Convergence to asymptotic shapes in threshold growth models

Janko Gravner (Univ. of California, Davis)
David Griffeath (Univ. of Wisconsin, Madison)

Threshold Growth Model (TGM).

The occupied set $A_t \subset \mathbf{Z}^2$ evolves in discrete time $t = 0, 1, \ldots$, with A_0 some (large enough) *finite* set. The rule is determined by

- a finite *neighborhood* $\mathcal{N}$,
- a *threshold* $\theta \geq 1$, and
- update probabilities $0 < p_\theta \leq p_{\theta+1} \leq \cdots \leq p_{|\mathcal{N}|-1}$.

Then:

(1) $A_t \subset A_{t+1}$,

(2) $x \notin A_t$ belongs to A_{t+1} with probability $p_{|A_t \cap (x+\mathcal{N})|}$.

For example, $\mathcal{N}$ could consist of the 4 nearest sites to the origin (*von Neumann* neighborhood) or 8 nearest sites (*Moore* neighborhood). Large neighborhoods $\mathcal{N} = \mathcal{N}_\rho$ will, for simplicity, be $(2\rho + 1) \times (2\rho + 1)$ boxes centered at 0. Often, $p_i \equiv p$.

Deterministic dynamics.

Here $p_\theta = 1$. There are only 2 possibilities (Bohman, 1999):

- either A_t stop growing: $A_{t+1} = A_t$ for some t,
- or $A_\infty = \mathbf{Z}^2$ and A_t/t converges, as $t \to \infty$, to the limiting shape L.

The shape $L = L_1$ is determined by the *Wulff transform*. Imagine that $\bar{\mathcal{T}}$ acts exactly as $\mathcal{T}$ on subsets of $\mathbf{R}^2$. $\bar{\mathcal{T}}$ translates any half-space

$$H_u^- = \{x \in \mathbf{R}^d : \langle x, u \rangle \leq 0\}$$

into

$$\bar{\mathcal{T}}(H_u^-) = H_u^- + w(u) \cdot u, \qquad \text{for some } w(u) \geq 0.$$

Set

$$K_{1/w} = \cup\{[0, 1/w(u)] \cdot u : u \in S^{d-1}\},$$

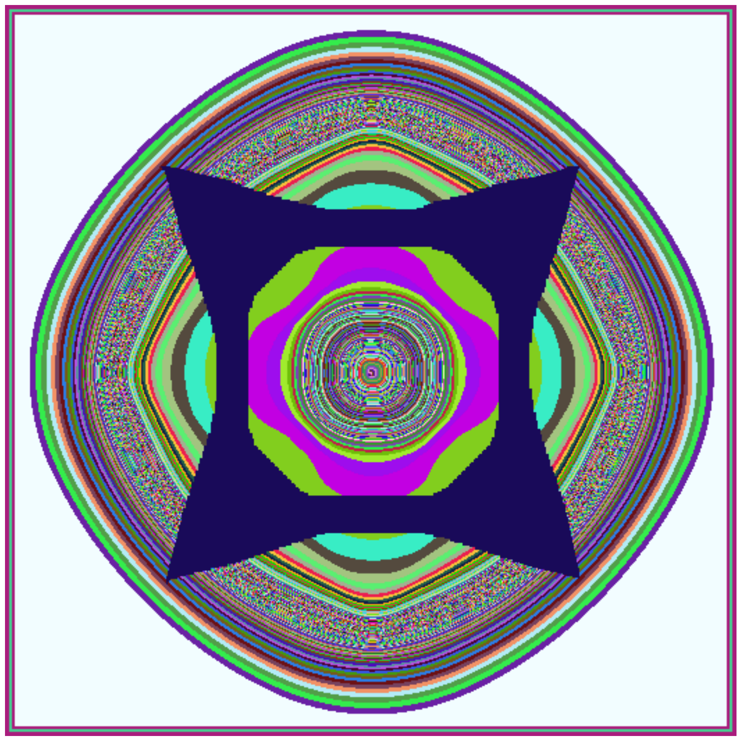
then L is a polygon given by

$$L = K_{1/w}^* = \{x \in \mathbf{R}^d : \langle x, u \rangle \leq w(u)\}.$$

The case when $w > 0$ is *supercritical* and is equivalent to $\theta \leq \rho(2\rho + 1)$.

In the supercritical case a A_0 which grows forever exists. When $\theta \sim \lambda\rho^2$, the cardinality of smallest such sets is $\sim \gamma_E(\lambda)\rho^2$. In fact, one can show that $\gamma_E(\lambda) = \lambda$ up to some critical $\lambda_c \in (1.61, 1.66)$.

The convergence to limiting set L_1 is very fast. In fact, there exists a constant $C = C(\mathcal{N}, A_0)$ so that A_t differs from tL by C (Willson, 1978). This C is, for appropriate initial sets, of order ρ.



Initial set which proves that $\lambda_c > 1.61$: an initial set (dark blue) of size θ for TGM with $\rho = 150$, $\theta = 36,760$.

Random TGM shapes.

Regularity of growth (Bohman-G, 1999): If $x \in A_t$ is at distance at least $C\rho^4$ from A_0, then there is a set G within distance $\mathcal{O}(\rho^4)$ from x which is occupied and grows forever by itself.

Therefore, the dynamics can be successfully restarted from x, and the shape theorem follows by classic subadditive arguments.

Theorem. *If A_0 grows forever, then $A_t/t \to L$ as $n \to \infty$. Here, $L = L_p$ is a bounded convex set with a non–empty interior.*

Not known whether $w(u)$ exist in all cases, so Wulff characterization is not available.

What can be said about the random shape?

A classic result (Durrett–Liggett, 1981) states that, for the *additive* (that is, $\theta = 1$) nearest neighbor TGM,

(1) $L_p \to L_1$ as $p \to 1$ and has a flat edge in the diagonal direction as soon as p is close enough to 1.

(2) However, L_p is for $p < 1$ *not* equal to L_1 due to the fact that its extent in coordinate direction is below 1.

The reason for (1) is that the growth at the boundary of tL_1 does exactly 1d oriented percolation, which survives for large enough p. This part can be generalized to arbitrary supercritical TGM, which, for p close enough to 1, retain a portion of every flat edge of L_1.

Is it possible that $L_p = L_1$?

Exact stability.

This holds if the TGM is as far from additive as possible. Note that for additive TGM $K_{1/w} = \mathcal{N}^*$ is convex. Rules other than additive have convex $K_{1/w}$; this property simplifies many interaction properties. Accordingly, such rules are called *quasi–additive*. (An example: Moore neighborhood with $\theta = 2$.)

Let $K_0 = \partial(K_{1/w}) \cap \partial(\text{co}K_{1/w})$.

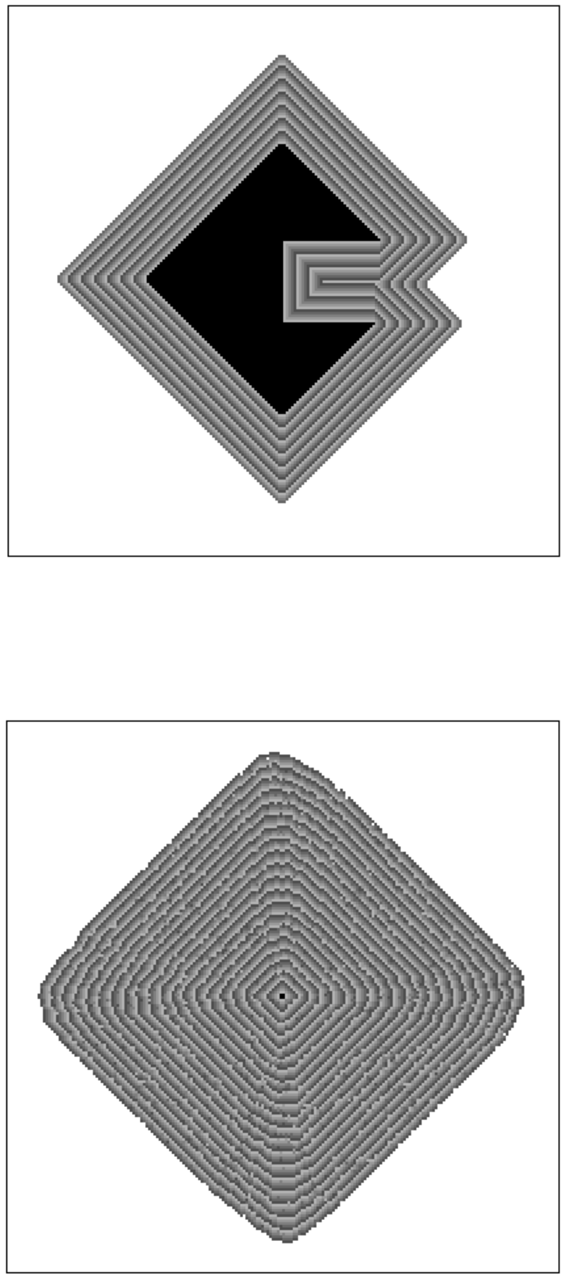
Theorem. *Consider a supercritical TGM.*

(1) Assume that K_0 consists of isolated points. Then, for p close enough to 1, $L_p = L_1$.

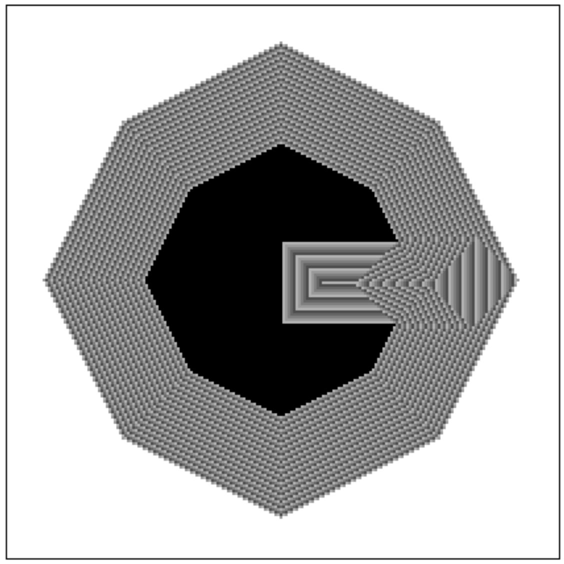
(2) Conversely, if K_0 includes a line segment, then $L_p \neq L_1$ for every $p < 1$.

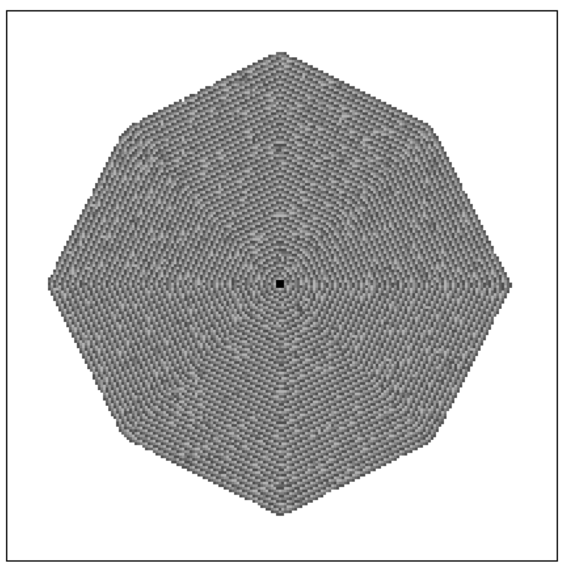
In case (1) A_t is logarithmically close to L_1, in the sense that, a.s., $L_1 \subset (A_t + C \log t) + B_\infty(0, 1/2)$ eventually in t.

Moore neighborhood with $\theta = 3$ TGM falls into case (1).



Moore neighborhood, $\theta = 2$. Top figure: perturbation of the invariant set in the determnistic TGM. Bottom figure: random TGM with $p = 0.9$.





Moore neighborhood, $\theta = 3$. Top figure: perturbation of the invariant set in the determnistic TGM. Bottom figure: random TGM with $p = 0.9$.

Reasons for exact stability.

If K_0 consists of isolated points, then the deterministic TGM is able to fix (or "erode") any finite perturbation. Therefore, the random dynamics can be favorably compared to a Toom rule.

Another approach is the analysis of K_{1/w_p}, for p close to 1. As K_{1/w_p} is not known to exist, the following statements have to be interpreted as appropriate bounds.

The corners of K_0 remain fixed, and otherwise K_{1/w_p} is close to $K_{1/w}$. In fact, the slopes of the boundary of K_{1/w_p} are close to the boundary of $K_{1/w}$ near K_0.

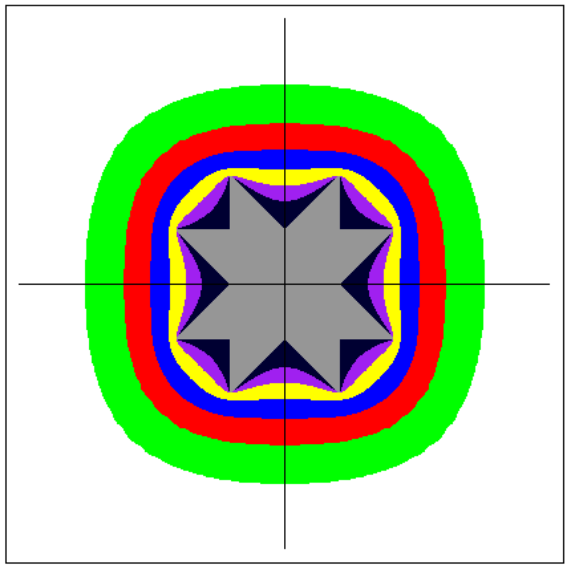
Any half–space corresponding to a flat edge of L_1 has the following property: a bump on its boundary expands faster than tL_1. Therefore such a half–space expands similarly to the following one–dimensional interface dynamics $h_t : \mathbf{Z} \to \mathbf{Z}$.

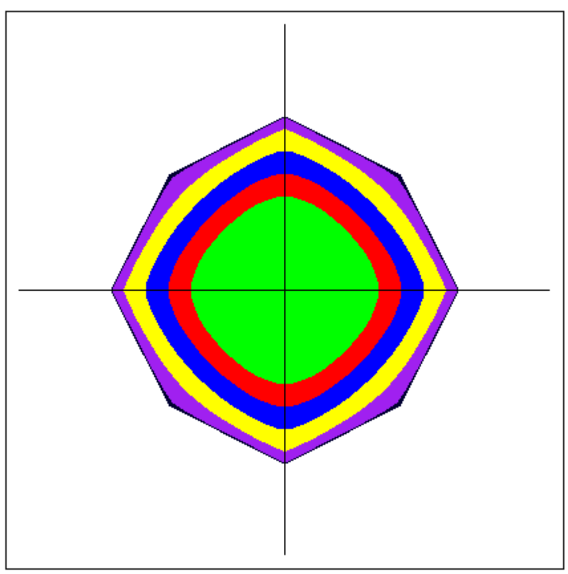
Toss, independently for every x and t, a p–coin. If the toss is unsuccessful, $h_t(x+1) = h_t(x) - 1$, otherwise,

- If $h_t(x) = h_t(x-1) < h_t(x-2)$, then $h_{t+1}(x) = h_t(x) + 1$.
- Otherwise, $h_t(x) = \max\{h_t(x-1), h_t(x)\}$.

Fix an $\epsilon > 0$ and assume that the initial interface has slope $-\alpha \leq 0$. Then, for p close to 1, h_t advances with speed $\geq (1-\epsilon)2\alpha$. (Note that this is true for $\alpha = 0$ due to the oriented percolation comparison.)

Appropriate rescaling translates into the required slope condition.





K_{1/w_p} and L_p for $p = 1, \ldots, 0.4$.

Approximations for small update probabilities.

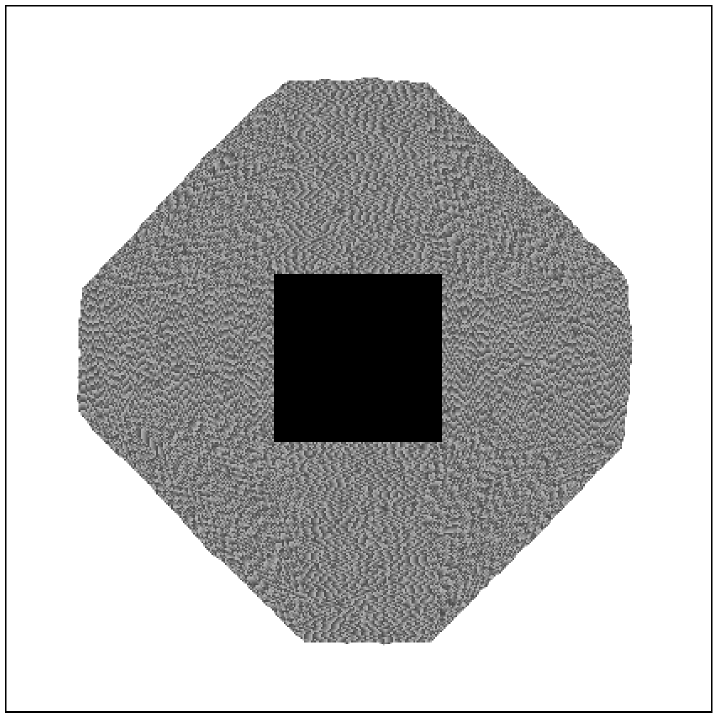
Consider the Moore $\theta = 3$ case, but with $p_3 = p$, $p_4 = 1$. The exact stability near $p = 1$ is obviously implied, but what is the geometry for $p \to 0$? (A similar problem was introduced by Kesten–Schonmann, 1995.)

Coordinate and diagonal directions u are the only ones for which $w_p(u) \to 0$. Those are, therefore, the only ones likely to be "seen" in the limit. In fact, the following precise result can be proved.

Theorem. *As* $p \to 0$, $p^{-1/2} L_p \to \{x \in \mathbf{R}^2, ||x||_1 \le \sqrt{2}\}$.

The limiting set can be precisely identified because an appropriately rescaled version of this rule approximates Hammersley's process, and longest increasing subsequence in a random perturbation of $\{1, \ldots, n\}$ is known to be $\sim 2\sqrt{n}$ (Logan–Shepp, Vershik–Kerov, 1977; Aldous–Diaconis, 1995).

Also, analysis of K_{1/w_p} shows that L_p must have *at least one corner* for small p. (A flat edge is very unlikely, but not ruled out at this point.)



TGM from the previous page, with $p_3 = 0.01$, started from 100×100 square, run until time 1050.

A TGM for which all the answers are known?

Almost. Consider the TGM with

$$x + \mathcal{N} = \begin{array}{cc} \bullet & x \\ & \bullet \end{array}$$

$\theta = 1$, $p_1 = p$, and $p_2 = 1$. This TGM is called *oriented digital boiling (ODB).*

This model (Seppäläinen 1998; Johansson, 1999; G–Tracy–Widom, 2000) is, in a sense, explicitly solvable when $A_0 = \{(x, y) \in \mathbf{Z}^2 : x \geq 0, y \leq -x\}$, yielding explicit asymptotic shape L'_p given by $\{(x, y) \in \mathbf{R}^2 : x \geq 0, y \leq c_1(x)\}$, where $c_1 : [0, \infty) \to \mathbf{R}$ as follows:

$$c_1(\gamma) = \begin{cases} 2(1-\gamma)p - p + 2\sqrt{p(1-\gamma)(1-p)\gamma}, & \text{if } \gamma < 1 - p, \\ 1 - \gamma, & \text{if } \gamma > 1 - p. \end{cases}$$

In fact, fluctuations about this shape are known.

Fluctuations.

Let A_t be given by $\{y \le h_t(x)\}$. There are four asymptotic regimes.

(1) *Square–root regime.* Keep x fixed and let $t \to \infty$. Then

$$\frac{h_t(x) - pt}{\sqrt{p(1-p)t}} \xrightarrow{d} M_x,$$

a Brownian functional.

(2) *Universal regime.* Assume $\gamma = x/t < 1 - p$ and let

$$\begin{aligned} c_2 = & \left(\gamma(1-\gamma) \right)^{1/6} \left(p(1-p) \right)^{1/2} \\ & \left(1 + \sqrt{(1-p)\gamma p^{-1}(1-\gamma)^{-1}} \right)^{2/3} \\ & \left(\sqrt{(1-\gamma)\gamma^{-1}} - \sqrt{p(1-p)^{-1}} \right)^{2/3}. \end{aligned}$$

Then

$$P\left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} \le s \right) \to F_2(s),$$

where

$$F_2(s) = \exp\left(- \int_s^\infty (x-s) q(x)^2 \, dx \right)$$

and q solves

$$q'' = sq + 2q^3, \quad q \sim Ai(s) \text{ as } s \to \infty.$$

(3) *Critical regime.* If $x/t = (1-p) + o(t^{-1/2})$, then $h_t(x) - (t-x)$ converges in distribution.

(4) *Deterministic regime.* If $\gamma = x/t > 1-p$ then $P(h_t(x) = t-x)$ converges to 1 exponentially fast.

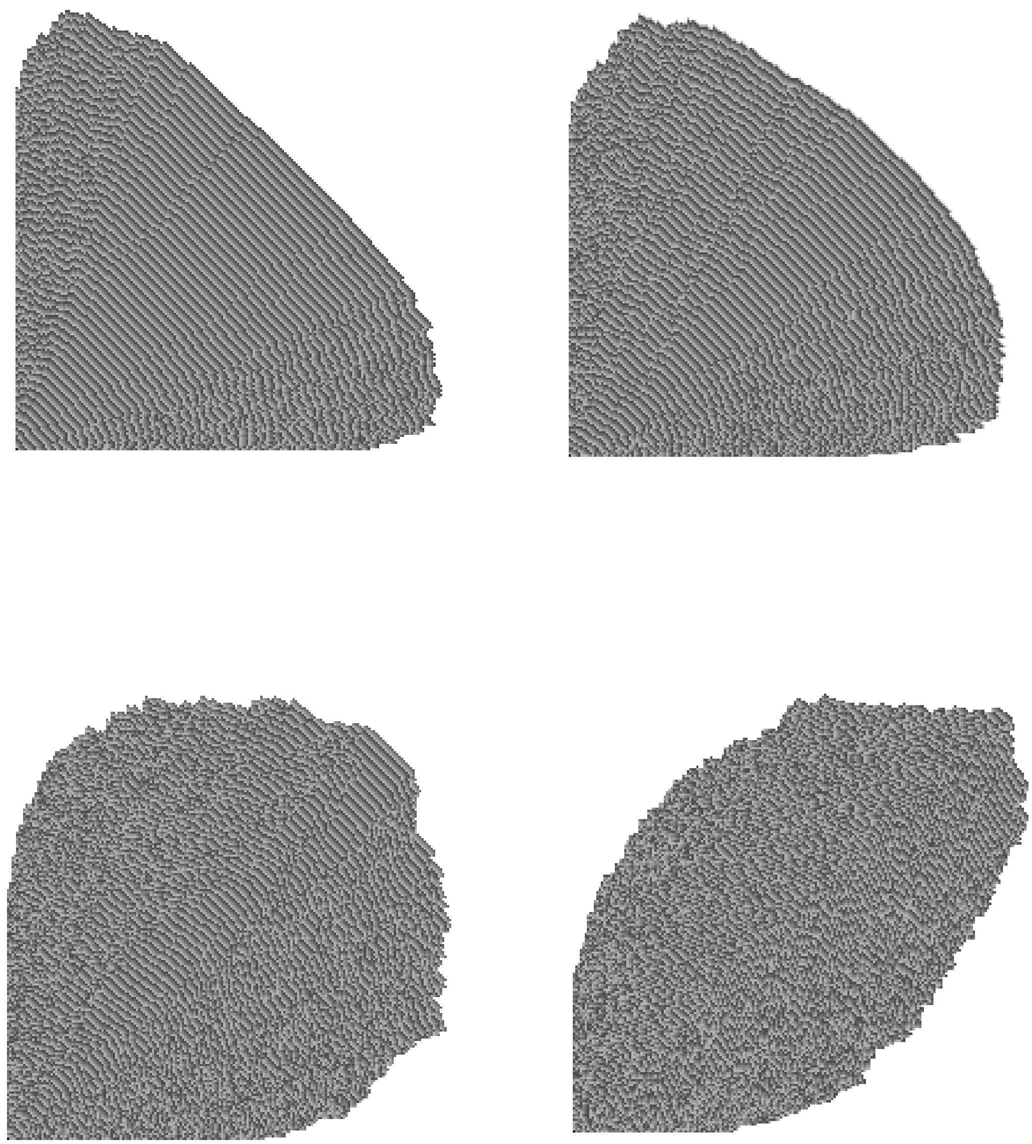
Shape of ODB from finite sets.

Assume (without loss of generality) that $A_0 = \{0\}$.

Then the shape L_p is simply given as the intersection of L'_p and L'_p reflected over the line $y = x$. Therefore,

- L_p has a flat edge, but no corners, when $p > 1/2$,
- L_p is smooth when $p = 1/2$,
- L_p has a corner (in the diagonal direction), but no flat edge when $p < 1/2$,
- $p^{-1/2} L_p \to \{(x, y) \in \mathbf{R}^2 : x^2 \leq 4y(1-y), y^2 \leq 4x(1-x)\}$ as $p \to 0$.

The fluctuations can be determined only when $\gamma = x/t < p$, thus we have complete description of fluctuations when $p > 1/2$. It not clear what happens in the middle directions when $p \leq 1/2$.



ODB shapes for $p = 0.6, 0.5, 0.3, 0.1$.