LECTURES ON G_2-Geometry

— Naichung Conan Leung.

(M^7, g) holonomy $\subset G_2$

- Octonion Geometry.

- M-Theory. $M^7 \times \mathbb{R}^{3,1}$.

$\mathbb{R}$

Oriented Riemannian manifolds

2   3   4   5   ......

[Donaldson-Floer theory,
Chern-Simon theory,
Seiberg-Witten theory, ...] $\mathbb{H}$

$\mathbb{C}$

Calabi-Yau manifolds

4   6   8   10   ......

$8 = 7+1$

CY^3 G_2-mfd

$\mathbb{O}$

[String / M-theory].

$$\begin{tikzcd}
\mathbb{R} & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & 4 \arrow[d] & 5 \arrow[d, "\substack{\text{Special}\\ \text{Lagrangian.}\\ \text{(+ flat U(1)-conn).}}"] & \cdots\cdots & \text{mfd.} \\
\mathbb{C} & 2 & 4 & 6 & 8 & 10 & \cdots\cdots & \text{CY mfd.}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{H} & 3 \arrow[d, swap, "\substack{\text{SLag.}\\ \text{(+flat conn)}}"] & 4 \arrow[d, "\substack{\text{Coassociative.}\\ \text{(+ ASD-conn.)}}"] \\
\mathbb{O} & \substack{6\\ CY^3} & \substack{7+1.\\ G_2\text{-mfd.}}
\end{tikzcd}$$

§ Definition of G_2-manifolds

$$\begin{cases} M^m \text{ manifold (i.e. locally } \mathbb{R}^m) \\ g = \sum g_{ij}(x)\, dx^i \otimes dx^j \end{cases}$$

Riemannian metric.

$\Rightarrow$ $\exists !$ connection ∇

s.t. (i) Torsion $= 0$

$$\begin{tikzcd} \Gamma(T_M^*) \arrow[r, "\nabla"] \arrow[d, equal] & \Omega^1(T_M^*) \arrow[r, "\wedge"] \arrow[d, phantom, "\circlearrowright"] & \Gamma(\Lambda^2 T_M^*) \arrow[d, equal] \\ \Omega^1(M) \arrow[rr, "d"] & & \Omega^2(M) \end{tikzcd}$$

(ii) Metric compatible.

$$\nabla g = 0.$$

$$\nabla g = 0$$

$\Longleftrightarrow$ Parallel transport preserves inner product.

$$\rightsquigarrow \left\{\begin{array}{l}\text{loops} \\ \text{at } p \in M\end{array}\right\} \xrightarrow{\text{hol.}} O(T_pM) \cong O(m).$$

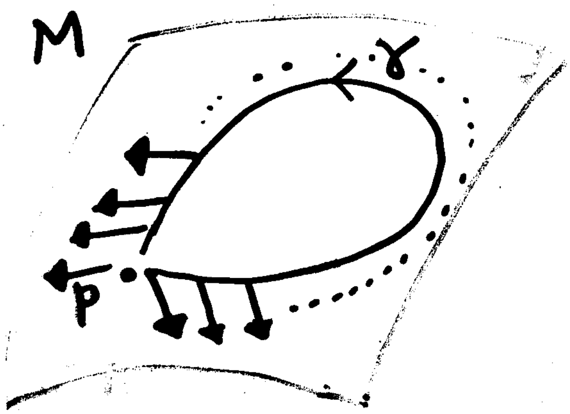

Image $=: \mathrm{Hol}_p(M)$
Holonomy group

Bergen Classifications of Holonomy Groups:

(M, g) non-symmetric

$Hol_p(M) \cong$

$O(m)$	$SO(m)$
$U(n)$ (Kählen)	$SU(n)$ (Calabi-Yau).
$Sp(n)Sp(1)$	$Sp(n)$
$Spin(7)$	G_2.

Manifolds defined over a field $\mathbb{R}$, $\mathbb{C}$.
+ Orientation

$/\mathbb{R}$ (loc. $\mathbb{R}^n$). (M, g) $\begin{cases} \mathrm{Tor}(\nabla) = 0 \\ \nabla g = 0 \end{cases}$

Oriented Riemannian mfd. $\mathrm{Hol} \subset SO(n)$.

$$\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(M).$$

Volume Form.

$/\mathbb{C}$ (loc. $\mathbb{C}^n$) (M, g, J), $J^2 = -I$ $\begin{cases} \mathrm{Tor}\,\nabla = 0 \\ \nabla g = 0 \\ \nabla J = 0 \end{cases}$

Calabi-Yau ~~Kähler~~ mfd. $\mathrm{Hol} \subset SU(n)$.

$$\Omega \in \Omega^{n,0}(M)$$

Holomorphic Volume Form.

Example: $M = \mathbb{C}^n$

(M, g) $\qquad \nabla g = 0$ $\qquad$ Riemannian $O(m)$.

$$g = \sum (dx^j)^2 + (dy^j)^2$$

$J : T_M \to T_M$, $J^2 = -1$ $\qquad \nabla J = 0$ $\qquad$ Kähler $U(n)$

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}$$

$$\omega(X, Y) := g(JX, Y)$$

$$\nabla \omega = 0$$

$$\omega = \sum dx^j \wedge dy^j$$
$$= \frac{i}{2} \sum dz^j \wedge d\bar{z}^j$$

$\Omega \in \Omega^{n,0}(M)$ $\qquad \nabla \Omega = 0.$ $\qquad$ Calabi-Yau $SU(n)$.

$\Omega \bar{\Omega} = c\, \omega^n$ $\qquad \Omega = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n$.

$(|\det A_{\mathbb{C}}|^2 = \det A_{\mathbb{R}})$

$\Downarrow$ More Structures [Algebra] $\qquad$ $\Downarrow$ Smaller Holonomy.

$\mathbb{A}$-Manifolds over Algebras (normed)

(M, g) $\qquad$ $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

i.e. $M \sim$ "Curved" $\mathbb{A}^n$.

$$\mathrm{Hol}(M) \subset \left\{ \begin{array}{l} \text{twisted automorphisms} \\ \text{of } \mathbb{A}^n \end{array} \right\} = G_{\mathbb{A}}(n) \ni \varphi.$$

$$\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n \quad \text{isometry}$$

$v \in \mathbb{A}^n$

$$\boxed{\varphi(v \cdot x) = \varphi(v)\, \theta(x)}$$

$\exists\, \theta \in SO(\mathbb{A}), \ \forall x \in \mathbb{A}$

$\mathbb{A}$-Orientation ;

<u>Special</u> $\mathbb{A}$-manifolds.

$$(M\ ,\ g)$$

$$\mathcal{H}ol(M) \subset H_{\mathbb{A}}(n) \leq G_{\mathbb{A}}(n) \xrightarrow{\lambda \ (\det_{\mathbb{A}})} \mathbb{A}$$

$\parallel$

Isotropic subgp. of $1 \in \mathbb{A}$.

Manifolds Over Algebras:

	$\mathbb{A}$-mfd. $G_{\mathbb{A}}(n)$	Special $\mathbb{A}$-mfd. $H_{\mathbb{A}}(n)$.
$\mathbb{R}$	Riemannian $O(n)$	Oriented Riem. $SO(n)$.
$\mathbb{C}$	Kähler $U(n)$	Calabi-Yau $SU(n)$
$\mathbb{H}$	Quaternionic-Kähler $Sp(n)\,Sp(1)$	Hyperkähler $Sp(n)$
$\mathbb{O}$	Spin(7)-mfd $Spin(7)$	G_2-mfd. G_2

- Unified treatment for holomomy.
- $\dim_{\mathbb{O}} M = 1$ only.
- G_2-manifolds : <u>RIGHEST STRUCTURES</u>

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$$\boxed{\begin{array}{l} G_2\text{-manifold} \\ \equiv \text{Special } \mathbb{O}\text{-manifold} \end{array}} \qquad \begin{array}{c}(M^8, g) \\ \parallel \\ S^1 \times M^7\end{array}$$

i.e. $\mathrm{Hol}(M,g) \subset \left\{ \varphi \in O(\mathbb{O}) : \begin{array}{l} \exists \theta \in SO(\mathbb{O}), \forall v, x \\ \varphi(v \cdot x) = \varphi(v) \cdot \theta(x) \\ \varphi(1) = 1. \end{array} \right\}$

$\Rightarrow \quad \varphi = \theta \qquad$ (take $v = 1$).

i.e. $\varphi(v \cdot x) = \varphi(v) \cdot \theta(x)$

i.e. $\varphi \in \mathrm{Aut}(\mathbb{O}) = G_2 . \quad \subseteq SO(7).$

(1) $\quad \times \; : \mathrm{Im}\,\mathbb{O} \times \mathrm{Im}\,\mathbb{O} \longrightarrow \mathrm{Im}\,\mathbb{O}$

$$x \times y \; := \; \mathrm{Im}(x \cdot y) = x \cdot y + \langle x, y \rangle$$

(2) $\quad \Omega(x, y, z) := \langle x \times y, z \rangle$

$\Omega \in \Lambda^3 (\mathrm{Im}\,\mathbb{O})^*$

Both $\times$ and Ω preserved by G_2.

In fact,

$$G_2 = \{ \varphi \in GL(7, \mathbb{R}) : \varphi^* \Omega = \Omega \}.$$

$$\boxed{G_2 = \mathrm{Aut}\,(\mathbb{R}^7, \Omega).}$$

$$\Omega = dx^{123} - dx^1(dy^{01} + dy^{23}) - dx^2(dy^{02} + dy^{31}) - dx^3(dy^{03} + dy^{12})$$

<u>Reasons:</u> (1) Ω determines orientat^{n}. μ.

(2) $\langle x, y\rangle = \frac{1}{6}\; \iota_x\Omega \wedge \iota_y\Omega \wedge \Omega \,/\, \mu$

$\Rightarrow$ $\mathrm{Aut}\,(\mathbb{R}^7, \Omega) \subseteq SO(7)$

(3) $\Omega(x, y, z) = \langle x \times y,\ z\rangle$

(4) $x * y = x \cdot y + \langle x, y\rangle.$

$\Rightarrow$ $\mathrm{Aut}\,(\mathbb{R}^7, \Omega) = \mathrm{Aut}\,\mathbb{O} = G_2$

REMARK:

ANY generic 3-form $\Omega \in \Lambda^3(\mathbb{R}^7)^*$ is conjugate to either

$$dx^{123} - dx^1(dy^{01} + dy^{23}) - \quad \text{(positive)}$$

or

$$dx^{123} + dx^1(dy^{01} + dy^{23}) + \quad \text{(negative).}$$

Back to G_2-manifolds,

(M^7, Ω) ← positive 3-form

$\rightsquigarrow$ metric $g(v,w) = \frac{1}{6} \frac{\iota_v\Omega \wedge \iota_w\Omega \wedge \Omega}{\nu}$

$S^1 \times M^7$: Special Φ-manifold / G_2-mfd.

$\Longleftrightarrow$ $\mathrm{Hol}(M^7, g) \subset G_2$

$\Longleftrightarrow$ $\nabla\Omega = 0$

$\Longleftrightarrow$ $d\Omega = d^*\Omega = 0$

Remark: Similar to Kähler manifolds,

(X^{2n}, g, ω) ← non-degen. 2-form.

X : Kähler (i.e. $\mathrm{Hol} \subset U(n)$)

$\Longleftrightarrow$ $\nabla\omega = 0$

$\Longleftrightarrow$ $d\omega = 0$ $(\Rightarrow d^*\omega = 0)$

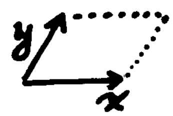
Remark:

(M^7, g, Ω) : G_2-manifold.

$\rightsquigarrow \quad \times : \Lambda^2 T_M \longrightarrow T_M$

$$g(x \times y, z) := \Omega(x, y, z)$$

(i) $(x \times y) \perp x$

(ii) $|x \times y| =$ Volume (

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i.e. <u>Vector Cross Product</u> (2-fold).

- <u>Classification</u> (Brown-Gray).

of r-fold vector cross product on (M^n, g)

(1). $r = 1$ $\quad M^{2m}$: Kähler (symplectic).

(2) $r = n-1$ $\quad M^n$: Oriented

(3) $r = 2$ $\quad M^7$: G_2-mfd (almost G_2)

(4) $r = 3$ $\quad M^8$: Spin(7)-mfd

§ Examples. of G_2-manifolds.

(0) Calabi-Yau Manifolds.

<u>Theorem</u> (Yau) : X^{2n} compact Kähler

$$c_1(X) = 0 \Rightarrow \exists \text{ metric } g$$
$$\mathcal{H}ol(M, g) \subset SU(n)$$

i.e. Calabi-Yau manifolds.

Eg. $X = \{ f(z_0, \dots, z_{n+1}) = 0 \} \subset \mathbb{CP}^{n+1}$

$\deg f = n+2.$

Eg. (1). $M^7 = \underset{CY^3}{\underset{\wr}{X^6}} \times S^1$ $\ni \theta$

Reason: $SU(3) \subset G_2$.

More Details:

$$G_2 \subset SO(7) \curvearrowright \underset{\substack{\Psi \\ \text{ANY } v}}{S^6} \subset \mathbb{R}^7$$

- $J := \times v : \underset{\substack{S\text{II} \\ \mathbb{R}^6}}{\langle v \rangle^\perp} \circlearrowleft$

  $J^2 = -id.$ (i.e. complex structure).
- $\Omega + i\, \iota_v (*\Omega)$ on $\langle v \rangle^\perp$: holo. volume.

$\Rightarrow$ isotropy of $v = SU(3) \subset G_2$.

Relations:

$$\Omega_M = \mathrm{Re}\,\Omega_X + \omega_X \wedge d\theta.$$

Eg(2). [Bryant]. $\Lambda^2_+ C^4$.

C : Oriented Riemannian 4-manifold (analytic).

$\Rightarrow$ Neighborhood of C in $\Lambda^2_+ C^4$ admits a <u>incomplete</u> G_2-metric.

provided $\Lambda^2_+ C \xrightarrow{\mathbb{R}^3} C$ topologically trivial.

[Method : Cartan - Kähler].

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Eg (2) [Bryant - Salamon]

$$M^7 = \Lambda^2_+(C), \qquad C = S^4 \text{ or } \overline{\mathbb{CP}}^2$$

(i.e. Einstein A.S.D.; $Ric = 3g_C$)

G_2-metric (complete):

$$\frac{dr^2}{1-\frac{\rho^4}{r^4}} + \left(1-\frac{\rho^4}{r^4}\right)\frac{r^2}{4}|d_A u|^2 + \frac{r^2}{2} g_C$$

where u : fiber coordi. for $\mathbb{R}^3 \to \Lambda^2_+(C) \to C$

ρ : positive const.

$r \geq \rho$ radial coordi.

Remark: As $\rho \to 0^+$, we obtain <u>cone</u> on $Z(C)$ (twistor).

Approach:

$g_C \rightsquigarrow$ Induced metric on $\Lambda^2_+(C)$.

$\cancel{dr^2} \rightsquigarrow$ Rescale fiber metrics by a function of $|u|$ only.

$d\Omega = d{*}\Omega = 0 \Rightarrow$ O.D.E.

Solve it.

$$M^7 = \$(S^3) \cong \mathbb{R}^4 \times S^3 .$$

$\left(\begin{array}{l} S^3 : \text{Einstein} \\ Ric = g_{S^3} \end{array} \right)$

Note: $\mathbb{R}^4 \times S^3 \simeq \text{Cone}(S^3) \times S^3$

$$S^3 \times S^3 = \{ (a, b, c) : \; abc = 1 \in S^3 \}$$

(using $S^3 = SU(2)$)

$$g_{S^3 \times S^3} = - \operatorname{Tr}\left[(a^{-1}da)^2 + (b^{-1}db)^2 + (c^{-1}dc)^2 \right] .$$

($SU(2)^3 \times \Sigma_3$ - invariant $\Rightarrow$ Einstein.).

$$g_M = \frac{dr^2}{1 - \frac{\rho^3}{r^3}} + \left(1 - \frac{\rho^3}{r^3} \right) \frac{r^2}{36} \left[g_{S^3 \times S^3} + 3 \operatorname{Tr}(b^{-1}db)^2 \right] \pm \frac{r^2 \operatorname{Tr}(b^{-1}db)}{24}$$

where $\rho > 0$: const.

$r \geq \rho$: radial coordi.

Remark: As $\rho \to 0^+$, we have cone on $S^3 \times S^3$

Remark: Triality (Atiyah-Witten).

Remark: Earlier similar construction for metrics with holonomy $SU(2)$ on T^*S^2. [Eguchi - Hanson] [Gibbons - Hawking].

g_ρ with $\rho > 0$.

As $\rho \to 0^+$, we obtain flat metric on $\mathbb{C}^2/\mathbb{Z}_2$ (i.e. cone on $\mathbb{RP}^3$.)

Eg (3) [Joyce]

<u>Compact</u> G_2 manifolds

via Singular Perturbations non-explicit

Method Start T^7 (flat)

T^7/G orbifold

$\in$ finite group.

Singularity Fix point set

$\overset{\text{nbd}}{\sim} T^3 \times (\mathbb{C}^2/\mathbb{Z}_2)$

$\big\downarrow$ Gluing Eguchi-Hansen

$T^3 \times (T^* S^2)$

Use Implicit Function Theorem

perturb to G_2 metric on

$M \quad \widetilde{T^7/G} \quad g_\rho$

As $\rho \to 0^+$ we obtain flat orbifold metric on T^7/G.

(Expect T^7/G is of <u>finite distance</u> to M)

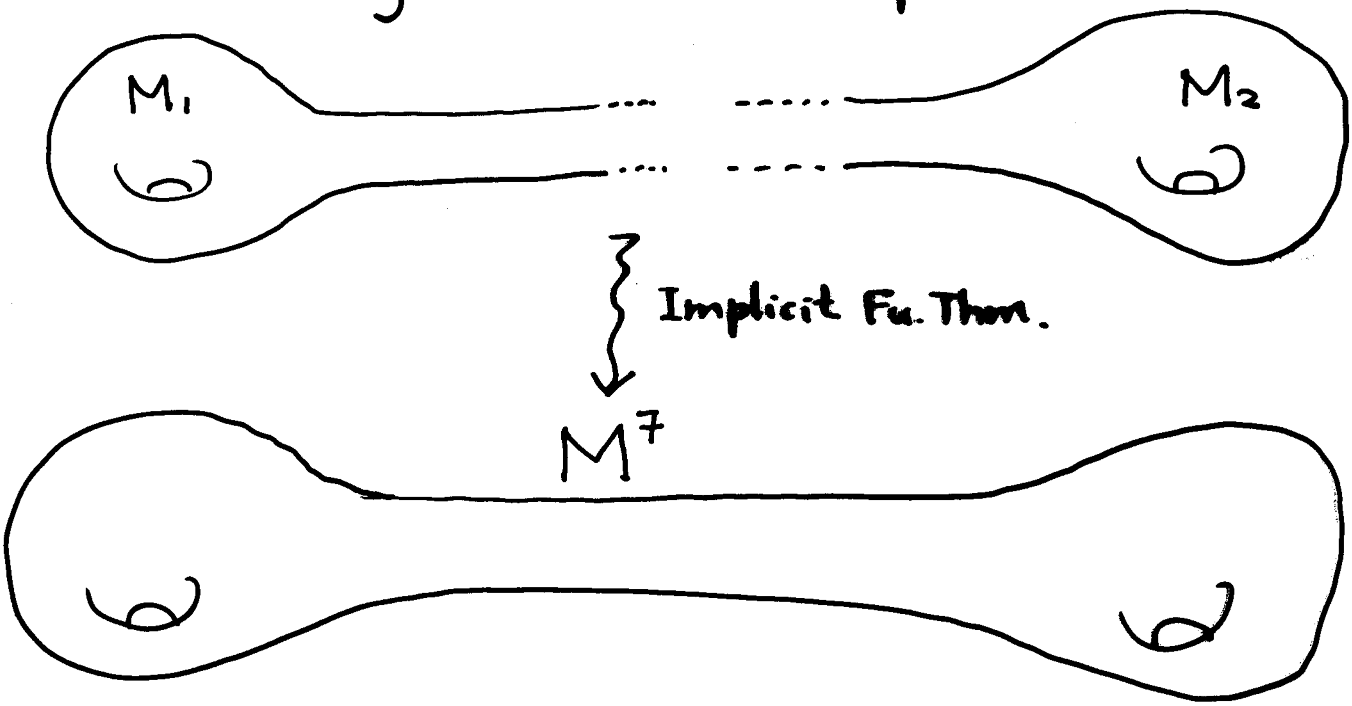
Eg (4). [Kovalev]

Again <u>compact</u> examples of G_2-mfds, but <u>infinite distance</u> to $\partial\mathcal{M}$.
(expect).

Method:

1° Construct Asym. Cylindrical G_2-mfd.
($X^6 \times S^1$ with X^6: asym. cylindrical CY^3)
[~ Tian - Yau.]

2° Gluing 2 such spaces



Remark: Topological Quantum Field Theory.

§ Basic Properties.

(Hodge theory for G_2-manifolds)

RECALL : HODGE THEORY.

① (X^{2n}, g, J) complex Hermitian manifold.

$$\Rightarrow \Lambda^k \otimes \mathbb{C} := \Lambda^k T_X^* \otimes \mathbb{C}$$

$$= \bigoplus_{p+q=k} \Lambda^{p,q}$$

$$0 \to \Lambda^{0,0} \xrightarrow{\bar\partial} \Lambda^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Lambda^{0,n} \longrightarrow 0$$ Dolbeault complex

② X Kähler, i.e. $\nabla J = 0$
(equivalently $\nabla \omega = 0$, $\omega(v,w) = g(Jv,w)$)

$$\Rightarrow H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$$

space of harmonic forms of type (p,q).

Note: $\nabla \omega = 0 \quad \Leftrightarrow \quad d\omega = 0$

③ (X^{2n}, ω) Kähler

If $\exists\ \Omega \in \Lambda^{n,0}$, $\nabla\Omega = 0$.

$\Rightarrow$ Ricci $\equiv 0$ (i.e. Calabi-Yau).

In this case,

$$\int_X p_1(X) \wedge \frac{\omega^{n-2}}{(n-2)!} = \frac{-1}{8\pi^2} \int_X |Rm|^2 \, dv_X$$

$$\leqslant 0 .$$

"=" iff X flat.

BACK to Hodge theory for $\underline{G_2\text{-manifolds}}$ (M^7, g, Ω).

① (M^7, g, Ω) : G_2-manifold.

(Only Need G_2-structure on TM.)

$$\Rightarrow \begin{cases} \Lambda^1 = \Lambda^1_7 \\ \Lambda^2 = \Lambda^2_7 + \Lambda^2_{14} \\ \Lambda^3 = \Lambda^3_1 + \Lambda^3_7 + \Lambda^3_{27} \\ \Lambda^k_\ell = *\Lambda^{7-k}_\ell, \quad k \geq 4 \end{cases}$$

Reasons:

- $G_2 \subseteq SO(7) \curvearrowright \mathbb{R}^7$: str. gp. of TM.

  $\rightsquigarrow \quad \Lambda^1 \simeq T^*_M \xrightarrow{g} T$
  $\simeq \Lambda^1_7$

- $\mathfrak{g}_2 \subset \underline{so}(7) \simeq \Lambda^2 \mathbb{R}^{7*}$

  $\rightsquigarrow \quad \Lambda^2_{14} \subset \Lambda^2$

- $T_M \simeq \Lambda^1_7 \hookrightarrow \Lambda^2$

  $v \mapsto \iota_v \Omega$

  $\Bigg\} \Rightarrow \Lambda^2 = \Lambda^2_7 + \Lambda^2_{14}$.

- $T_M \simeq \Lambda^1_7 \hookrightarrow \Lambda^3$

  $v \mapsto \iota_v(*\Omega)$

- $\Omega \in \Lambda^3$

- $\mathrm{Sym}^2_0 \mathbb{R}^7 = T_g(\mathcal{M}et) \longrightarrow \Lambda^3$

  $\delta g \mapsto \delta(\Omega(g))$

$\Rightarrow \Lambda^3 = \Lambda^3_1 + \Lambda^3_7 + \Lambda^3_{27}$

06. A (Option)

(2) (M^7, g, Ω)

G_2-structure on TM.

- $0 \to \Lambda^0 \xrightarrow{(d)_7} \Lambda^1_7 \xrightarrow{(d)_7} \Lambda^2_7 \xrightarrow{(d)_1} \Lambda^3_1 \to 0$

is an <u>elliptic complex</u>

$\Longleftrightarrow \quad d^*\Omega = 0$.

Proof: Elliptic Complex

$\Leftrightarrow \quad (d)_1 : \Lambda^2_{14} \longrightarrow \Lambda^3_1$ <u>zero</u>. $(\Lambda^2 = \Lambda^2_7 + \Lambda^2_{14})$

$\Leftrightarrow \quad \forall \varphi \in \Lambda^2_{14}$ implies $(d\varphi)_1 = 0$

i.e. $\varphi \wedge *\Omega = 0 \in \Lambda^6 = \Lambda^6_7$ i.e. $(d\varphi)\wedge *\Omega = 0 \in \Lambda^7 = \Lambda^7_1$.

$$d(\varphi \wedge *\Omega) = (d\varphi)\wedge *\Omega + \varphi \wedge d*\Omega$$

$\Leftrightarrow \quad d*\Omega = 0$

$\Leftrightarrow \quad d^*\Omega = 0$ #.

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\# ② (M^7, g, Ω)

G_2-structure on TM.

Lemma: M <u>G_2-manifold</u>

i.e. $\nabla\Omega = 0$

$\Leftrightarrow \quad d\Omega = d^*\Omega = 0.$

Reason: [$\Rightarrow$] obvious.

[$\Leftarrow$]. Holonomy Reduction:

$$\text{Obstruction} \in T^* \otimes \mathfrak{g}^\perp \subset T^* \otimes \Lambda^2.$$
$$\mathfrak{g}^\perp \cong \Lambda_7 \otimes \Lambda_7$$

$$\Rightarrow \nabla\Omega \in T^* \otimes \Lambda^3_7 \subset T^* \otimes \Lambda^3$$

$$T^* \otimes \Lambda^3_7 \cong T^* \otimes T^*$$

$$\Lambda^2 T^* + \mathrm{Sym}^2 T^*$$

$$\Lambda_{14} + \Lambda_7 + \Lambda_1 + \Lambda_{27}.$$

$$\underbrace{\Lambda_{14} + \Lambda_7}_{\Lambda^2 \ni d^*\Omega} \qquad \underbrace{\Lambda_7 + \Lambda_1 + \Lambda_{27}}_{\Lambda^4 \ni d\Omega}$$

\#.

(M^7, g, Ω) : G_2-manifold

(i.e. $\nabla\Omega = 0$, equiv. $d\Omega = d^*\Omega = 0$)

$$\Rightarrow \begin{cases} H^1(M) = H^1_7 \\ H^2(M) = H^2_7 \oplus H^2_{14} \\ H^3(M) = H^3_1 \oplus H^3_7 \oplus H^3_{27} \end{cases}$$

$$H^k_\ell \xrightarrow[*]{\cong} H^{7-k}_\ell, \quad k \geq 4.$$

- $H^3_1 = \mathbb{R}\langle\Omega\rangle$
- $H^1_7 \cong H^2_7 \cong H^3_7$.

[Reason: Δ commutes with $\Lambda^k \xrightarrow{\text{proj}} \Lambda^k_\ell$.]

Cor: $\mathcal{H}ol(M^7, g) = G_2$

$$\Rightarrow H^*_7(M) = 0.$$

[Reason: Ricci $= 0$ + Bochner].

Remark: $\mathcal{H}ol \underset{\text{"="}}{\subseteq} G_2 \iff \pi_1(M)$ finite.

[deRham decomposition].

③ (M^7, g, Ω) : G_2-manifold.

Claim: Ricci $\equiv 0$ (Einstein).

Reason:

- $\forall$ Riemannian manifold,

$$Rm \in \mathrm{Sym}^2(\Lambda^2 T_M^*) \subset \bigotimes^4 T_M^*$$

$\Rightarrow$ No component in $\underset{\nwarrow \text{irred. } O(n)\text{-mod.}}{\Lambda^4 T_M^*}$

- For G_2-manifold,

$$Rm \rightsquigarrow \xrightarrow{Tr_g} \text{Ricci} \in \mathrm{Sym}^2 T_M^*$$
$$\cong$$
$$\Lambda_1 + \Lambda_{27} \subset \Lambda^4 T_M^*$$

$\Rightarrow$ Ricci $\equiv 0$ #.

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Characteristic Numbers Inequality :

Prop: (M^7, g, Ω) : G_2-manifold

$$\Rightarrow \int_M p_1(M) \wedge [\Omega] \leqslant 0$$

"=" iff M is flat.

Proof:

$$Rm \in \Omega^2(M, \mathrm{ad}\, T_M)$$

$$\Rightarrow Rm \in \Omega^2_{14}(M, \mathrm{ad}\, T_M) \quad \left(\begin{array}{c} \because \mathrm{ad}\, T_M \simeq \Lambda^2_{14} \\ R_{ijkl} = R_{klij} \end{array} \right)$$

$$\Rightarrow \mathrm{Tr}(Rm)^2 \wedge \Omega = -|Rm|^2 \, d\nu_M$$

(linear alg. ; Riemann Bilinear Relation in Kähler)

$$\Rightarrow \int_M p_1(M) \wedge [\Omega] = \frac{-1}{8\pi^2} \int_M |Rm|^2 \, d\nu_M \leqslant 0$$

#

§ Moduli of G_2-manifolds.

M^7

$$\mathcal{M}_{G_2} := \frac{\{ G_2\text{-metrics on } M \}}{\mathrm{Diff}_0(M)}$$

Recall

g : G_2-metric $\longleftrightarrow$ $\Omega \in \Omega^3(M)$
$\Omega > 0$
$d\Omega = d^*\Omega = 0$.

$$\left[g(v,w) = \iota_v \Omega \wedge \iota_w \Omega \wedge \Omega \,/\, dv_g \right]$$

Variational Approach:

$$\mathcal{F} : \left\{ \Omega \in \Omega^3(M) : \begin{matrix} \Omega > 0 \\ d\Omega = 0 \end{matrix} \right\} \longrightarrow \mathbb{R}$$

$$\mathcal{F}(\Omega) \triangleq \int_M |\Omega|_g^2 \, dv_g = \int_M \Omega \wedge * \Omega$$

1st variation:

$$\delta \mathcal{F}(\Omega)(\Phi) = \frac{7}{3} \int_M \langle \Omega, \Phi \rangle_g \, dv_g .$$

Cor: In a FIXED class $[\Omega] \in H^3(M)$,

Ω : critical point to $\mathcal{F}$

$\Leftrightarrow \; d^* \Omega = 0 \qquad (\because \Phi = d(2\text{-form}))$.

i.e. (M, G) : G_2-manifold.

Gradient Flow : $\dfrac{d\Omega}{dt} = \Delta \Omega$

2^{nd} variation for $\mathcal{F}(\Omega) = \int_M \Omega \wedge *\Omega$

$$\delta^2 \mathcal{F}(\Omega)(\Phi, \Phi)$$

$$= \frac{7}{3} \int_M \left(\frac{4}{3} |\Phi_1|^2 + |\Phi_7|^2 - |\Phi_{27}|^2 \right) dv_g$$

where $\underset{\overset{\parallel}{\Phi_1 + \Phi_7 + \Phi_{27}.}}{\Phi} \in \Omega^3(M) = \Omega^3_1 + \Omega^3_7 + \Omega^3_{27}$

<u>CLAIM</u>: $\delta^2 \mathcal{F}$ is <u>non-degenerate</u> at any critical point Ω. (mod. Diff(M))

<u>Cor</u> (Local Torelli Theorem).

<u>period</u> map:

$$\tau : \mathcal{M}_{G_2} \longrightarrow H^3(M, \mathbb{R})$$

$$\Omega \mapsto [\Omega]$$

is <u>local</u> <u>diffeo</u>morphism.

[reason: Implicit Function Thm.].

- Action by $\mathrm{Diff}(M)$:

$$v \in \Gamma(M, T_M).$$

$$\mathcal{L}_v \Omega = d(\underbrace{\iota_v \Omega}_{\in \Lambda^2_7(M)}) + \iota_v(\cancel{d\Omega})$$

i.e. $\Omega \mapsto \Omega + d\varphi$, $\varphi \in \Lambda^2_7(M)$
is generated by ~~Diff~~ symmetry

- $\Rightarrow$ consider

$$\delta^2 \mathcal{F}(\Omega)(d\varphi, d\varphi) \quad , \quad \varphi \in \Lambda^2_{14}(M).$$

with $d^*\varphi = 0$. ($\because$ Hodge).

- Recall $\varphi \in \Lambda^2_{14} \Rightarrow (d\varphi)_1 = 0 \in \Lambda^3_1(M)$.
($\because d{*}\Omega = 0$).

- $\varphi \in \Lambda^2_{14} \iff {*}\varphi = -\varphi \wedge \Omega$

$$\Rightarrow (d\varphi) \wedge \Omega = 0 \quad (\because d^*\varphi = 0).$$

$$\Rightarrow \quad (d\varphi)_7 = 0 \in \Lambda^3_7(M)$$

$$\Rightarrow \delta^2 \mathcal{F}(\Omega)(d\varphi, d\varphi) = -\frac{7}{3} \int_M |d\varphi|^2_g \, dv_g \ll$$

Non-degenerate !

#.

Cubic Structure on $\mathcal{M}_{G_2}$

(M^7, Ω) G_2-manifold

$$T_\Omega \mathcal{M}_{G_2} \cong H^3(M, \mathbb{R})$$

Define: symmetric tri-linear form

$$\mathcal{C}_\Omega : \bigotimes^3 H^3(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$\varphi \in \Omega^3(M) \rightsquigarrow \hat{\varphi} \in \Omega^1(M, TM).$$

$$\left[\begin{array}{l} \text{Define } \chi \in \Omega^3(M, TM) \\ \quad \text{by } \chi(v) = \imath_v(*\Omega) \in \Lambda^3 \\ \quad \hat{\varphi} = *(\varphi \wedge \chi). \end{array}\right.$$

$$\mathcal{C}_\Omega(\varphi_1, \varphi_2, \varphi_3)$$

$$:= \int_M \Omega(\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3) \wedge *\Omega$$

Remarks:

(1) $\mathcal{C}_\Omega(\Omega,\Omega,\Omega)$

$= C\cdot \int_M \Omega\wedge *\Omega. \qquad = c\ \mathcal{F}(\Omega).$

(2) $\mathcal{C}_\Omega(\Omega,\Omega,\varphi) \qquad \forall\varphi\in H^3_{27}(M,\mathbb{R}).$

$= 0$

(3) $\mathcal{C}_\Omega(\Omega,\varphi,\varphi)$

$= c' \int_M \varphi\wedge *\varphi \qquad = c''\ \delta^2\mathcal{F}(\Omega)(\varphi,\varphi).$

(4) $\mathcal{C}_\Omega(\varphi,\varphi,\varphi)$ <u>Yukawa Coupling</u>

$M = X\times S^1$

- If M: irreducible, then

$$T_\Omega \mathcal{M}_{G_2} = \underbrace{\mathbb{R}\langle\Omega\rangle}_{\text{volume.}} \oplus H^3_{27}(M)$$

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Degenerations of G_2-manifolds.

$$\partial \mathcal{M}_{G_2}$$

Speculations:

(M_t, Ω_t) $\quad t \neq 0$, diameter $= 1$

Suppose

$$\lim_{t \to 0} M_t = \overset{\swarrow \text{singular}}{M_0} \quad \text{(Gromov-Hausdorff Limit.)}$$

1. Volume $\geq c > 0$ $\quad \forall\, t \neq 0$.

$\Rightarrow \dim M_0 = 7$.

$\sim$ conical singularities.

$\sim$ M_0 of <u>finite</u> <u>distance</u> to interior of $\mathcal{M}_{G_2}$.

$Rc = 0$, diam $= 1 \Rightarrow$ vol $\leq C$.

Degenerations of G_2-manifolds.

$$\partial \mathcal{M}_{G_2}$$

Speculations:

$$(M_t, \Omega_t) \qquad t \neq 0, \quad \text{diameter} = 1$$

Suppose

$$\lim_{t \to 0} M_t = \overset{\text{singular}}{M_0} \qquad \text{(Gromov-Hausdorff Limit.)}$$

(2) $\mathrm{Vol}(M_t) \longrightarrow 0$ as $t \to 0$.

YES: Associative T^3-fibration

$\mathcal{V}_{\Omega_t}(H^3) \to 0$

NO: Coassociative fibration.

$$\int_{M_t} p_1(M) \wedge \Omega_t \longrightarrow 0$$

YES: fibers T^4

NO. fibers K3

§ Minimal Submanifolds in G_2-manifolds.

- <u>Special Lagrangian</u> Submanifolds in G_2-manifold (M^7, Ω).

$A^3 \subset M^7$ SLag. of type I (i.e. associative submfd.)

$C^4 \subset M^7$ SLag. of type II (i.e. coassociative submfd.).

RECALL:

(X^{2n}, ω, Ω) : Calabi-Yau manifold
(i.e. Special $\mathbb{C}$-manifold).

$C^n \subset X$: (1) <u>Lagrangian</u>
if $\omega|_C = 0$.

(2) Special Lagrangian <u>(SLag)</u>
of <u>type I</u>
if $\omega|_C = 0 = \mathrm{Im}\,\Omega|_C$

(3) <u>SLag.</u> of <u>type II</u>
if $\omega|_C = 0 = \mathrm{Re}\,\Omega|_C$.

Remark: (Special) Lagrangians of type I, II are defined on any <u>(Special) $\mathbb{A}$-manifold</u>, $\mathbb{A} = \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Now (M^7, Ω) : G_2-manifold

(i.e. $S^1 \times M$ a Special Φ-manifold)

Prop. : $D^4 \subset S^1 \times M^7$

① D: SLag, type I

$\Leftrightarrow D = S^1 \times A^3$ with $A^3 \subset M^7$

$\underline{\text{preserved by } X}$.

(i.e. associative submanifold).

② D : SLag. of type II

$\Leftrightarrow D = \{p\} \times C^4$ with $C^4 \subset M^7$

satisfying $\underline{\Omega|_C = 0}$

(i.e. co-associative submanifold).

- <u>SLag.</u> are <u>Calibrated</u>.

<u>Recall</u>: $C \subset (M, g)$

C : <u>minimal submanifold</u>

$\overset{\text{def.}}{\Longleftrightarrow}$ C is critical w.r.t. $Vol(C, g|_C)$

$\Longleftrightarrow$ Mean Curvature Vector $\vec{H} \equiv 0$.

Difficult to determine whether C is ABSOLUTE minimal or not.

CALIBRATION (Harvey – Lawson).

(M^n, g) Riemannian manifold.

Def^{n}: $\Omega \in \Omega^k(M)$ Calibrating Form

if (i) $d\Omega = 0$

(ii) $\Omega|_C \leqslant d\nu_C \quad \forall\, C^k \subset M$

Def^{n}: $C^k \subset M$ Calibrated Submfd

if $\Omega|_C = d\nu_C$

Key Lemma:

① $C \subset M$ Calibrated $\Rightarrow$ C has absolute min. volume within $[C] \in H_k(M)$.

② $\mathrm{Vol}(C') = \mathrm{Vol}(C)$, $[C'] = [C]$ $\Rightarrow$ C' also calibrated.

Proof: $[C'] = [C]$

$$\Rightarrow \int_{C'} \Omega = \int_C \Omega \qquad (\because d\Omega = 0)$$

$$\int_{C'} \Omega \leqslant \int_{C'} d\nu_{C'}, \qquad \int_C \Omega = \int_C d\nu_C$$

#

Examples of Calibration:

(1) (X, ω) Kähler manifold

(i) $\frac{\omega^k}{k!} \in \Omega^{2k}(M)$ calib. form.

(ii) $C \subset X$ Calib. submfd.

$\Leftrightarrow C$ <u>complex submanifold.</u>

Reason $(k=1)$:

$C \subset \mathbb{C}^n$ real 2-plane.

$|\omega(C)|^2 = |\langle C, JC \rangle| \leqslant |C|^2$

"$=$" $\Leftrightarrow JC = C$

ie. complex.

(Wirtinger).

(2). (X^{2n}, ω, Ω) Calabi-Yau mfd.

(i) $\mathrm{Re}\,\Omega \in \Omega^n(M)$ calib. form.

(ii) $C^n \subset X^{2n}$ calib. submfd.

$\Longleftrightarrow C$: <u>SLag. of type I.</u>

i.e. $\omega|_C = 0 = \mathrm{Im}\,\Omega|_C$.

$\mathrm{Im}\,\Omega$: calib. $\rightsquigarrow$ type II.

[Reason: $\mathbb{C}^n$ $\quad \Omega = dz^1 \wedge dz^2 \wedge \dots \wedge dz^n$

$C \subset \mathbb{C}^n$ real n-plane

$$|\Omega(C)|^2 = |C \wedge JC| \leqslant |C|^2$$

"$=$" iff $JC \perp C$

i.e. Lagr.

$$|\Omega(C)|^2 = |\mathrm{Re}\,\Omega(C)|^2 + |\mathrm{Im}\,\Omega(C)|^2.$$

(3) (M^7, Ω) G_2-manifold.

(i) $\Omega \in \Omega^3(M)$ Calib. form

(ii) $A^3 \subset M^7$ calib. submfd

$\Leftrightarrow$ A : <u>Associative</u> submfd.

(i.e. preserved by $\times$).

$$\Omega(u, v, w) = g(u \times v, w).$$

Reason: $A \subset \mathrm{Im}\,\mathbb{O} \simeq \mathbb{R}^3$ 3-plane

$A = \mathrm{Span}(u, v, w)$, o.n.

$$|\Omega(A)| = |g(u \times v, w)|$$
$$\leqslant |u| \cdot |v| \cdot |w|$$

"$=$" $\Leftrightarrow$ $w = \pm u \times v$

i.e. A preserved by $\times$.

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[illegible].

(4) (M^7, Ω) G_2-manifold.

(i) $*\Omega \in \Omega^4(M)$ calib. form.

(ii) $C^4 \subset M$ calib. submfd.

$\Leftrightarrow$ C : <u>Co-associative</u> submfd.

i.e. $\Omega|_C = 0$.

Reason: $C' \subset \mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ real 4-plane

$\Omega|_C = 0 \Leftrightarrow (u, v \in C$ implies $u \times v \perp C)$.

$\Leftrightarrow -4\,\mathrm{Alt}\,\langle w, u \times v\rangle x = 0$
$\forall\, w, u, v, x \in C$.

Linear alg:
$\exists$ 4-fold cross product on $\mathbb{O}$
s.t. when $x, w, u, v \in \mathrm{Im}\,\mathbb{O}$

$$x \times w \times u \times v = \underbrace{(*\Omega)(x\,w\,u\,v)}_{\in \mathbb{R}} - \underbrace{4\,\mathrm{Alt}\langle w, u \times v\rangle x}_{\in \mathrm{Im}\,\mathbb{O}}.$$

$$\Rightarrow |x \times w \times u \times v|^2 = |x \wedge y \wedge u \wedge v|^2.$$

$$\| \qquad\qquad$$

$$|*\Omega|^2 + |4\,\mathrm{Alt}|^2$$

Remark: (M^7, Ω) G_2-manifold.

① $A_1, A_2 \subset M$ Associative submfds.

$\Rightarrow \dim(A_1 \cap A_2) \neq 2$

($\because$ preserved by $\times$).

② $C_1, C_2 \subset M$ Co-associative submfds.

$\Rightarrow \dim(C_1 \cap C_2) \neq 3$

(Reason: ① and next remark.).

Remark: In the linear situation,

$$A^3 \subset \operatorname{Im}\mathbb{O}$$

A : associative $\iff$ $C = A^{\perp}$: coassociative.

e.g. $\operatorname{Im}\mathbb{O} = \langle \underbrace{x_1, x_2, x_3}_{A}, \underbrace{y_0, y_1, y_2, y_3}_{C} \rangle$.

Reason: $\Omega|_A = d\nu_A$

$\iff (*\Omega)|_{A^{\perp}} = d\nu_{(A^{\perp})}$.

Prop: $A^3 \subset (M^7, \Omega)$

Associative (i.e. preserved by $\times$).

$$\Leftrightarrow \quad \chi|_A \equiv 0$$

where $\chi \in \Omega^3(M, TM)$ is

$$\langle \chi, v \rangle = \iota_v(*\Omega) \in \Lambda^3. \quad \forall v \in TM.$$

Reason: On $\mathbb{R}^7 \simeq \mathrm{Im}\,\mathbb{O}$

$$|\underbrace{\Omega(u,v,w)}_{\in \mathbb{R}}|^2 + |\underbrace{(*\Omega)(u,v,w)}_{\in \mathrm{Im}\,\mathbb{O}}|^2 = |u \wedge v \wedge w|^2$$

In fact, $\exists$ ~~vector~~ 3-fold cross product on ~~Im~~ $\mathbb{O}$, s.t. when $u, v, w \in \mathrm{Im}\,\mathbb{O}$

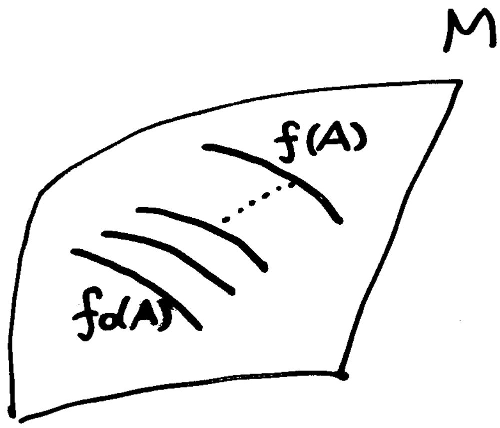
$$u \times v \times w = \iota_{u \wedge v \wedge w}(\Omega + *\Omega) \in \mathbb{O}.$$

$\rightsquigarrow$ gives above equality.

Cor. $f : A^3 \longrightarrow (M^7, \Omega)$ parametrizes an associative submfd.

$\Leftrightarrow$ $f \in \mathrm{Map}(A, M)$ is a critical point of the functional

$$\int_{A\times[0,1]} F^*(*\Omega)$$



$F : A\times[0,1] \longrightarrow M$

$\begin{cases} F(\cdot, 1) = f \\ F(\cdot, 0) =: f_0 \leftarrow \text{Fix map}. \end{cases}$

[Reason: Varying F make $\iota_v(*\Omega)|_{f(A)} = 0$]

Remark: $\operatorname{Im}\mathbb{O} \simeq \mathbb{R}^7$.

(i) $\{\text{linear associative subspaces}\} \simeq \dfrac{G_2}{SO(4)}$

(ii) $\left\{\begin{array}{l}\text{linear} \\ \text{coassociative subspaces}\end{array}\right\} \simeq \dfrac{G_2}{SO(4)}$

Reason: (i) $\Longleftrightarrow$ (ii) $\quad (\because\ A^\perp = C)$

$C^4 \subset \operatorname{Im}\mathbb{O}$ co-ass. iff $\Omega|_C = 0 \in \Lambda^3 C$

$$\begin{array}{ccc} G_2 & \curvearrowright & \left\{\begin{array}{l}\text{linear} \\ \text{co-ass.}\end{array}\right\} \subset Gr_4(4,7) \\ & & \uparrow \qquad\qquad \dim = 12 \\ \text{isotropic} & & \dim = 12 - \operatorname{rank}(\Lambda^3 C) \\ \cup\!\! \mid & & = 8 \\ SO(4). & & \\ \dim G_2/SO(4) = 8 & & \end{array}$$

$\Rightarrow \dfrac{G_2}{SO(4)} \simeq \left\{\begin{array}{l}\text{linear} \\ \text{co-ass}\end{array}\right\}^0$ (In fact, connected).

[Better proof: Via linear algebra.]

$$\begin{array}{c} SO(4) \subset G_2 \subset SO(\operatorname{Im}\mathbb{O}) \\ \| \\ Sp(1)\,Sp(1). \\ \Downarrow \\ (p, q) \qquad\qquad (a, b) \in \operatorname{Im}\mathbb{H} + \mathbb{H} = \operatorname{Im}\mathbb{O} \end{array}$$

via $(p, q)\cdot(a, b) = (p a \bar{p}, q b \bar{p})$.

$(M^7; \Omega)$: G_2-manifold

• Example of (Co-)associative submfd.

(1) $S^3 \underset{\text{zero section}}{\subset} S^3 \times \mathbb{R}^4 \simeq \$(S^3)$ Spinor bundle.

ass. submfd.

$S^4 \underset{\text{zero section}}{\subset} \Lambda^2_+(S^4)$

coass. submfd

$\mathbb{C}P^2 \underset{\text{zero section}}{\subset} \Lambda^2_+(\mathbb{C}P^2)$.

(2). $\underset{G_2\text{-mfd.}}{M^7} = \underset{\text{CY 3-fold.}}{X^6} \times S^1$

$$\Omega = \mathrm{Re}\,\Omega_X + \omega_X \wedge d\theta.$$

$$\begin{cases} \Sigma^2 \times S^1 \text{ associative} \Leftrightarrow \Sigma \subset X & \text{holomorphic curve.} \\ L^3 \times \{p\} \text{ associative} \Leftrightarrow L \subset X & \text{SLag., type I} \end{cases}$$

$$\begin{cases} L^3 \times S^1 \text{ coassociative} \Leftrightarrow L \subset X & \text{SLag. type II} \\ C^4 \times \{p\} \text{ coassociative} \Leftrightarrow C \subset X & \text{complex surface.} \end{cases}$$

(3) $\sigma : M \longrightarrow M \qquad \sigma^2 = id.$

σ isometry, i.e. $\sigma^* g = g$

$\Rightarrow \sigma^* \Omega = \pm \Omega.$

$M^\sigma := \{ x \in M : \sigma(x) = x \}$ fixed set

is a calibrated submfd.

- $\sigma^* \Omega = \Omega$

$\Rightarrow \dim M^\sigma = \underset{\text{associative}}{3} \quad \text{or} \quad \underset{M^\sigma = M}{7}$

- $\sigma^* \Omega = -\Omega$

$\Rightarrow \dim M^\sigma = \underset{M^\sigma = \{p\}.}{0} \quad \text{or} \quad \underset{\text{coassociative}}{4}$

• <u>Deformation</u> of $\underset{\text{co-ass.}}{C^4} \subset \underset{G_2\text{-mfd.}}{(M^7, \Omega)}$

$$\Omega|_C = 0$$

$$\Rightarrow \quad \begin{aligned} N_{C/M} &\longrightarrow \Lambda^2 T_C^* \\ v &\longmapsto \iota_v \Omega \end{aligned}$$

<u>Claim:</u> $\underline{N_{C/M} \xrightarrow{\cong} \Lambda^2_+ T_C^*}$

[Reason: Since $G_2 \curvearrowright \left\{ \begin{matrix} \text{linear} \\ \text{coass} \end{matrix} \right\}$ transitive

$\Rightarrow$ Enough to check the case $C = \mathbb{H}$ in $M = (\mathrm{Im}\,\mathbb{H}) + \mathbb{H}$. (Exercise)

($\because$ $\Omega = dx^{123} - dx^1 (\underbrace{dy^{01} + dy^{23}}_{\text{self-dual form on } \mathbb{H}}) - [1\,2\,3]$)

Co-ass. $C^4 \subset (M^7, \Omega)$

$$\begin{array}{ccccc} & \left\{\begin{matrix}\text{nearby}\\ \text{submfds}\end{matrix}\right\} & & \\ & \wr\!\| & & \\ \Omega^2_+(C) \cong & \Gamma(\mathcal{N}_{C/M}) & \xrightarrow{\ F\ } & \Omega^3(C) \\ 2_v\Omega \leftrightarrow & v & \longmapsto & \exp_v^*\Omega. \end{array}$$

$$F^{-1}(0) = \left\{\begin{matrix}\text{nearby}\\ \text{co-ass. submfds}\end{matrix}\right\}. \subset \mathcal{B}^{\text{coass.}}$$ moduli

$$dF(0)(v) = (\mathcal{L}_v\Omega)|_C$$
$$= d(2_v\Omega)|_C$$

$$\Rightarrow \text{"}T_C\text{"}\,\mathcal{B}^{\text{coass}} = H^2_+(C).$$

i.e. infinitesimal deformations of coassociative submfd.s are parametrized by $H^2_+(C)$.

Claim (McLean). : No obstructions to deformations i.e. $\mathcal{B}^{\text{coass}}$ smooth

and $T_C \mathcal{B}^{\text{coass}} \simeq H^2_+(C)$.

Reason: Implicit Function Theorem.

$$F : \Omega^2_+(C) \simeq \Gamma(N_{C/M}) \longrightarrow \Omega^3(C)$$

$dF(0) = d$, $\Rightarrow$ $\operatorname{Im} dF(0) = \underline{d\Omega^2_+(C)}$

In fact, $\operatorname{Im} F \subset \underline{d\Omega^2_+(C)}$

$[\because \ \Omega|_C = 0$, $C \sim C' \Rightarrow \Omega|_{C'} \in d\Omega^2$.

$\Rightarrow$ Surjectivity $\Rightarrow$ Apply Imp. Fn. Thm.

<u>Comparison</u> :

(X^{2n}, ω, Ω) Calabi-Yau mfd.

$L^n \subset X$ SLag. submfd.

$\omega|_L = \mathrm{Im}\,\Omega|_L = 0.$

- $\mathcal{N}_{L/X} \simeq T^*_L$

  $v \mapsto \iota_v \omega$

- $\left\{\begin{array}{l}\text{infinitesimal} \\ \text{deformation of SLag}\end{array}\right\} \simeq H^1(L, \mathbb{R}).$

- No obstruction (McLean).

(L, D_E) : <u>SLag. cycle</u>. (SUSY A-cycle)

SLag. flat U(1)-connection/L.

(e.g. in S.Y.Z. mirror conjecture).

$\mathcal{M}^{SLag.}$: moduli of SLag. cycles.

- smooth (non-compact) Kähler manifold of $\dim_{\mathbb{R}}$ $2 b_1(L)$.

Coassociative Cycle: (C, D_E)

[Supersymmetric cycles in G_2-manifolds]

Definition: $C^4 \subset (M^7, \Omega)$ coass. submfd.

+ D_E : ASD connection on

$$\mathbb{C}^r \longrightarrow E \longrightarrow C$$

i.e. $\quad F_E^+ = 0 \in \Omega_+^2(C, \mathrm{ad}E).$

Denote $\mathcal{M}^{\mathrm{coass}}$: moduli of (C, D_E).

$$T_{(C,D_E)}\mathcal{M}^{\mathrm{coass}} = \frac{\left\{(B,\varphi) \in \Omega^1(C,\mathrm{ad}E) \times \Omega_+^2(C) : \begin{array}{l} D_E^+ B = \varphi \\ d\varphi = 0 \end{array}\right\}}{\{(D_E c, 0)\}}$$

$$0 \longrightarrow \underbrace{H^1(C, \mathrm{ad}E)}_{\substack{\text{deform } D_E \\ \text{over fix } C.}} \longrightarrow T\mathcal{M}^{\mathrm{coass}} \longrightarrow \underbrace{H_+^2(C)}_{\substack{\text{deform} \\ C}} \longrightarrow H_+^2(C, \mathrm{ad}E) \longrightarrow$$

- Natural 3-form on $\mathcal{M}^{coass}$ (Correlation function).

$$\underline{\Omega_{\mathcal{M}^{coass}} \in \Omega^3(\mathcal{M}^{coass}).}$$

At (C, D_E), $(B_i, \varphi_i) \in T\mathcal{M}^{coass}$.

$$\Omega_{\mathcal{M}^{coass}}\big((B_1,\varphi_1),\ (B_2,\varphi_2),\ (B_3,\varphi_3)\big)$$

$$:= \int_C [\varphi_1, \varphi_2]_{H^2_+} \wedge \varphi_3 - \mathrm{Tr}\, \varphi_1 B_2 B_3 \quad + [123].$$

Well-defined? Say $(B_3, \varphi_3) = (D_E C, 0) \in \Omega^1(\mathrm{ad}E) \times \Omega^2_+$

$$\Omega_{\mathcal{M}^{coass}} = -\int \mathrm{Tr}\, \varphi_1 B_2 (DC) + \int \mathrm{Tr}\, \varphi_2 B_1 (DC)$$

$$\underset{\text{by part}}{=} \int \mathrm{Tr}\, \varphi_1 (DB_2)_+ C - \int \mathrm{Tr}\, \varphi_2 (DB_1)_+ C$$

$(\because d\varphi_i = 0)$.

$$= \int \mathrm{Tr}\, \varphi_1 \varphi_2 C - \int \mathrm{Tr}\, \varphi_2 \varphi_1 C \quad (\because DB_i = \varphi_i)$$

$$= 0.$$

- Cubic structure on $\mathcal{M}^{coass}$.
  (Yukawa coupling).

$$\mathcal{Y}_{\mathcal{M}^{coass}} \in \Gamma(\mathcal{M}^{coass}, \mathrm{Sym}^3 T^*_{\mathcal{M}^{coass}}).$$

$$\mathcal{Y}_{\mathcal{M}^{coass}}\big((B_1, \varphi_1), (B_2, \varphi_2), (B_3, \varphi_3)\big)$$

$$= \int_C \mathrm{Tr}\, \varphi_1 [B_2, B_3]_{ad(E)} + [123].$$

Well-defined (Exercise).

An important case:

(C, D_E) coass. cycle with $\underline{\text{rank}(E) = 1}$

(For simplicity, assume $b_1(C) = 0$, i.e. $\nexists$ nontrivial flat line bundle $/C$).

$$\dim \mathcal{M}^{\text{coass}} \text{ "=" } 0$$

Reason:

(i) Deforming $C \sim H^2_+(C)$.

(ii) $c_1(E) \in H^2_-(C) \cap H^2(C, \mathbb{Z})$

i.e. $U(1)$-ASD occur in codim b^2_+.

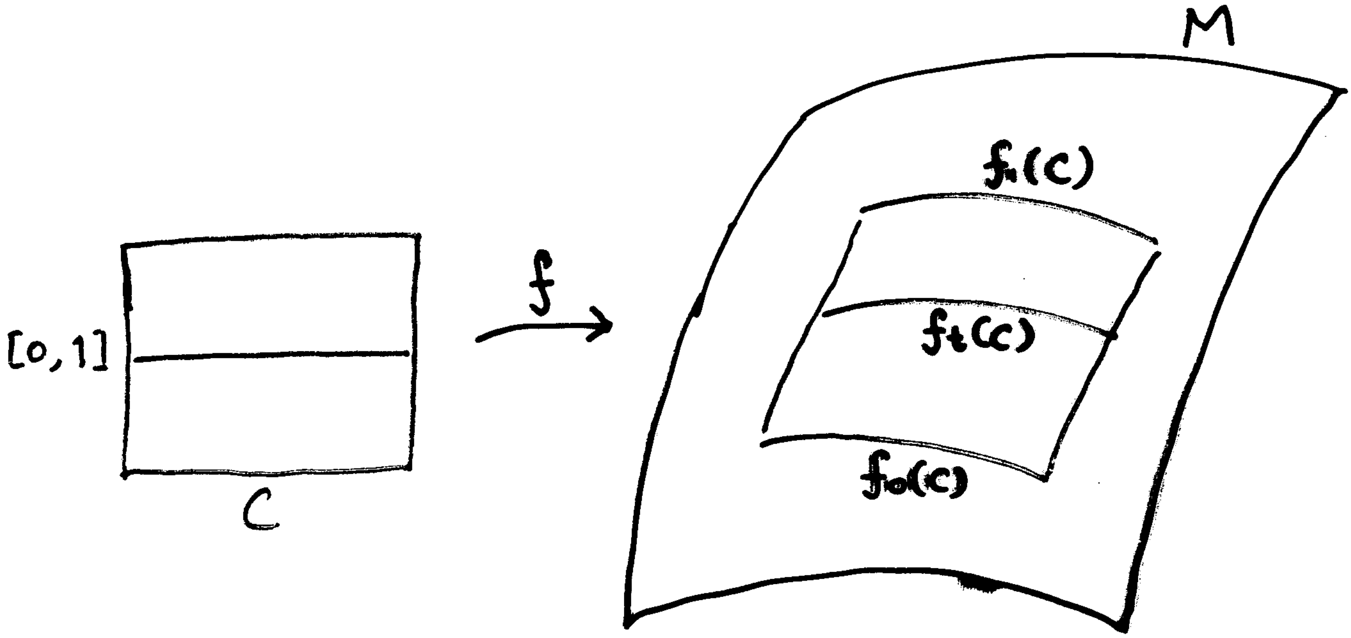
$\# \mathcal{M}^{\text{coass}}$?

TQFT ?

(Parametrized) coassociative cycles are <u>critical points</u> of the functional:

$$\mathcal{F} : \mathrm{Map}(C^4, M) \times \mathcal{A}(E) \longrightarrow \mathbb{R} / \mathbb{Z} .$$

$$\mathcal{F}(f_1, D_{1E}) = \int_{C \times [0,1]} F \wedge f^* \Omega$$



(f_0, D_{0E}) : some fixed background configuration.

Symmetry / Gauge Group. $\mathcal{G}$

$$1 \to \mathrm{Aut}(E) \to \mathcal{G} \longrightarrow \mathrm{Diff}(C) \to 1$$

$$\mathcal{G} \ni g$$

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ C \arrow[r, "g_C"] & C \end{tikzcd}$$

$$\mathcal{G} \curvearrowright \mathrm{Map}(C,M) \times \mathcal{A}(C)$$

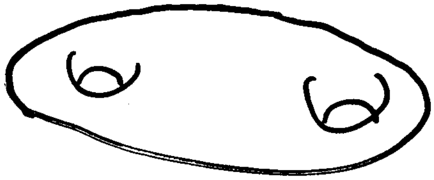
$$g \cdot (f, D_E) := (f \circ g_C, g^* D_E).$$

- $\mathcal{F}$ is $\mathcal{G}$-invariant
- $\mathcal{M}^{\text{coass}} = d\mathcal{F}^{-1}(0)/\mathcal{G}$
- Witten's Morse theory

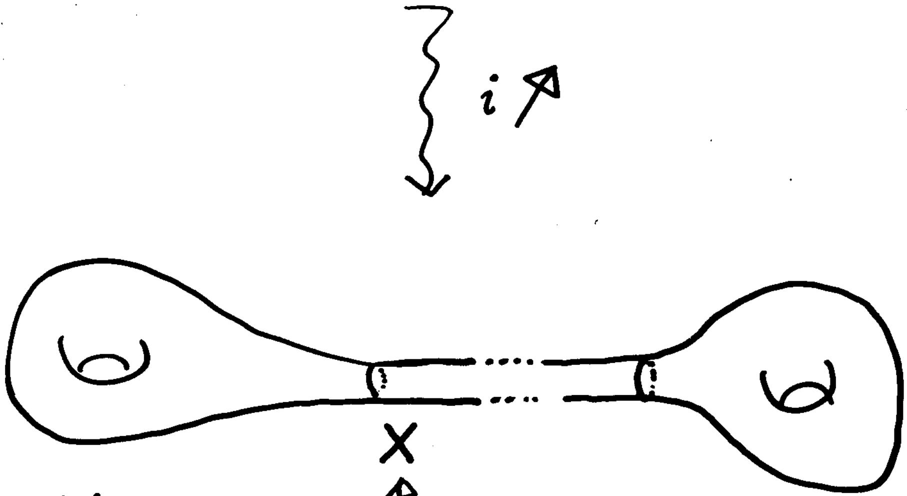
"$\rightsquigarrow$" Homology $H_\bullet^{\text{coass}}(M)$ s.t.

$$\#\mathcal{M}^{\text{coass}} = \chi(\text{—}\ \text{"}\ \text{—}).$$

- <u>Moduli</u> of <u>Coassociative Cycles</u>. under <u>Degeneration</u>



(M^7, Ω_i) $\quad d\Omega_i = 0$.



M_1

Calabi-Yau 3-fold.

M_2 : asympotically cylindrical G_2-mfd.

How does $H^{coass}(M)$ recover from M_1 and M_2 ?

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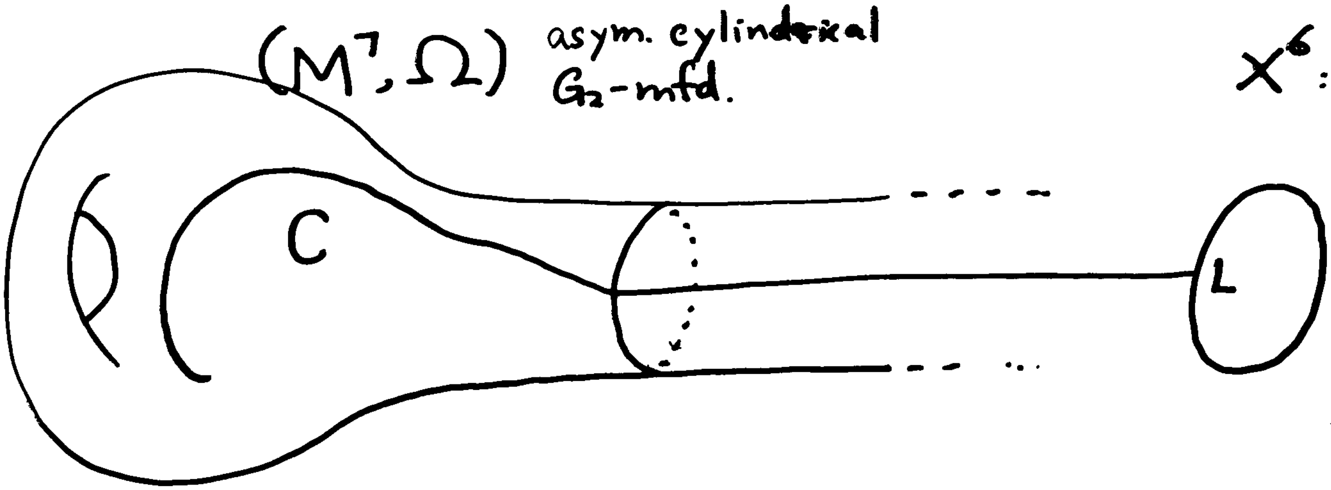
$X^6 : CY^3$



(C, D_E) coass. cycle on M

$\Rightarrow \partial(C, D_E)$ SLag of type I in X.

$$\begin{array}{c} \partial(C, D_E) \\ \shortparallel \\ (\underset{\substack{\uparrow \\ \text{SLag.}}}{L}, \underset{\substack{\uparrow \\ \text{flat bdl.}}}{D_{E'}}) \end{array}$$

Claim (Conjecture) $\mathcal{M}^{\text{coass}}(M) \xrightarrow[\text{value}]{\text{boundary}} \mathcal{M}^{\text{SLag}}(X)$

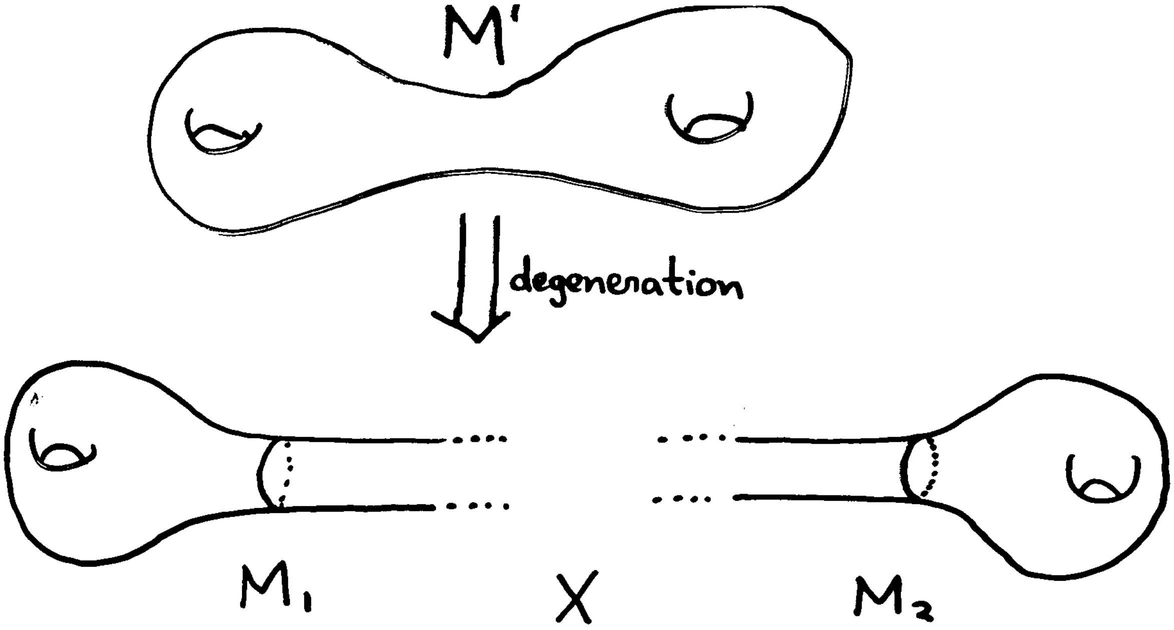
is a <u>Lagrangian</u> immersion.

Reason: ① Linearize $\rightsquigarrow$

$$\begin{array}{ccc} H^2_+(C) & \alpha & H^2(L) \\ \oplus & \longrightarrow & \oplus \\ H^1(C) & \beta & H^1(L) \end{array}$$

②

$$\begin{tikzcd}
H^2_+(C) \arrow[r, "\alpha"] & H^2(L) \arrow[r] & H^3(C, L) \\
\otimes & \otimes & \otimes \\
H^2_+(C, L) \arrow[d] & H^1(L) \arrow[l] \arrow[d, "\substack{\text{sympl. str} \\ \text{on } \mathcal{M}^{\text{SLag}}}"] & H^1(L) \arrow[l, "\beta"'] \arrow[d] \\
\mathbb{R} & \mathbb{R} & \mathbb{R}
\end{tikzcd}$$



$$\mathcal{M}^{coass}(M_1) \xhookrightarrow[\text{Lagr.}]{} \mathcal{M}^{SLag}(X) \xhookleftarrow[\text{Lagr.}]{} \mathcal{M}^{coass}(M_2)$$

"<u>Conjecture</u>" (Gluing Theorem).

$$\underline{H^{coass}(M) = HF^{\mathcal{M}^{SLag}(X)}_{Lag.}\left(\mathcal{M}(M_1), \mathcal{M}(M_2)\right)}$$

Floer homology group for Lagrangian Intersections.

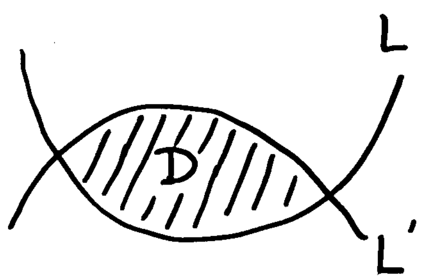
$\rightsquigarrow$ TQFT (Topological Quantum Field Theory).

§ Intersection Theory for Coassociative Submanifolds.

Recall: Floer's Lagrangian Intersection.

(X^{2n}, ω) symplectic manifold.

$L^n \subset X$ Lagrangian submfd (i.e. $\omega|_L = 0$)



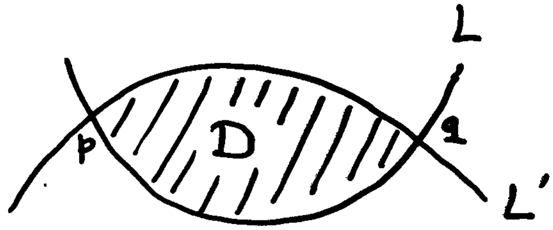
<u>$HF(L, L')$</u> <u>Floer homology group</u>

$\chi(\text{—}\,''\,\text{—}) = \# \, L \cap L'$.

By counting of <u>holomorphic</u> disks D :

$$\begin{cases} D^2 \subset X \\ \partial D \subset L \cup L' \\ D : \text{preserved by } J \end{cases}$$

$$\begin{cases} CF := \mathbb{Z}^{\#|L \cap L'|} \\ \delta : CF \longrightarrow CF \\ \delta p := \sum_{q \in L \cap L'} n(p,q)\, q \\ \qquad\qquad n(p,q) = \#\text{ holo. disks joining } p \text{ to } q. \\ \delta^2 = 0. \end{cases}$$



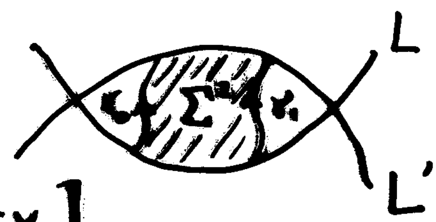
(Require compactness of moduli of holo. disks.)

$$HF(L, L') := \frac{\mathrm{Ker}(\delta)}{\mathrm{Im}(\delta)}.$$

Formally, $HF(L, L') = H^{\frac{\infty}{2}}(\mathcal{L}_{L,L'}X)$.

$$\mathcal{F} : \mathcal{L}_{L,L'}X \longrightarrow \mathbb{R}/\mathbb{Z}$$

$$\mathcal{F}(\gamma) = \int_{\Sigma} \omega$$



[Witten-Morse theory.]

Remark: Fukaya-Floer category;

Mirror Symmetry for C.Y. mfds.

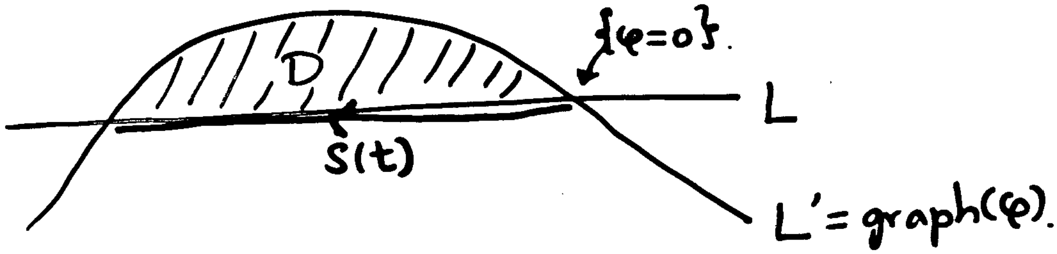
Remark (Nearby Lagrangian):

$$L^n \subset (X^{2n}, \omega) \quad \text{Lagr.}$$

$\Rightarrow N_{L/X} \cong T^*L$

Nearby Lagr. $\longleftrightarrow$ (small) Closed 1-form on L.

$L' = \text{graph}(\varphi)$ $\qquad \varphi \in \Omega^1(L)$, $d\varphi = 0$.



Holo. disks D bound $L + L'$ $\longleftrightarrow$ Gradient flow line $S(t)$ on X for φ. (+ bubbling)

[via singular perturbation].

BACK TO

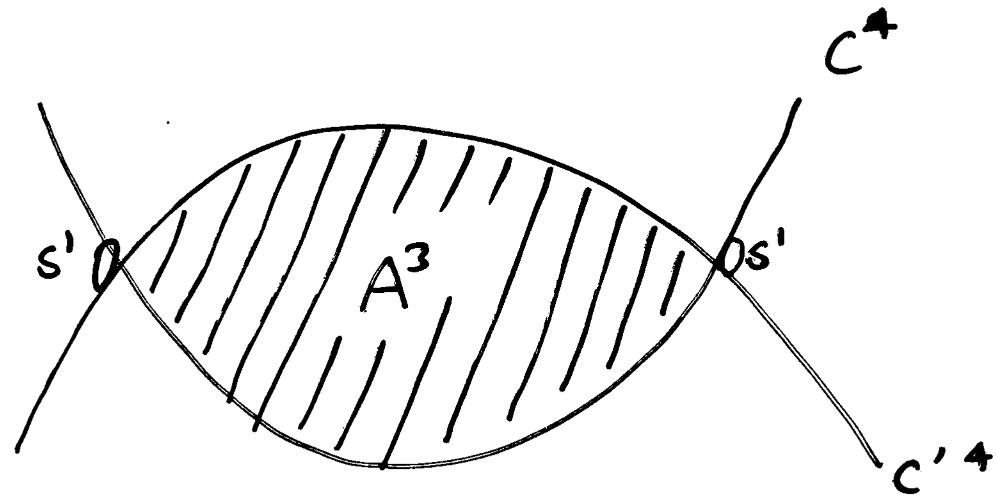
(M^7, Ω) G_2-manifold.

$C^4 \subset M$ Coassociative submfd.
(i.e. $\Omega|_C = 0$).

<u>Propose</u> to define $H(C, C')$,
analog to Floer homology,
by counting <u>associative</u> submfd.

$A^3 \subset M$ (i.e. A: preserved by $\times$)

with $\partial A \subset C \cup C'$.

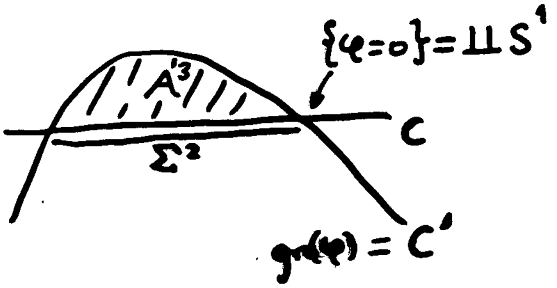


Remark: (Nearby coass. submfds.).

$C^4 \subset M$ coass.- submfd.

$$\Rightarrow \begin{cases} N_{C/M} \cong \Lambda^2_+(C) \\ \text{nearby coass. submfd. } C' = \mathrm{graph}(\varphi) \xleftrightarrow{(1^{st}\text{ order})} \varphi \in \Omega^2_+(C),\ d\varphi = 0. \end{cases}$$

$$C \cap C' = \{\varphi = 0\}$$



★ φ : Degenerated Sympl. form on C^4.

"Conjecture":

$\#$ ass. submfd. $A^3 \subset M^7$ bounding C & C'

Singular perturbat$^{\text{n}}$. $\#$ holo. curve $\Sigma^2 \subset C^4$ w/ bdy. $\{\varphi = 0\}$.

Taube Seiberg-Witten of C^4.

Comparison:

X^{2n}: Sympl. mfd.

$\omega \in \Omega^2(X)$ non-degen.

$d\omega = 0$

M^7: G_2-mfd.

$\Omega \in \Omega^3(M)$. positive

$d\Omega = 0$ (almost G_2).

$L^n \subseteq X$ Lagr.

$\omega|_L = 0$

$C^4 \subseteq M$ Coass.

$\Omega|_C = 0$.

- $\mathcal{N}_{L/X} \simeq \Lambda^1(L)$
- $\mathcal{N}_{C/M} \simeq \Lambda^2_+(C)$

- Nearby Lagr.

  $\sim \varphi \in \Omega^1(L)$, $d\varphi = 0$

- Nearby Coass. submfd.

  $\underset{\text{1st order}}{\sim} \varphi \in \Omega^2_+(C)$, $d\varphi = 0$

  (Degen. Sympl. 4-mfd).

- SUSY cycle

  (L, D_E)

  $F_E = 0$

  (i.e. flat connection)

- SUSY cycle.

  (C, D_E)

  $F_E^+ = 0$

  (i.e. ASD connection).

Comparison (Continued).

(Symplectic)	(G_2)
$\Sigma^2 \subseteq X$ Holo. curve	$A^3 \subseteq M$ Associative submfd.
Σ preserved by J	A preserved by $\times$
$\omega(v,w) = g(Jv, w)$	$\Omega(u,v,w) = g(u \times v, w)$.
• Disk (instanton)	• Handlebody (instanton).
• $S^2 = D^2 \cup_{S^1} D^2$	• $S^3_{\mathbb{Q}} = A_1^3 \cup_{\Sigma} A_2^3$ (handlebody decomposition).
• $\partial\Sigma \subseteq L$	• $\partial A \subseteq C$ (∂A has a canonical cpx. str.).

$HF(L, L')$ Floer homology group	"$H(C, C')$" ?

Loop Space Interpretation:

(M^7, Ω) G_2-manifold

$\mathcal{L}M$: Unparametrized Loop space.

$$\Omega \in \Omega^3(M) \xrightarrow{\int_\gamma \Omega} \omega_{\mathcal{L}M} \in \Omega^2(\mathcal{L}M)$$

$\Longrightarrow (\mathcal{L}M, \omega_{\mathcal{L}M})$: Symplectic mfd. (∞ dim.).

- $\mathcal{L}C \subset \mathcal{L}M$ Lagrangian
  $\Leftrightarrow$ $C^4 \subset M$ Coassociative
- $\Sigma^2 \subset \mathcal{L}M$ Holomorphic curve
  $\Leftrightarrow$ $\Sigma \times S^1 \subset M$ Associative

Thus

$$\underset{\text{for } M^7}{H(C, C')} \overset{\text{formal}}{\equiv} \underset{\text{for } \mathcal{L}M.}{HF(\mathcal{L}C, \mathcal{L}C')}$$

A

§ Gauge Theory on G_2-manifolds

$$\mathbb{R}^r \longrightarrow E \longrightarrow M^n$$

Connection $\quad D_E : \Omega^0(M;E) \longrightarrow \Omega^1(M,E)$

Curvature $\quad F_E = (D_E)^2 \in \Lambda^2(M) \otimes \mathrm{ad}E$.

$F_E = 0$ (flat bundle) $\leftrightsquigarrow \rho : \pi_1(M) \longrightarrow GL(r,\mathbb{R})$.

When M is (special) $\mathbb{A}$-manifold.

$(\mathbb{A} = \mathbb{C}, \mathbb{H}, \mathbb{O})$.

$\mathrm{Hol}(M,g)$ is $(H_{\mathbb{A}} \subset)\ G_{\mathbb{A}} \subset O(n)$.

$$\mathfrak{h}_{\mathbb{A}} \subset \mathfrak{g}_{\mathbb{A}} \subset \underline{o}(n) = \Lambda^2.$$

$\rightsquigarrow$ (Pantial Flat bundle)

Def.

① $F_E \in \mathfrak{g}_{\mathbb{A}}(M) \otimes \mathrm{ad}E$: <u>$\mathbb{A}$-connection</u>

② $F_E \in \mathfrak{h}_{\mathbb{A}}(M) \otimes \mathrm{ad}E$: <u>special $\mathbb{A}$-connection</u>

$\mathbb{A}$	$\mathbb{A}$-conn.	Special $\mathbb{A}$-conn.
$\mathbb{C}$	$F^{0,2}=0$ (Holomorphic Bdl.)	$F^{0,2}=\Lambda F=0$ (Herm. YM bdl.).
$\mathbb{H}$	$F\in sp(n)sp(1)(M,adE)$	$F^{02}_I=F^{02}_J=F^{02}_K=0$ (A.S.D. / Hyperholomorphic)
$\mathbb{O}$	$F\wedge\Theta+*F=0$ (Donaldson-Thomas bdl.)	$F\wedge *\Omega=0$ (Donaldson-Thomas bdl.)

Remark: All Special $\mathbb{A}$-connections minimize the Yang-Mills energy $\int_M |F|^2$. (via YM-calibration).

(M^7, Ω) G_2-manifold

D_E : D.T. G_2-connection/Special $\mathbb{O}$-conn.

$\left(\begin{array}{ll} \text{i.e.} & F_E \wedge *\Omega = 0 \\ \text{or} & F_E \in \Omega^2_{14}(M, \mathrm{ad}E) \end{array}\right)$

<u>Deformed</u> D.T.-conn. (SUSY cycles).

$$F_E \wedge *\Omega + F_E^3/6 = 0.$$

These are critical points of the C.S. functional $\mathcal{F}$

$$\mathcal{F}(D_E) = \int_{M^7} CS(D_0, D_E)\, e^{*\Omega}.$$

$$= \int_{M\times[0,1]} \mathrm{Tr}\left(e^{*\Omega + \tilde{F}}\right)$$

$\mathcal{M}^{\mathrm{bdl.}}$ denotes their moduli space.

(relevance to Mirror Symmetry for G_2-mfd.

'06.

Natural Geometric Structures on $\mathcal{M}^{bdl}$:

① 3-form / Correlation function

$$\Omega_{\mathcal{M}^{bdl}}(B_1, B_2, B_3) = \int_M \mathrm{Tr}\, B_1 B_2 B_3\, e^{*\Omega + F_E} + [123].$$

② Cubic form / Yukawa coupling

$$\mathcal{Y}_{\mathcal{M}^{bdl}}(B_1, B_2, B_3) = \int_M \langle B_1, [B_2, B_3]\rangle_{\text{Killing}}\, e^{*\Omega + F_E} + [123].$$

(Optional)

Remark: (M^7, Ω) : G_2-mfd.

$\Rightarrow$ T_M is a D.T.-bdl.

When $M^7 = \underset{CY^3}{X^6} \times S^1$, then

$\mathcal{Y}_{m^{bdl}(M)}$ on $H^1(M, \mathfrak{g}_2(T_M))$

$$\begin{array}{c} H^1(M, \mathfrak{g}_2(T_M)) \\ \wr\Vert \\ H^{1,1}(X) + H^{2,1}(X) + H^1(X, \mathrm{End}_0 T_X) \end{array}$$

becomes Yukawa couplings for CY 3-fold X

on (i) $H^{1,1}(X)$,

(ii) $H^{2,1}(X)$,

(iii) $H^1(X, \mathrm{End}_0 T_X)$.

§ Mirror Symmetry for G_2-mfd.s (Acharya).

Recall SYZ Mirror Conjecture for Calabi-Yau manifolds (X^{2n}, ω, Ω)

(1). SLag. fibration ($\Rightarrow$ fiber is T^n).

$$T^n \longrightarrow X^{2n} \longrightarrow S^n$$

(semi-flat limit $\sim$ large cpx. str. limit).

(2) Mirror manifold Y^{2n} : dual fibration

$$\check{T}^* \longrightarrow Y^{2n} \longrightarrow S^n$$

(3) Symplectic Geometry of X $\xleftrightarrow[\text{Transformation}]{\text{Fourier - Legendre}}$ Complex Geometry of Y

(e.g. Floer's theory on Lagrangian intersection.)

(e.g. Category of coherent sheaves, HYM-bundles.)

A.

(M^7, Ω) G_2-manifold.

- Coassciative fibration

$$C^4 \longrightarrow M^7 \longrightarrow A^3.$$

Expect $C = T^4$ or $K3$.

in the ~~semi-flat &~~ Adiabetic Limit.

Reason: ① (M, Ω_t) with $C \subset M$

2nd. fund. form $\longrightarrow 0$.

$\Rightarrow$ C^4 : Einstein.

$\Rightarrow$ Hitchin ineqt.

$$3 \, \mathrm{Sign}(C) + 2 \, \mathrm{Euler}(C) \geqslant 0.$$

② Coass. fibration $\Rightarrow$ $\Lambda^2_+(C)$ trivial

$\Rightarrow$ "=" in Hitchin ineqt.

$\Rightarrow$ C : T^4 or $K3$.

$$C = T^4 \sim \int_M p_1(M) \wedge \Omega_t \longrightarrow 0$$

$$C = K3 \sim \int_M p_1(M) \wedge \Omega_t \not\longrightarrow 0$$

Suppose

$$T^4 \longrightarrow M^7 \longrightarrow A^3$$

4-torus

Co-associative fibration.

$\rightsquigarrow$ Mirror manifold W^7 : Dual fibration.

$$\hat{T}^4 \longrightarrow W^7 \longrightarrow A^3.$$

SYZ G_2-mirror conjecture:

Coassociative Geometry of M	$\xleftrightarrow[\text{Transform}]{\text{Fourier}}$	Associative Geometry of W

(e.g. Coass. submfd. + ASD
Intersection theory)

(e.g. D.T.-bundle,
Ass. submfd. + flat bdl.
?)

(e.g. Can be verified in the flat case).

Suppose

$$K3 \longrightarrow M^7 \longrightarrow A^3$$ coassociative fibration.

"Mirror" manifold X^6 is CY 3-fold

with $E_8 \times E_8$ - bundle $/X$

$$T^3 \longrightarrow X^6 \longrightarrow S^3.$$

(e.g. GYZ).

$(T^3 = H^2_+(K3)/\Lambda$ in certain limit$)$.

§ Conclusions / Questions

- Special $\mathbb{O}$-geometry
  (Triality ?).

- (M^7, Ω)

  $d\Omega = 0$: G_2-symplectic Geometry.

  $d^*\Omega = 0$: G_2-complex Geometry.

- $\mathbb{R}$-manifold L

  (i) Any dim. Flat Bundles

  (ii) 3/4 dim. CS theory / Donaldson theory

  Complexify

  (i) $L \to TL$ complex geometry

  $\ L \to T^*L$ sympl. geometry

  mirror symmetry

  (ii) 6/7 dim. CY^3 / G_2-geometry.

- M-theory.