(ALMOST) SPECIAL HOLONOMY & M-THEORY

- hep-th/0203060, Nucl. Phys. B638 (2002) 186.

+ other works by

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M-THEORY AND STRING THEORY

- M-theory is an 11-dimensional theory that is believed to arise as a strong-coupling regime for 10-dimensional string theory.

- The simplest manifestation of this can be seen in the "low-energy limit," which is supergravity.

- The bosonic sector of 11-dimensional supergravity is comprised of Gravity + 4-form field, and described by an action principle $I = \int \hat{\mathcal{L}}$,

$$\boxed{\hat{\mathcal{L}} = \hat{R} \,\hat{*} 1 - \tfrac{1}{2} \hat{*} \hat{F}_4 \wedge \hat{F}_4 + \tfrac{1}{6} \hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3}$$

$$\hat{F}_4 = d\hat{A}_3$$

- If the 11-dimensional fields have a U(1) isometry $\frac{\partial}{\partial z}$, the theory becomes effectively 10-dimensional, reduced according to the "Kaluza-Klein ansatz"

$$\boxed{\begin{aligned} d\hat{s}^2 &= e^{-\frac{1}{6}\phi} ds^2 + e^{\frac{4}{3}\phi} (dz + A_1)^2 \\ \hat{A}_3 &= A_3 + dz \wedge A_2 \end{aligned}}$$

ds^2, A_1, ϕ, A_3, A_2 all independent of z

This gives the 10-dimensional reduced Lagrangian

$$\boxed{\begin{aligned} \mathcal{L} = R *1 &- \tfrac{1}{2} *d\phi \wedge d\phi - \tfrac{1}{2} e^{\frac{3}{2}\phi} *F_2 \wedge F_2 - \tfrac{1}{2} e^{-\phi} *F_3 \wedge F_3 \\ &- \tfrac{1}{2} e^{\frac{1}{2}\phi} *F_4 \wedge F_4 + \tfrac{1}{2} dA_3 \wedge dA_3 \wedge A_2 \end{aligned}}$$

Type IIA Supergravity

$$F_4 = dA_3 - dA_2 \wedge A_1 \qquad F_3 = dA_2 \qquad F_2 = dA_1$$

LARGE CIRCLE = STRONG COUPLING

- The dilaton field ϕ has exponential couplings to the form fields. In the Type IIA string, the potential A_2 couples directly to the string worldsheet, and the inverse prefactor of the kinetic term for $F_3 = dA_2$ acquires the interpretation of the "string coupling constant" g_{string}

$$\mathcal{L} = \cdots - \tfrac{1}{2} e^{-\phi} * F_3 \wedge F_3 \quad \Rightarrow \quad \boxed{g_{string} = e^{\phi}}$$

- In the Kaluza-Klein metric reduction,

$$d\hat{s}^2 = e^{-\frac{1}{6}\phi} ds^2 + e^{\frac{4}{3}\phi} (dz + A_1)^2,$$

the radius of the circle is $\boxed{R = e^{\frac{2}{3}\phi}}$

- Hence we have $\boxed{R = g_{string}^{2/3}}$

- Weak coupling, $g_{string} \to 0$, is the perturbative regime, for which we see $R \to 0$, and hence perturbative string theory is 10-dimensional.

- Strong coupling, $g_{string} \to \infty$, is the non-perturbative regime, for which $R \to \infty$. The non-perturbative regime gives rise to 11-dimensional M-theory.

REDUCTION OF M-THEORY SOLUTIONS WITH U(1) ISOMETRIES

- In view of the M-theory/string theory relation, it becomes of interest to study M-theory solutions where there is a U(1) isometry, since these admit a re-interpretation in string theory.

- Especially interesting are solutions which are supersymmetric (admit Killing spinors); i.e. solutions with special holonomy.

- Amongst these, two classes of example are:

1) $$(\text{Minkowski})_3 \times (\text{Spin } 7 \text{ manifold})$$

2) $$(\text{Minkowski})_4 \times (G_2 \text{ manifold})$$

- An explicit family of Spin 7 metrics with a U(1) isometry whose radius stabilises asymptotically is known. ALC

- Analogous families of G_2 metrics with stabilising U(1) isometries also exist.

Spin 7 METRICS ON CHIRAL SPIN BUNDLE OF S^4

- Isolated example of a complete Spin 7 metric, which is Asymptotically Conical (cone over S^7) was found by Bryant & Salamon, and Gibbons, Page & Pope.

- Recently, a 1-parameter family of complete Spin 7 metrics for the same topology was found — Asymptotically Locally Conical (asymptotic to circle bundle over cone over $\mathbb{CP}^3$). Cvetič, Gibbons, Lü & Pope. hep-th/0103155 Nucl. Phys. B620 (2002) 29 math.DG/0105119

The Construction: $\boxed{S^7 \cong SO(5)/SO(3)}$

- Left-invariant 1-forms of $SO(5)$: $L_{AB} = -L_{BA}$

$$dL_{AB} = L_{AC} \wedge L_{CB} \qquad 0 \leq A \leq 4$$

$A = (a, 4)$ $\qquad \boxed{L_{ab} \quad SO(4) \text{ subgroup}} = SU(2)_L \times SU(2)_R \qquad 1 \leq i \leq 3$

$$R_i = \tfrac{1}{2}(L_{0i} + \tfrac{1}{2}\epsilon_{ijk} L_{jk}) \quad SU(2)_R$$

$$L_i = \tfrac{1}{2}(L_{0i} - \tfrac{1}{2}\epsilon_{ijk} L_{jk}) \quad SU(2)_L$$

$$\boxed{P_a \equiv L_{a4} \qquad SO(5)/SO(4) = S^4 \text{ Coset}}$$

- Cohomogeneity one metrics with S^7 principal orbits:

$$\boxed{ds_8^2 = dt^2 + 4a^2(R_1^2 + R_2^2) + 4b^2 R_3^2 + c^2 P_a^2}$$

- Associative 4-form (self-dual):

$$\boxed{\Phi = -e^0 \wedge e^1 \wedge e^2 \wedge e^3 - e^8 \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + \tfrac{1}{2}\epsilon_{ijk}\, e^{\hat{i}} \wedge e^{\hat{j}} \wedge \hat{J}^k + e^8 \wedge e^{\hat{i}} \wedge \hat{J}^i}$$

$$e^8 = dt, \quad e^{\hat{1}} = 2aR_1, \quad e^{\hat{2}} = 2aR_2, \quad e^{\hat{3}} = 2bR_3 \qquad e^a = cP_a$$

$$\hat{J}^i = c^2(P_0 \wedge P_i + \tfrac{1}{2}\epsilon_{ijk} P_j \wedge P_k)$$

- The metric has Spin 7 holonomy if $d\Phi = 0$, which implies

$$\boxed{\dot{a} = 1 - \frac{b}{2a} - \frac{a^2}{c^2}\,, \qquad \dot{b} = \frac{b^2}{2a^2} - \frac{b^2}{c^2}\,, \qquad \dot{c} = \frac{a}{c} + \frac{b}{2c}}$$

- These equations can be solved explicitly, giving the general result

$$\boxed{ds_8^2 = \frac{v f\, dz^2}{4z(1-z^2)(1-z)(v-2)} + \frac{4(v-2)zf}{(1+z)v}\left(R_1^2 + R_2^2\right) + \frac{16(v-2)zf}{(1+z)v^3} R_3^2 + f P_a^2}$$

"B_8 Metrics"

Cvetic, Gibbons, Lü, Pope

where $$v = \frac{2k\sqrt{z}}{(1-z^2)^{1/4}} - 2z\; {}_2F_1\!\left(1, \tfrac{1}{2}; \tfrac{5}{4}; 1-z^2\right)$$

$$f = \left(\frac{1+z}{1-z}\right)^{1/2} \exp \int^z \frac{dz'}{v(z')(1-z'^2)}$$

- Detailed analysis shows that this metric is extendable on the chiral spin bundle of S^4. The non-trivial parameter k determines the radius of a stabilising circle (R_3 direction) at large distance, relative to the size of the degenerate S^4 orbit at short distance.

- The radius of the stabilising circle can be adjusted freely;

$$0 \leq \text{Radius} \leq \infty$$

$\uparrow$ Gromov-Hausdorff Limit $\equiv$ Weak Coupling

$\uparrow$ Original Asymptotically Conical metric

- In the Gromov-Hausdorff limit, the metric becomes S^1 times the G_2 metric (AC) on R^3 bundle over S^4.

$$(\text{Minkowski})_3 \times (\text{Spin } 7) \longrightarrow (\text{Minkowski})_3 \times G_2$$

M-theory String theory

Remarks:

- One member of the 1-parameter family of Spin 7 metrics has a simple form:

$$\boxed{ds_8^2 = \frac{(r-\ell)^2\,dr^2}{(r-3\ell)(r+\ell)} + \frac{4\ell^2(r-3\ell)(r+\ell)}{(r-\ell)^2}R_3^2 + (r-3\ell)(r+\ell)(R_1^2+R_2^2) + \tfrac{1}{2}(r^2-\ell^2)P_a^2}$$

(ℓ = trivial scale constant) $\qquad r \geq 3\ell$

$$\boxed{\frac{\text{Radius of } S^1 \text{ at infinity}}{\text{Radius of } S^4 \text{ at } r=3\ell} = 1}$$

- Looked at "from the other side" $\quad -\infty \leq r \leq -\ell$

this metric extends smoothly to a different manifold, with topology $\mathbb{R}^8$. $\qquad$ "$\mathbb{A}_8$ Metric"

The $\mathbb{A}_8$ metric is isolated, — not a 1-parameter family.

Asymptotically Locally Conical G_2 metrics

- There exist similar 1-parameter families of Asymptotically Locally Conical metrics in seven dimensions, with G_2 holonomy.

- Of particular interest are G_2 metrics of cohomogeneity one, and $S^3 \times S^3$ principal orbits.

σ_i , Σ_i Left-invariant 1-forms for two copies of $SU(2) \simeq S^3$

- Consider the metrics

$$\boxed{ds_7^2 = dt^2 + a^2\left[(\Sigma_1 + g\sigma_1)^2 + (\Sigma_2 + g\sigma_2)^2\right] + b^2\left[(\Sigma_1 - g\sigma_1)^2 + (\Sigma_2 - g\sigma_2)^2\right] + c^2(\Sigma_3 - \sigma_3)^2 + f^2(\Sigma_3 + g_3\sigma_3)^2}$$

$e^0 = dt \quad e^1 = a(\Sigma_1 + g\sigma_1) \quad e^2 = a(\Sigma_2 + g\sigma_2) \quad e^3 = c(\Sigma_3 - \sigma_3)$

$e^4 = b(\Sigma_1 - g\sigma_1) \quad e^5 = b(\Sigma_2 - g\sigma_2) \quad e^6 = f(\Sigma_3 + g_3\sigma_3)$

Associative 3-form:

$$\boxed{\Phi = e^0 \wedge (e^1\wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6) - (e^1\wedge e^2 - e^4 \wedge e^5)\wedge e^3 + (e^1 \wedge e^5 - e^2 \wedge e^4)\wedge e^6}$$

$$d\Phi = 0, \quad d{*\Phi} = 0 \quad \Rightarrow \quad G_2 \text{ holonomy}$$

$$\dot{a} = \frac{c^2(a^2-b^2) + [4a^2(a^2-b^2) - c^2(5a^2-b^2) - 4abcf]g^2}{16a^2bcg^2}$$

$$\dot{b} = -\frac{c^2(a^2-b^2) + [4b^2(a^2-b^2) + c^2(5b^2-a^2) - 4abcf]g^2}{16ab^2cg^2}$$

$$\dot{c} = \frac{c^2 + (c^2 - 2a^2 - 2b^2)g^2}{4abg^2}$$

$$\dot{f} = -\frac{(a^2-b^2)[4abf^2g^2 - c(4abc + a^2f^2 - b^2f^2)(1-g^2)]}{16a^3b^3g^2}$$

$$\dot{g} = -\frac{c(1-g^2)}{4abg}$$

THE $\mathbb{D}_7$ METRICS

- INCLUDED WITHIN THESE G_2 METRICS IS SUBSET MORE SIMPLY WRITTEN AS

$$\boxed{ds_7^2 = dt^2 + a^2\left[(\Sigma_1 + g\sigma_1)^2 + (\Sigma_2 + g\sigma_2)^2\right] + b^2(\sigma_1^2 + \sigma_2^2) + c^2(\Sigma_3 + g_3\sigma_3)^2 + f^2\sigma_3^2}$$

WITH THE G_2 HOLONOMY CONDITIONS

$$\boxed{\begin{array}{l|l} g = -\dfrac{af}{2bc}\,, & g_3 = -1 + 2g^2 \\ \hline \dot{a} = -\dfrac{c}{2a} + \dfrac{a^5 f^2}{8b^4c^3}\,, & \dot{b} = -\dfrac{c}{2b} - \dfrac{a^2(a^2-3c^2)}{8b^3c^3}f^2 \\ \hline \dot{c} = -1 + \dfrac{c^2}{2a^2} + \dfrac{c^2}{2b^2} - \dfrac{3a^2f^2}{8b^4}\,, & \dot{f} = -\dfrac{a^4 f^3}{4b^4c^3} \end{array}}$$

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- THE ISOMETRY GROUP IS $\underbrace{SU(2)\times SU(2)}_{\text{Manifest left-invariance}} \times \underbrace{U(1)}_{\text{right action}}$

THE $U(1)$ ISOMETRY, WHICH WILL FORM THE M-THEORY CIRCLE, IS GENERATED BY $\boxed{K = \dfrac{\partial}{\partial\psi} + \dfrac{\partial}{\partial\tilde{\psi}}}$

right-acting $U(1)$'s in the two $SU(2)$'s.

- WE CANNOT SOLVE THE FIRST-ORDER EQUATIONS EXPLICITLY, BUT BY SERIES EXPANSION AT SHORT DISTANCE + NUMERICAL METHODS WE FIND PERSUASIVE EVIDENCE FOR A 1-PARAMETER FAMILY OF ALC SOLUTIONS:

SHORT DISTANCE :

$$\boxed{\begin{aligned} a &= \tfrac{1}{2}t + \cdots \\ b &= 1 + \cdots \\ c &= -\tfrac{1}{2}t + \cdots \\ f &= q + \cdots \end{aligned}}$$

- THE PARAMETER q IS FREELY ADJUSTABLE. NUMERICAL INTEGRATION SHOWS WE GET REGULAR NON-SINGULAR METRICS IF $0 < q \leq 1$.

- THE FUNCTION f BECOMES CONSTANT AT LARGE DISTANCE

$|K|^2 = f^2 + c^2(1+g_3)^2 \Rightarrow |K|$ EVERYWHERE FINITE AND NON-ZERO

- IN FACT $|K|$ INCREASE MONOTONICALLY FROM $\boxed{|K| = q}$ AT $t=0$ TO $\boxed{|K| = f(t=\infty)}$ AT $t=\infty$

- WE HAVE A 1-PARAMETER FAMILY OF ALC METRICS, WITH $S^3 \times S^3$ PRINCIPAL ORBITS; A DEGENERATE SQUASHED S^3 ORBIT AT $t=0$ $(ds^2 \sim \sigma_1^2 + \sigma_2^2 + q^2 \sigma_3^2)$; AND THEY ARE ASYMPTOTIC TO A CIRCLE BUNDLE OVER THE CONE OF $T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$.

Analogous to the exactly solved B_8 ALC metrics in 8 dims.

$q=0$ —— ALC —— $q=1$

GROMOV-HAUSDORFF LIMIT:

$S^1 \times \underbrace{\text{RESOLVED CONIFOLD}}$

$\mathbb{R}^4$ BUNDLE OVER S^2

"SMALL RESOLUTION" OF THE CONE OVER $T^{1,1}$

– CALABI-YAU

AC LIMIT

BRYANT-SALAMON

$\mathbb{R}^4$ BUNDLE OVER S^3

- IT IS NOW OF INTEREST TO PERFORM A KALUZA-KLEIN REDUCTION ON THE CIRCLE GENERATED BY $|K|$. (M-THEORY $\to$ TYPE IIA STRING)

KALUZA-KLEIN REDUCTION ON ISOMETRY CIRCLE

$$\boxed{d\hat{s}_7^{\,2} = e^{-2\phi}\, ds_6^2 + e^{8\phi}\,(dz + A)^2}$$

$K = \frac{\partial}{\partial z}$ ds_6^2, ϕ, $A = A_i\,dx^i$ all independent of z

- THE 7-DIMENSIONAL RICCI-FLAT CONDITION $\hat{R}_{AB} = 0$ BECOMES, IN TERMS OF THE 6-DIMENSIONAL QUANTITIES,

$$R_{ab} = 20\,\partial_a\phi\,\partial_b\phi + \frac{1}{2}\, e^{12\phi}\left(F_{ac}F_b{}^c - \frac{1}{8}F^2 g_{ab}\right)$$

$$d{*}d\phi = -\frac{3}{20}\, e^{12\phi} * F\wedge F$$

$$d(e^{12\phi} * F) = 0$$

$$\boxed{F = dA}$$

- THE ASSOCIATIVE 3-FORM $\hat{\Phi}_3$ FOR THE G_2 METRIC BECOMES

$$\boxed{\hat{\Phi}_3 = \Phi_3 + \Phi_2\wedge(dz + A)}$$

Φ_2, Φ_3 independent of z

- $d\hat{\Phi}_3 = 0 \Rightarrow$ $\boxed{d\Phi_2 = 0,}$ $\boxed{d\Phi_3 = -\Phi_2\wedge F}$

- $d\hat{*}\hat{\Phi}_3 = 0 \Rightarrow$ $\boxed{d(e^{4\phi} * \Phi_3) = 0,}$ $\boxed{d(e^{-6\phi} * \Phi_2) = e^{4\phi} * \Phi_3\wedge F}$

- IN GENERAL, THE CIRCLE REDUCTION OF A G_2 METRIC GIVES A SOLUTION OF EINSTEIN-MAXWELL-DILATON SYSTEM; THE 6-METRIC IS NOT RICCI-FLAT

- $\hat{\Phi}_{ABE}\,\hat{\Phi}_{CDE} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + \frac{1}{6}\,\epsilon_{ABCDE_1E_2E_3}\,\hat{\Phi}_{E_1E_2E_3}$

IMPLIES 1) $\hat{J}_{ab} \equiv \hat{\Phi}_{ab7}$ satisfies $\boxed{\hat{J}_{ab}\hat{J}_{bc} = -\delta_{ac}}$ Almost Complex Structure

2) $\hat{\Phi}_{abc} = \mathrm{Re}(e^{i\gamma}\epsilon_{abc})$

Holomorphic 3-form γ = constant phase

GROMOV- HAUSDORFF LIMIT

- IN THIS LIMIT THE CIRCLE BUNDLE BECOMES TRIVIAL

$$A \to 0 \qquad \phi \to \text{constant}$$

$$\boxed{\underbrace{d\hat{s}_7^{\,2} = ds_6^{\,2} + dz^2}_{Y_6 \times S^1}} \qquad \text{DIRECT SUM METRIC}$$

$$\boxed{R_{ab} = 0} \qquad Y_6 \text{ IS RICCI - FLAT}$$

- $$\hat{\Phi}_3 = \Phi_3 + \Phi_2 \wedge dz$$

$$d\hat{\Phi}_3 = 0 \quad \Rightarrow \quad \boxed{d\Phi_3 = 0 \qquad d\Phi_2 = 0}$$

$$d\hat{*}\hat{\Phi}_3 = 0 \quad \Rightarrow \quad \boxed{d*\Phi_3 = 0 \qquad d*\Phi_2 = 0}$$

- $J \equiv \Phi_2$ SATISFIES $\boxed{J^2 = -1\ , \quad dJ = 0}$

$\Omega_3 \equiv e^{-i\gamma}(\Phi_3 - i*\Phi_3)$ HOLOMORPHIC 3-FORM

SATISFIES $\boxed{d\Omega_3 = 0}$

- So Y_6 IS RICCI - FLAT, KÄHLER — CALABI-YAU

NOT QUITE THE GROMOV-HAUSDORFF LIMIT

- SUPPOSE NOW WE CONSIDER A "NEAR GROMOV-HAUSDORFF LIMIT" WHERE A IS TAKEN SMALL, OF ORDER ϵ, AND WE NEGLECT $O(\epsilon^2)$ TERMS.

- THE 6-DIMENSIONAL EQUATIONS BECOME

$$\boxed{R_{ab}=0}, \qquad \boxed{dF_2 = 0 = d*F_2} \qquad \boxed{\phi = \text{CONST.}}$$

$$\boxed{\begin{array}{c} d\Psi_2 = 0 \,, \\ \hline d\Psi_3 + \Psi_2 \wedge F_2 = 0 \end{array}} \quad \boxed{\begin{array}{c} d*\Psi_2 = *\Psi_3 \wedge F_2 \\ \hline d*\Psi_3 = 0 \end{array}}$$

- SO Y_6 IS $\boxed{\text{RICCI - FLAT}}$

- $J = \Psi_2$ SATISFIES $dJ=0 \Rightarrow$ $\boxed{\text{ALMOST KÄHLER}}$

- $d*\Psi_3 = 0$ BUT $d\Psi_3 \neq 0$, SO $\Omega_3 \equiv e^{-i\gamma}(\Psi_3 - i*\Psi_3)$ IS NOT CLOSED — Y_6 is <u>NOT</u> <u>KÄHLER</u>.

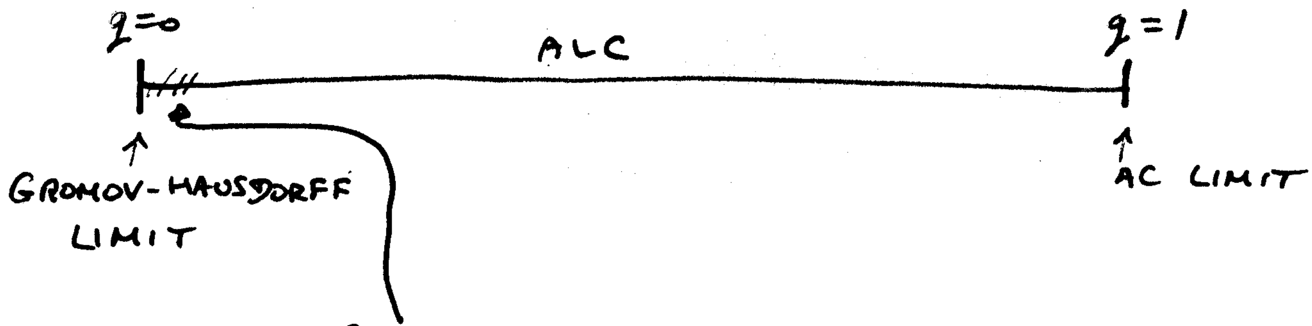
- AT THIS LINEARISED LEVEL, WE HAVE A DEFORMATION OF THE ORIGINAL RICCI-FLAT KÄHLER MANIFOLD THAT KEEPS IT RICCI-FLAT, BUT NOT KÄHLER.

- CONVERSELY, WE CAN REVERSE THE PROCEDURE, AND LIFT ~~...~~ A 6-METRIC SATISFYING THE ABOVE CONDITIONS TO GIVE A G_2 METRIC IN $D=7$.

- IF THE LENGTH OF THE CIRCLE IN THE G_2 METRIC IS EVERYWHERE FINITE AND NON-ZERO, WE GET A <u>NON-SINGULAR</u> PERTURBED METRIC IN 6 DIMENSIONS

EXISTENCE OF THE D_7 METRICS

- RECALLING THAT SO FAR WE ONLY HAVE A NUMERICAL "PROOF" OF THE EXISTENCE OF THE 1-PARAMETER FAMILY OF NON-SINGULAR D_7 METRICS, WE CAN NOW USE THE "LIFT" OF THE DEFORMED 6-DIMENSIONAL METRICS TO GIVE US A DEMONSTRATION OF THE EXISTENCE OF SMOOTH G_2 METRICS ON D_7 FOR SMALL VALUES OF THE PARAMETER q.



G_2 METRICS IN THIS REGION ARE EXPLICITLY COMPUTABLE, BY FIRST FINDING THE LINEARISED PERTURBATION OF THE RESOLVED CONIFOLD IN $D=6$ THAT MAKES IT ALMOST KÄHLER, RICCI FLAT, AND THEN LIFTING TO $D=7$.

ALMOST G_2 METRICS FROM SPIN 7

- WE CAN GIVE AN ANALOGOUS ANALYSIS FOR SPIN 7 METRICS WITH A CIRCLE ACTION:

$$\boxed{\begin{aligned} d\hat{s}_8^2 &= e^{-2\phi} ds_7^2 + e^{10\phi}(dz+A)^2 \\ \hat{\Phi}_4 &= \Phi_3 \wedge (dz+A) + e^{-6\phi} * \Phi_3 \end{aligned}}$$

Calibrating 4-form $\Rightarrow$ $\hat{\Phi}_4$; $\Leftrightarrow$ $\hat{\Phi}_4 = \hat{*}\hat{\Phi}_4$

- THE EQUATIONS FROM SPIN-7 HOLONOMY IN $D=8$, I.E. $d\hat{\Phi}_4 = 0$ AND HENCE $\hat{R}_{AB} = 0$, IMPLY THE $D=7$ EQUATIONS

$F = dA$

$$\boxed{d\Phi_3 = 0 \qquad d(e^{-6\phi} * \Phi_3) = \Phi_3 \wedge F}$$

$$\boxed{\begin{aligned} R_{ab} &= 30\,\partial_a\phi\,\partial_b\phi + \tfrac{1}{2} e^{14\phi}\left(F_{ac}F_b{}^c - \tfrac{1}{10}F^2 g_{ab}\right) \\ d*d\phi &= \tfrac{7}{60} e^{14\phi} * F\wedge F \\ d(e^{14\phi} * F) &= 0 \end{aligned}}$$

- GROMOV-HAUSDORFF LIMIT: $A=0$, $\phi = \text{constant}$

$\rightarrow$ $\boxed{d\Phi_3 = 0, \quad d*\Phi_3 = 0, \quad R_{ab} = 0}$ $\quad d\hat{s}_8^2 = ds_7^2 + dz^2$

G_2 Holonomy $\times$ S^1

- NEAR GROMOV-HAUSDORFF LIMIT: $A = O(\epsilon)$

WORK TO LINEAR ORDER IN ϵ:

$$\boxed{\begin{aligned} &d\Phi_3 = 0 \qquad d*\Phi_3 = \Phi_3\wedge F \qquad \phi = \text{const.} \\ &R_{ab} = 0 \end{aligned}}$$

"ALMOST G_2 METRIC"

- ANALOGOUSLY TO THE G_2/CALABI-YAU CASE, HERE WE HAVE A 1-1 CORRESPONDENCE, AT THE LEVEL OF LINEARISED PERTURBATIONS, BETWEEN

$$\boxed{\text{ALMOST } G_2 \text{ METRICS}} \underset{\text{REDUCE}}{\overset{\text{LIFT}}{\rightleftarrows}} \boxed{\text{SPIN-7 METRICS}}$$

- IF THE KILLING VECTOR K GENERATING THE CIRCLE ACTION HAS EVERYWHERE FINITE AND NON-ZERO LENGTH, THEN WE GET NON-SINGULAR ALMOST G_2 METRICS.

- IN OUR ORIGINAL EXAMPLES OF $\mathbb{B}_8$ ALC METRICS, $|K| \to 0$ AT SHORT DISTANCE.

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- WE ALSO FOUND ("NUMERICAL PROOF") A FAMILY OF ALC METRICS WITH $\frac{SU(3)}{U(1)}$ PRINCIPAL ORBITS, AND A (SQUASHED) S^5 DEGENERATE ORBIT AT SHORT DISTANCE. FOR THIS FAMILY, WE HAVE $|K|$ FINITE AND NON-SINGULAR EVERYWHERE.

$\varrho=0$ —— ALC —— $\varrho=1$

AC LIMIT

GROMOV-HAUSDORFF LIMIT

$= \underbrace{S^1 \times \mathbb{R}^3 \text{ bundle over } \mathbb{CP}^2}_{\text{Bryant, Salamon}}$

- WE HAVE EXPLICITLY SOLVED FOR NON-SINGULAR ALMOST-G_2 PERTURBATIONS OF THE BRYANT-SALAMON METRIC

$\Rightarrow$ LIFTS TO GIVE SPIN-7 NON-SINGULAR METRICS NEAR TO THE GROMOV-HAUSDORFF LIMIT

REMARKS

- Studying the almost-Kähler or almost-G_2 Ricci-flat deformations of Calabi-Yau or G_2 manifolds allows us to find analytic linearised "thickening out" of G_2 or Spin-7 families of ALC metrics, close to the Gromov-Hausdorff limit.

- The Ricci-flatness of the almost-special holonomy deformations holds only at the linearised level — at the quadratic order, $R_{ab} \neq 0$.

- This accords with a theorem by Sekigawa, in the almost-Kähler case:

  " The only complete, non-singular, ~~non~~ compact, Ricci-flat, almost-Kähler manifolds are Kähler"

  (Extension to non-compact AC metrics straightforward)

- Suggests a "G_2 Sekigawa theorem":

  " The only complete, non-singular, compact (or AC) Ricci-flat, almost-G_2 manifolds are G_2 manifolds"