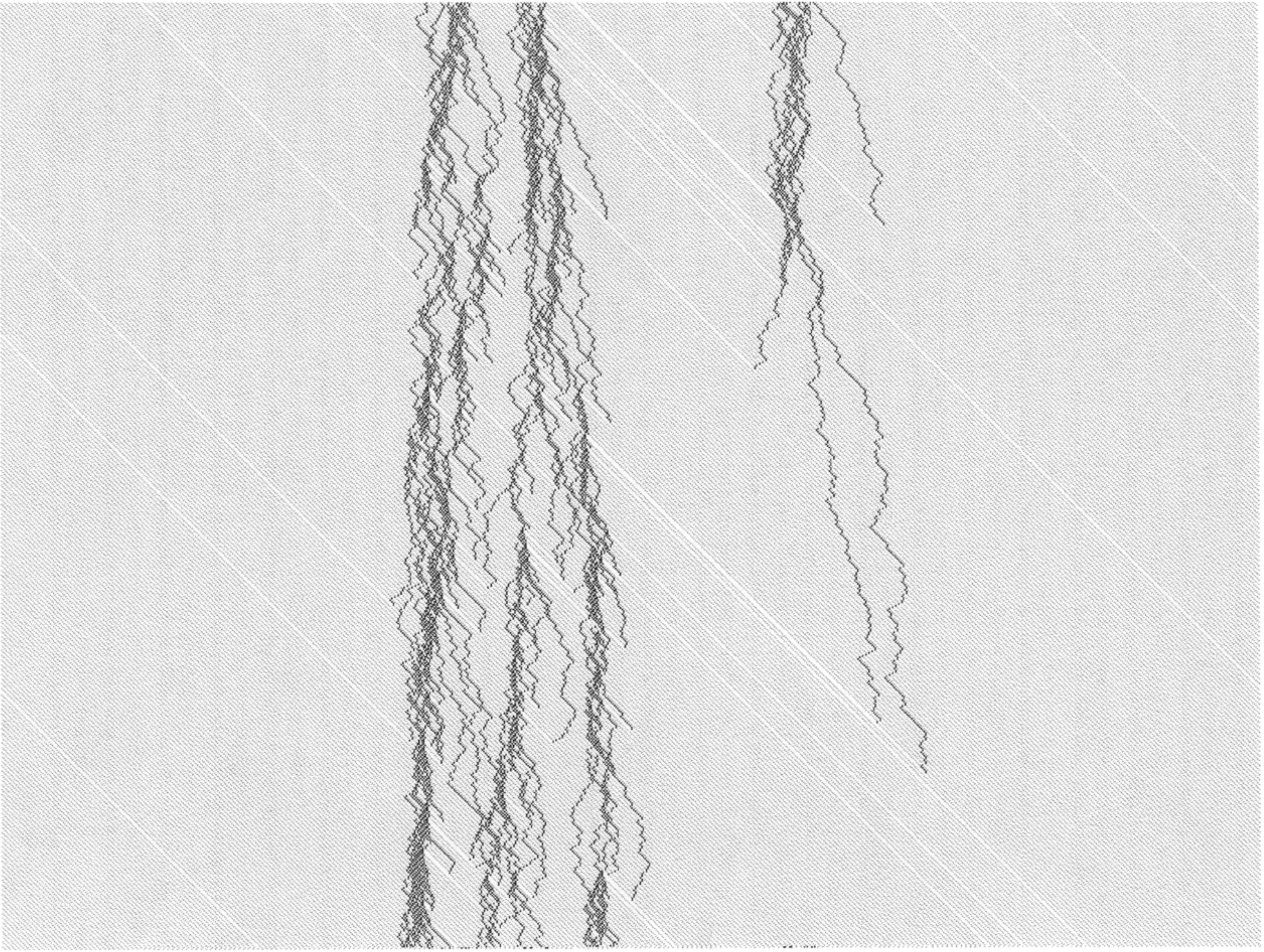


$\alpha = .11$, $\beta = .05$, $\gamma = \delta = .9$
25% cars (moderate traffic)

Possible Type-3 clustering. This is not the cruise-control case, so the free-flow state is nontrivial. Note nucleation and quick disappearance of small jam in the middle of the free-flow. Simulations of systems with up to 20,000 lattice sites, running for a day or two, show complete clustering.



The proof that $\rho_* < \rho^*$