Natural phase velocity of magnetic islands

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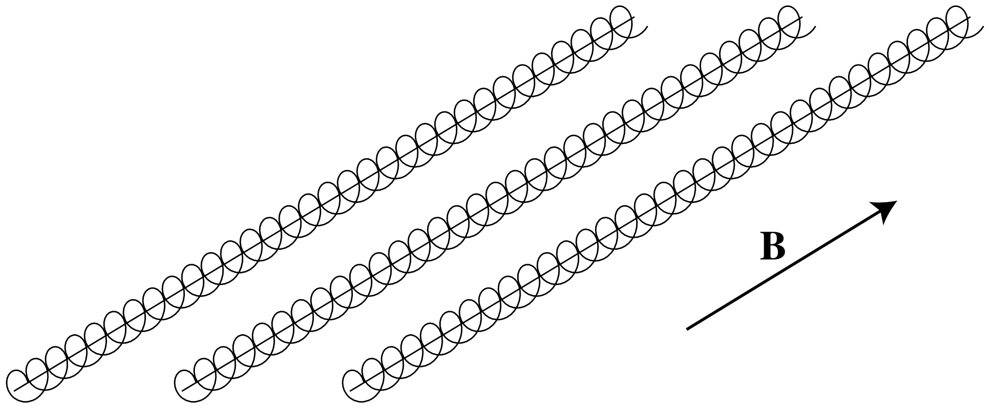
Particles and heat flow rapidly along magnetic field lines

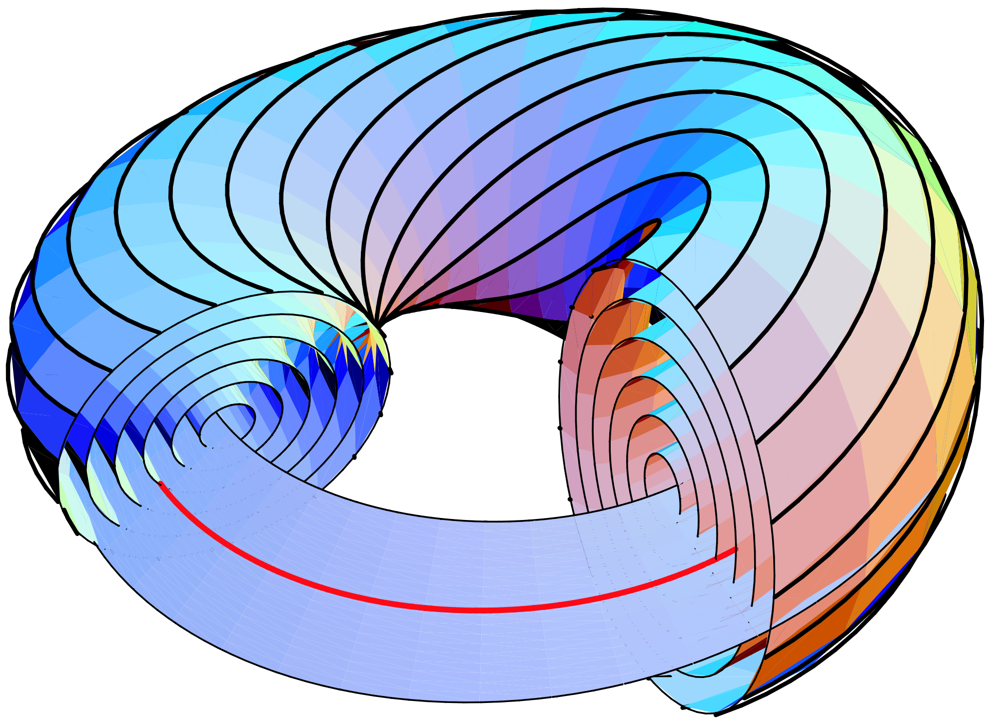


$$\omega_c = \frac{eB}{m}; \qquad \rho = \frac{v_{\text{th}}}{\omega_c}.$$

The ratio of thermal conductivities is very large:

$$\kappa_{\|}/\kappa_{\perp} = 10^8 - 10^{10}.$$

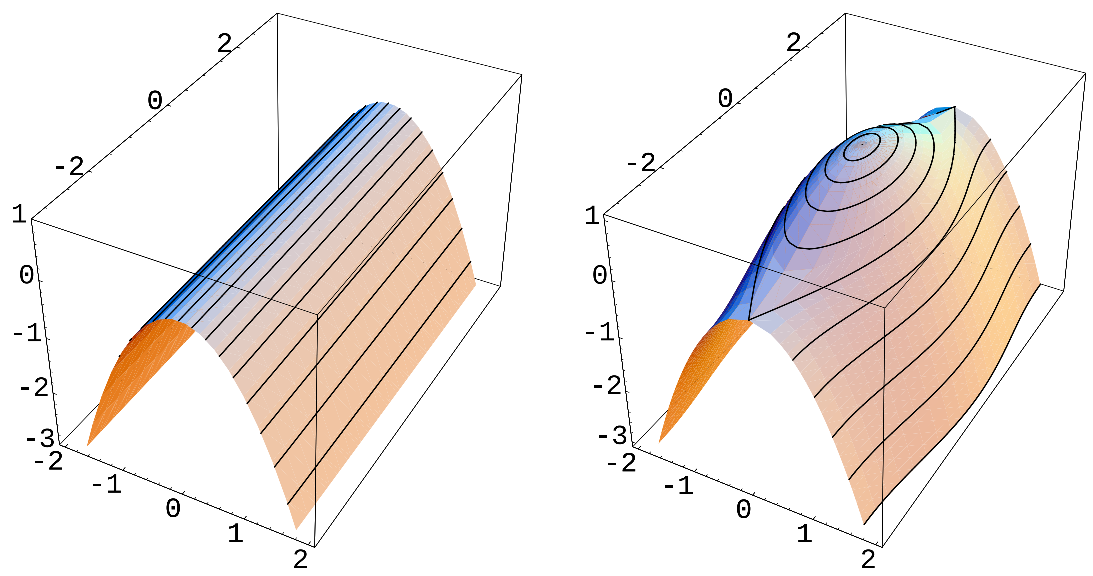
Magnetic confinement uses nested flux surfaces



There is a dense set of magnetic surfaces formed by field lines that close upon themselves.

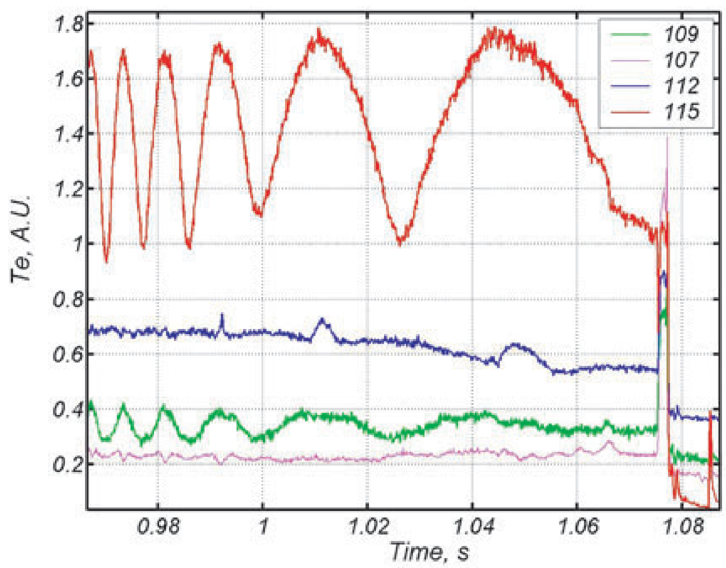
Symmetry-breaking perturbations create secondary magnetic axes

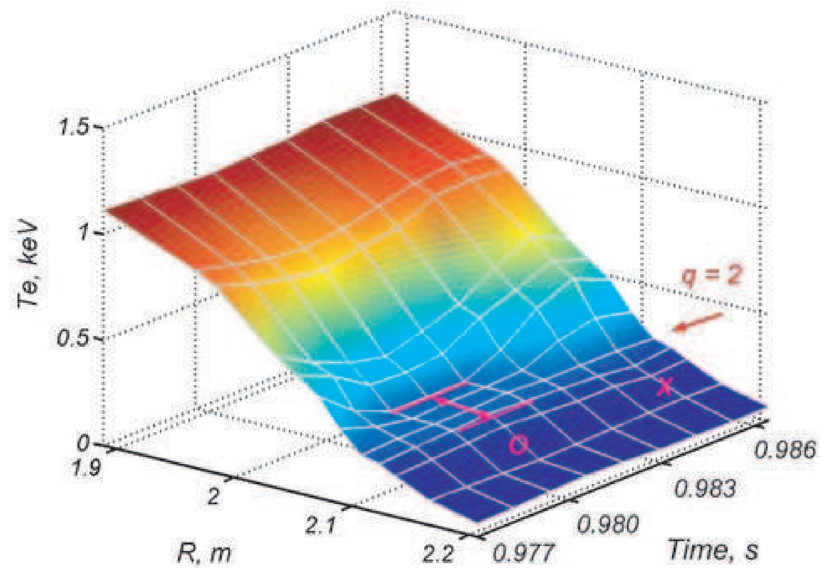
$$\mathbf{B} = \nabla\chi \times \nabla(\theta - \zeta/q) \quad \rightarrow \quad \mathbf{B}_h + \nabla\psi_h \times \nabla\zeta.$$



The perturbation allows rapid outward transport.

The flow of heat along magnetic field lines flattens the temperature inside the separatrix





(V. S. Udintsev et al., PPCF 2003)

Magnetic islands are a leading cause of anxiety

- Island overlap gives rise to magnetic stochasticity and (in tokamaks) disruptions.

- Experiments and theory show that increasing $\beta = 2\mu_0 p/B^2$ and decreasing ρ/L destabilizes neoclassical tearing modes.

- In stellarators, island minimization depends on precise shaping of current coils.

Island growth is controlled by the parallel current

The dominant factors affecting island growth are

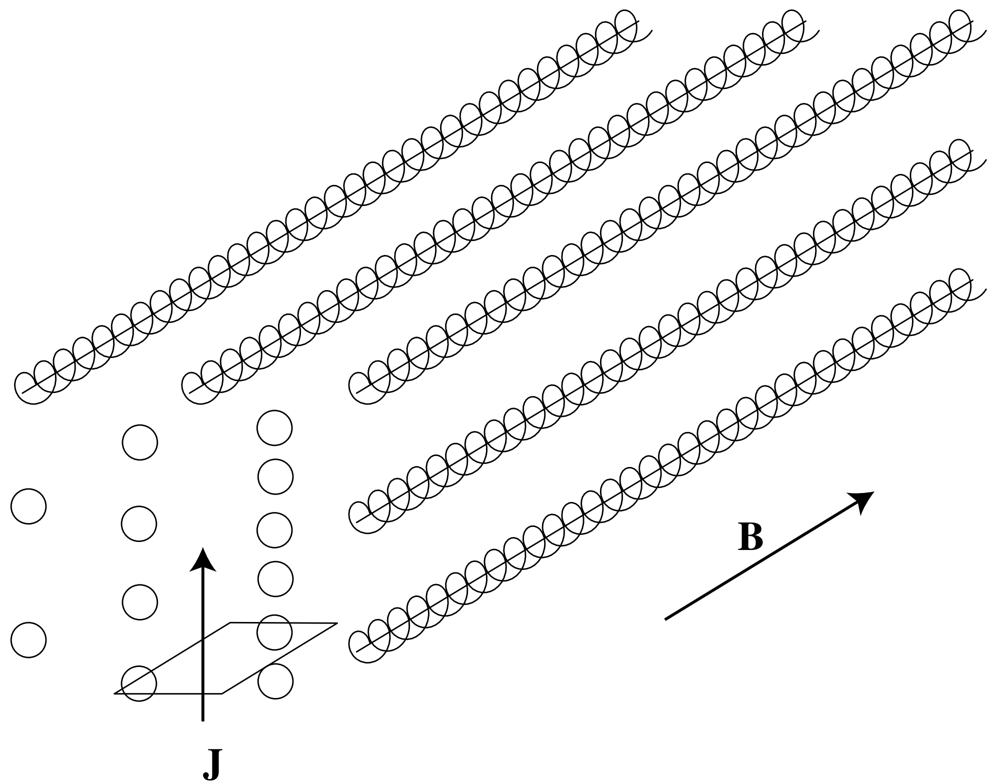
- The bootstrap current, proportional to the pressure gradient, and

- The polarization current, a quadratic function of the phase velocity of the island.

This talk will focus on the fluid theory of the polarization current.

The first question we must ask is why do islands rotate?

Reason 1: Inhomogeneous plasmas support diamagnetic flows



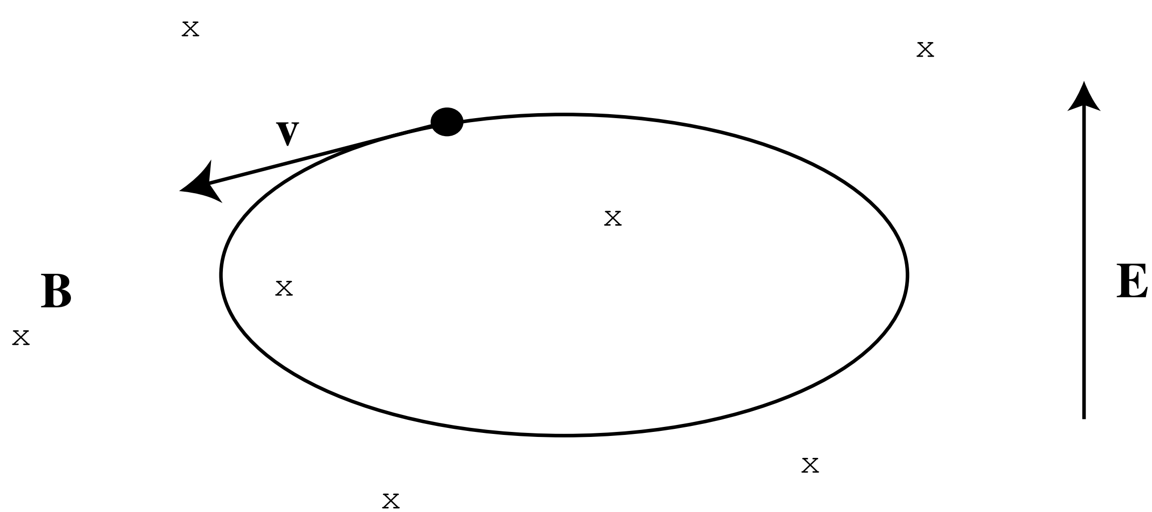
Electrons and ions have opposite diamagnetic flows.

Reason 2: Electric fields cause plasma flow

$$m\frac{d\mathbf{v}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

For $\omega \ll \omega_c = eB/m$, the solution is

$$\mathbf{v} = \frac{E}{B}\hat{\mathbf{x}} + \frac{1}{\omega_c}\frac{d}{dt}\left(\frac{E}{B}\right)\hat{\mathbf{y}}.$$

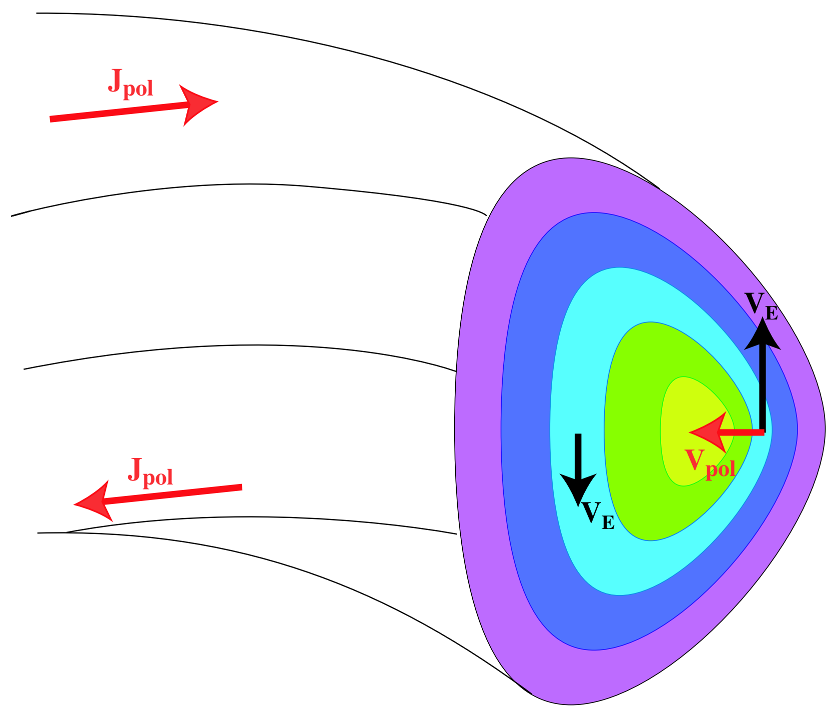


The inertial correction to the electric drift is called the polarization drift.

The polarization draws a parallel current

Quasi-neutrality gives rise to a return current flowing along the magnetic field:

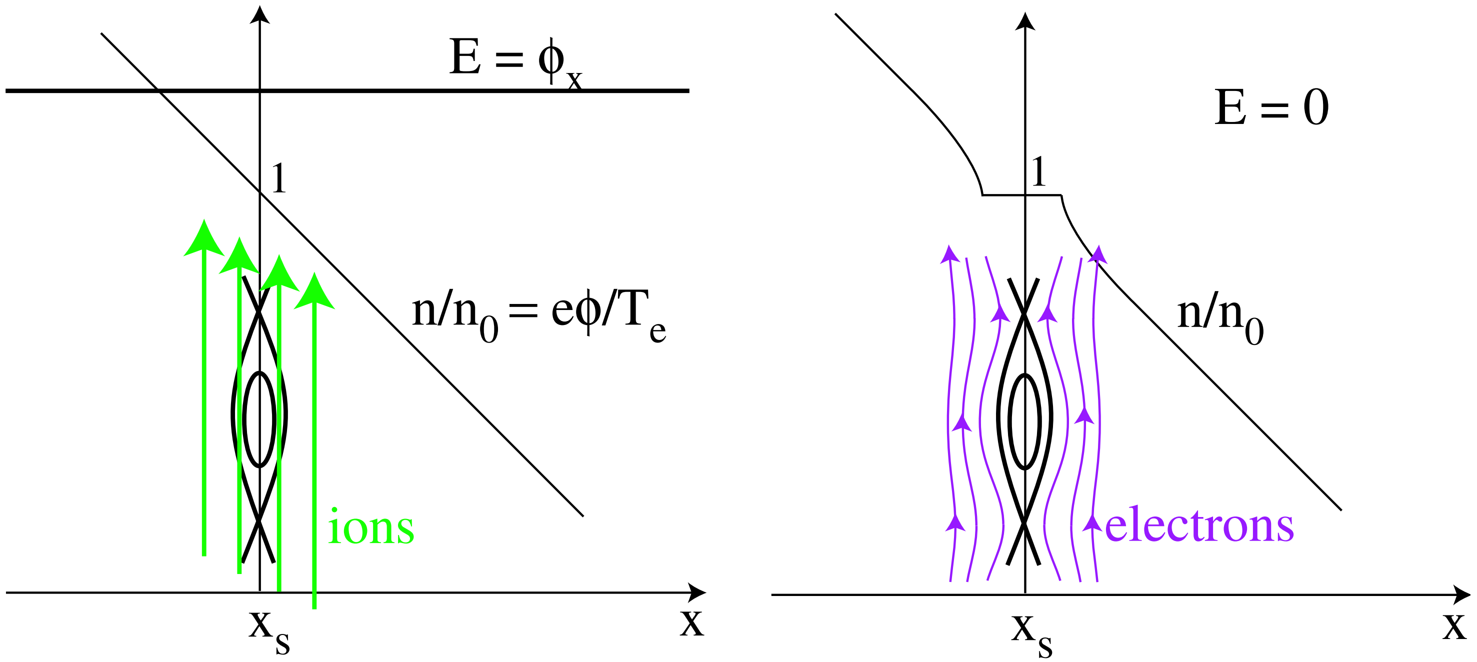
$$v_{\mathsf{pol}} \sim \phi'\phi'' \sim \phi/L^3$$



Do islands follow electrons or ions?

- The frozen-in law suggests the island should rotate with the electrons, $\omega = \omega_{*e}$

- Ion viscosity suggests the island should rotate with the ions.



Model

We consider a cold-ion ($T_i = 0$) incompressible ($\nabla_\| v_\| = 0$) fluid model in a slab.

$$
\begin{aligned}
\partial_t \psi + \mathbf{v}_E \cdot \nabla \psi + \nabla_\| n + \hat{\alpha} \nabla_\| T &= C j, & (1)\\
\partial_t U + \mathbf{v}_E \cdot \nabla U - \nabla_\| j &= \mu \nabla_\perp^2 U, & (2)\\
\partial_t n + \mathbf{v}_E \cdot \nabla n - \nabla_\| j &= D \nabla_\perp^2 n, & (3)\\
\tfrac{3}{2}(\partial_t T + \mathbf{v}_E \cdot \nabla T) - \hat{\alpha} \nabla_\| j &= \frac{\kappa}{C} \nabla_\|^2 T + \kappa_\perp \nabla_\perp^2 T, & (4)
\end{aligned}
$$

where $C = 0.51(\nu_e/\omega_*)(m_e/m_i)(L_s/L_n)^2$.

The obligatory multiscale slide

Key fact of island life is $k_{\|} = kx/L_s \sim kW/L_s$.

- Time scales:

	plasma	island
Alfven frequency ω_A	30 MHz	50 kHz
Drift frequency ω_*	50 kHz	10 kHz
Transport rate τ_E^{-1}	1 Hz	2 kHz

- Space scales:

plasma	1 m
$\omega_* = k_{\|} c_s$	10^{-1}m
island	$2 \cdot 10^{-2}$m
Ion gyro-radius ρ_i	10^{-3}m

Projective integration for thin islands

The flux variation across a thin island is negligible: $W \nabla \psi \ll \psi$. It follows that the shape of the island is fixed by its amplitude alone:

$$\psi = \frac{x^2}{2} + \tilde{\psi} \cos\left(y - \int \omega(t)\, dt\right).$$

The macroscopic equations are

$$\begin{aligned} \frac{dW}{dt} &= \Delta' + \widehat{\beta} \frac{\omega(\omega - \omega_*)}{W(W^2 + \rho^2)} && (5) \\ \frac{d\omega}{dt} &= g(W, \omega) && (6) \end{aligned}$$

where $W = 2\sqrt{\tilde{\psi}}$ is the half-width of the island.

Equilibrium solution

- Electron stream function

$$n = \varphi + H(\psi).$$

- Vorticity equation

$$\nabla^2 \varphi = K(\varphi) + H(\psi)$$

- Grad-Shafranov equation

$$J = I(\psi) + \varphi \frac{dH}{d\psi}.$$

Transport equations

- Particle transport equation:

$$\frac{dH}{d\psi} = \frac{\frac{D}{\kappa_\perp}\left(1 - \langle \partial^\psi \varphi \rangle_\psi\right)\Gamma + \frac{3}{2}\frac{C}{\kappa_\perp}\,\widehat{\alpha}\eta_e \Upsilon}{\frac{D}{\kappa_\perp}\left(\langle x^2 \rangle_\psi + \gamma \Upsilon\right)\Gamma + \frac{C}{\kappa_\perp}\,\widehat{\alpha}^2 \Upsilon \langle x^2 \rangle_\psi}. \tag{7}$$

where $\Upsilon = \langle \varphi^2 \rangle_\psi - \langle \varphi \rangle_\psi^2 / \langle 1 \rangle_\psi$ and $\Gamma = \langle x^2 \rangle_\psi + 3\frac{C}{\kappa_\perp}\Upsilon/2$.

- Potential vorticity transport equation

$$\frac{dK}{d\varphi} = \frac{D}{\mu}\left(1 - \frac{1}{\langle \varphi_x^2 \rangle_\varphi}\right) + \sigma\left(\frac{D}{\mu} - 1\right)\frac{\langle \varphi_x \partial_x H(\psi) \rangle_\varphi}{\langle \varphi_x^2 \rangle_\varphi}. \tag{8}$$

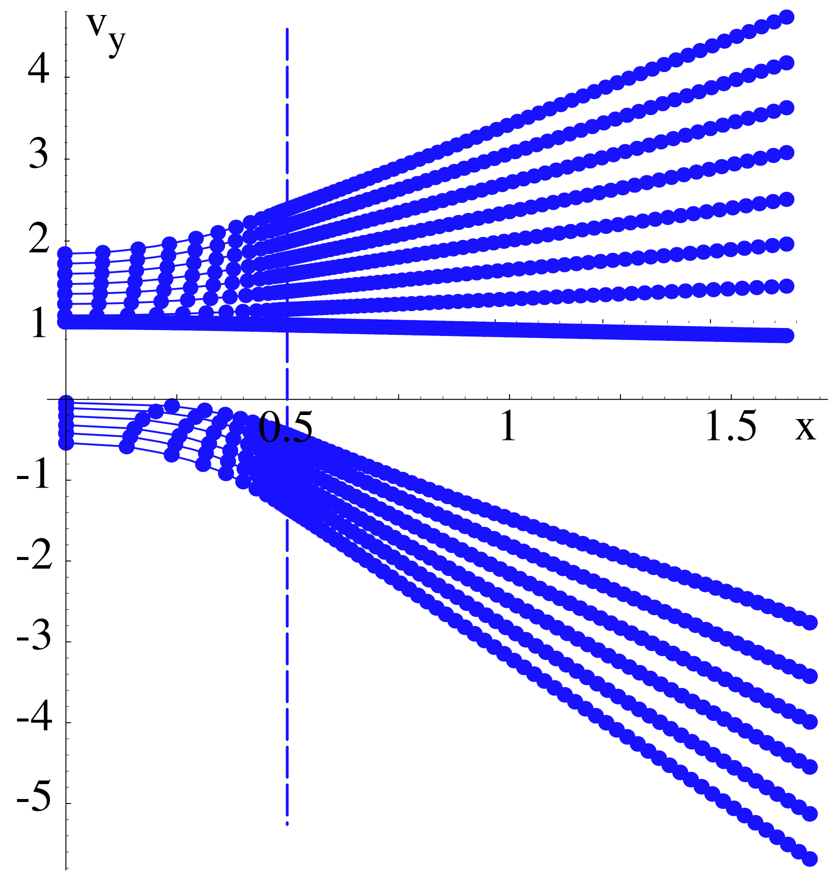
We solve the equilibrium and transport equations iteratively (H. Wilson)

$$\nabla^2 \varphi^{(n+1)} = K^{(n)}(\varphi^{(n+1)}) + H^{(n)}(\psi)$$

$$\frac{dK^{(n+1)}(\varphi)}{d\varphi} = F(\varphi^{(n)}; \varphi); \qquad \frac{dH^{(n+1)}(\psi)}{d\psi} = G(\varphi^{(n)}; \psi)$$

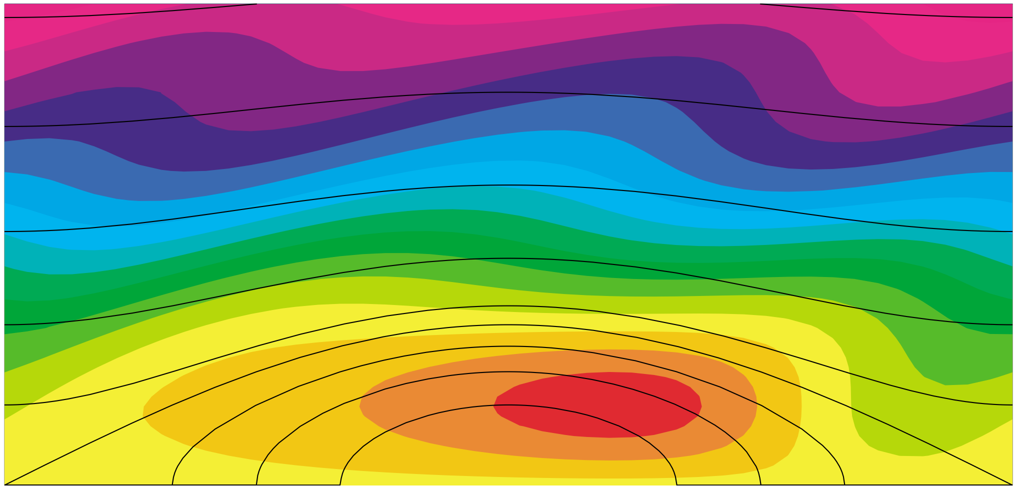
Velocity profiles are V-shaped

We determine the value of ω by requiring that the torque vanish, $\lim_{x\to\infty} \mu V_y' = 0$.



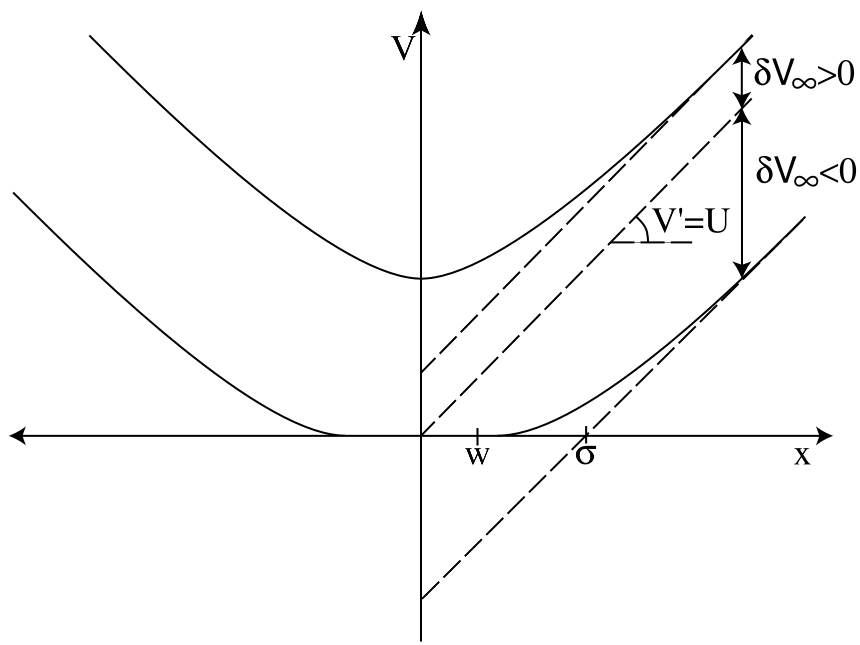
For $0 < \omega < \omega_{*e}$, the island excites drift waves

- The island emits a bow wave as it rotates.
- The island acts as a cavity resonator for the drift waves.
- Convection cells inside the island may act as ball bearings,

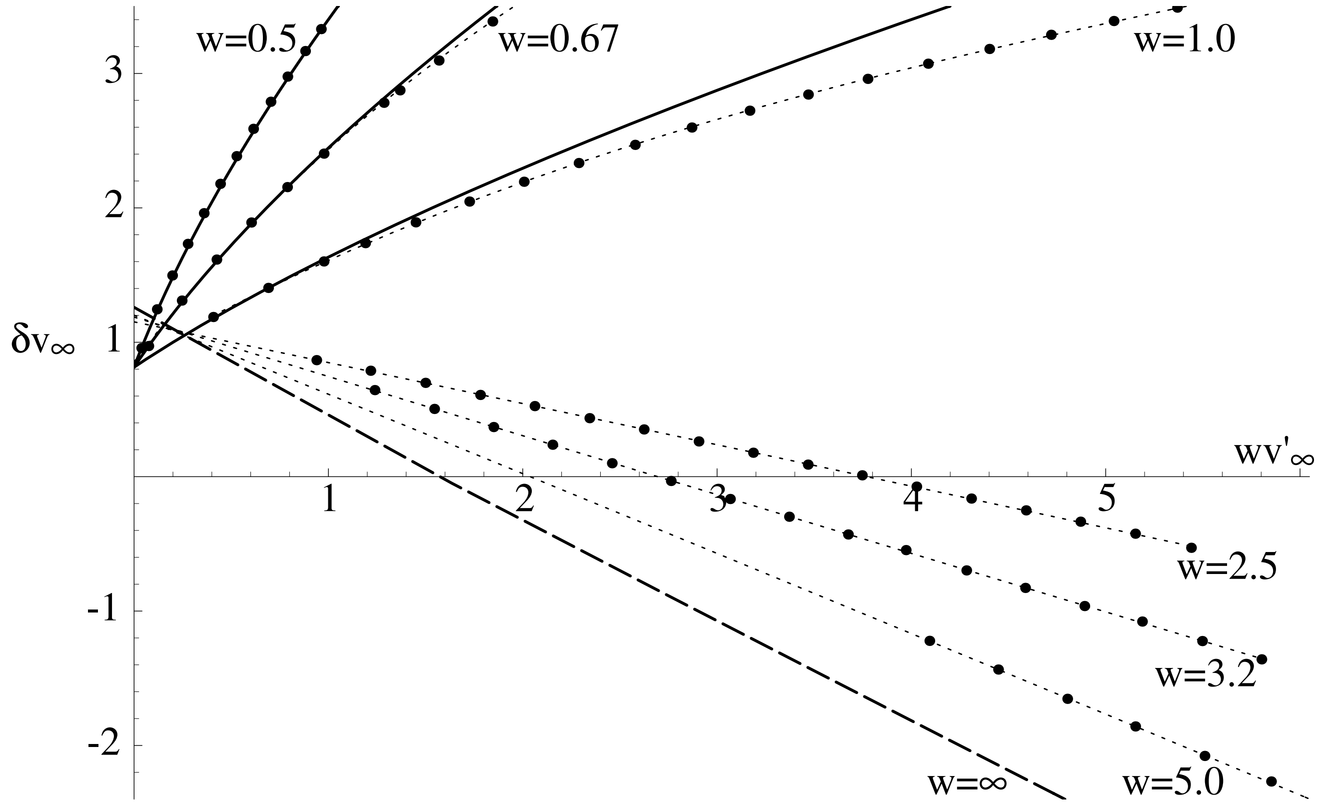


The slip velocity measures island permeability

- $\delta V_\infty > 0$ indicates that plasma is flowing through the island.
- $\delta V_\infty < 0$ indicates that the island is flattening the velocity profile. For $\delta V_\infty < 0$, we may define an effective island cross-section as $\sigma = |\delta V_\infty|/V'$



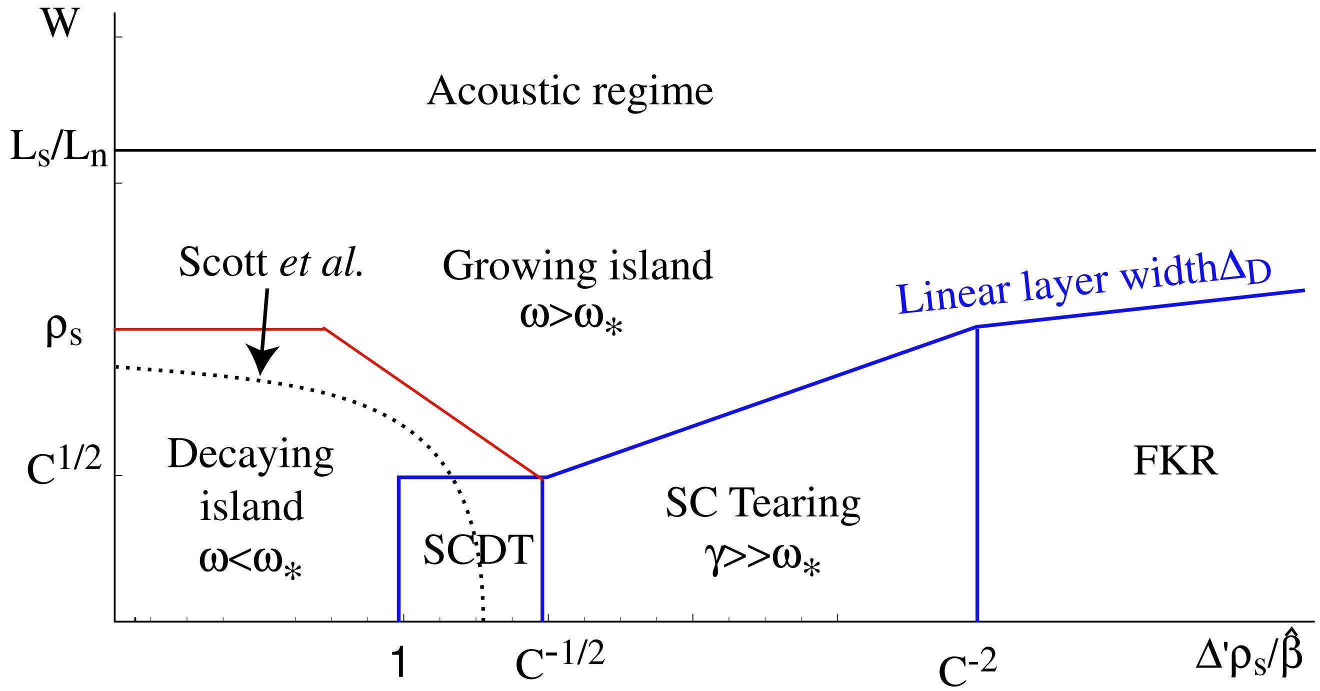
Islands become impermeable for $W > \rho_s$



Slip velocity as a function of the torque for intermediate size islands ($\rho_s \ll W \ll \rho_s L_s / L_n$).

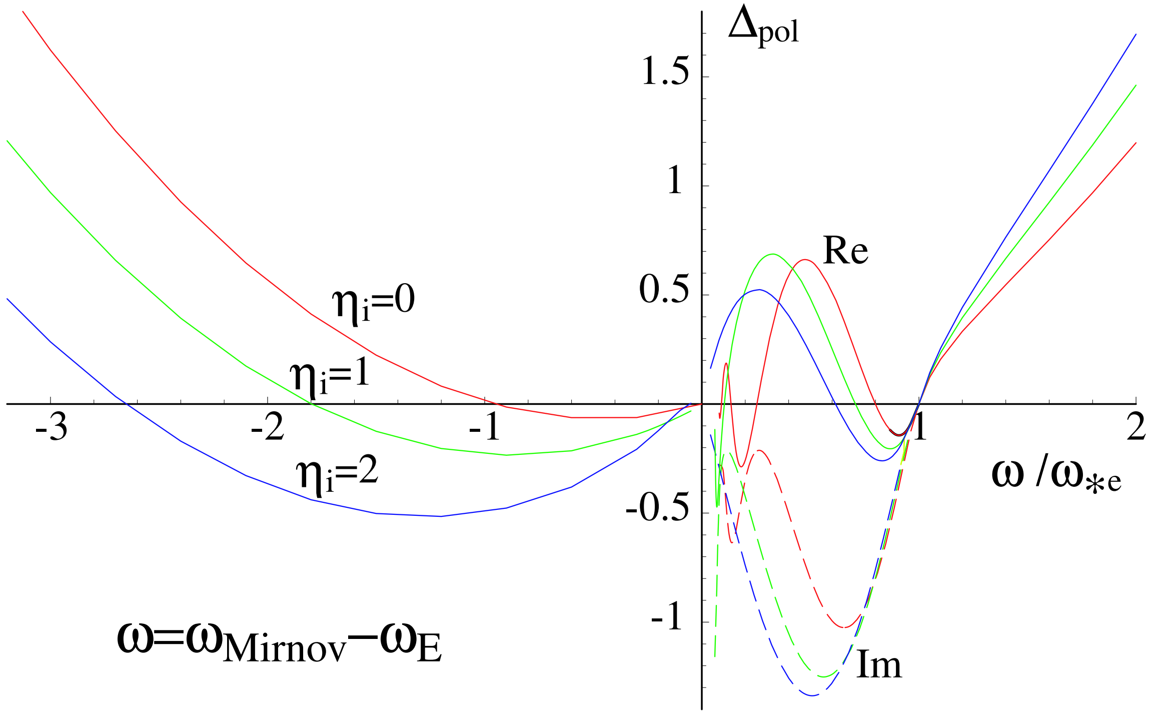
The polarization drift gives rise to an amplitude threshold for drift-tearing mode excitation

The stabilizing effect of the polarization drift for $W < \rho_s$ gives rise to a critical island width for excitation of drift-tearing modes. The scaling of the critical width found here agrees with that found numerically by Scott *et al.*



What about ion temperature?

The linear solution of ion gyrokinetic equation shows that the frequency band $\omega_i < \omega < 0$ is stable.



Unfortunately, fluid FLR models are not integrable.

Equilibrium integrability is a consequence of Hamiltonian dynamics

- Coherent structures are equilibrium solutions in a moving frame. For a system obeying Hamiltonian dynamics,

$$\frac{\partial \xi^j}{\partial t} = \{\xi^j, H\} = \{\xi^j, H + C\} = 0. \tag{9}$$

- The extrema of the functional $F = H + C$,

$$\delta F = 0, \tag{10}$$

are solutions of these equations.

- Equation (10) is a first-integral of Equation (9).

Consider BOUT model

$$\frac{\partial V_{\parallel e}}{\partial t}+(\mathbf{V_E}+V_{\parallel e}\mathbf{b}_0)\cdot\boldsymbol{\nabla} V_{\parallel e}=-\frac{e}{m_e}E_\parallel-\frac{1}{N_i m_e}(T_e\partial_\parallel N_i+1.71N_i\partial_\parallel T_e)+0.51\nu_{ei}(V_{\parallel i}-V_{\parallel e})$$
$$-\frac{1}{N_i m_e}\frac{2}{3}B^{3/2}\partial_\parallel(B^{-3/2}(P_\parallel-P_\perp)_e)+\frac{S^m_{\parallel e}}{N_i m_e}-\frac{S^p_e}{N_i}V_{\parallel e}, \tag{1}$$

$$\frac{\partial \varpi}{\partial t}+(\mathbf{V_E}+V_{\parallel i}\mathbf{b}_0)\cdot\boldsymbol{\nabla}\varpi=(2\omega_{ci})\mathbf{b}_0\times\boldsymbol{\kappa}\cdot\left(\boldsymbol{\nabla}P+\frac{1}{6}\boldsymbol{\nabla}(P_\parallel-P_\perp)_i\right)+N_iZ_ie\frac{4\pi V_A^2}{c^2}\boldsymbol{\nabla}_\parallel j_\parallel+\mu_{ii}\nabla_\perp^2\varpi$$
$$-(B\omega_{ci})\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{b}_0}{B}\times(\mathbf{S}^m_e+\mathbf{S}^m_i)\right)-\left(\frac{S^p_i}{N_i}\right)\varpi-(\omega_{ci}B)\boldsymbol{\nabla}\left(\frac{S^p_i}{N_i\omega_{ci}B}\right)\cdot(N_iZ_ie\boldsymbol{\nabla}\phi+\boldsymbol{\nabla}P_i)$$
$$-\tfrac{1}{2}\{N_iZ_ie\mathbf{V}_{\mathbf{Pi}}\cdot\boldsymbol{\nabla}(\nabla_\perp^2\phi)-m_i\omega_{ci}\mathbf{b}\times\boldsymbol{\nabla}N_i\cdot\boldsymbol{\nabla}V_E^2\}+\tfrac{1}{2}\{\mathbf{V_E}\cdot\boldsymbol{\nabla}(\nabla_\perp^2P_i)-\nabla_\perp^2(\mathbf{V_E}\cdot\boldsymbol{\nabla}P_i)\} \tag{2}$$

$$\frac{\partial N_i}{\partial t}+(\mathbf{V_E}+V_{\parallel i}\mathbf{b}_0)\cdot\boldsymbol{\nabla}N_i=\left(\frac{2c}{eB}\right)\mathbf{b}_0\times\boldsymbol{\kappa}\cdot(\boldsymbol{\nabla}P_e-N_ie\boldsymbol{\nabla}\phi)+\boldsymbol{\nabla}_\parallel\left(\frac{j_\parallel}{e}\right)-N_i\boldsymbol{\nabla}_\parallel V_{\parallel i}+S^p_e, \tag{3}$$

$$\frac{\partial T_i}{\partial t}+(\mathbf{V_E}+V_{\parallel i}\mathbf{b}_0)\cdot\boldsymbol{\nabla}T_i=\frac{4}{3}\left(\frac{cT_i}{N_ieB}\right)\mathbf{b}_0\times\boldsymbol{\kappa}\cdot\left(\boldsymbol{\nabla}P_e-N_ie\boldsymbol{\nabla}\phi-\frac{5}{2}N_i\boldsymbol{\nabla}T_i\right)+\frac{2}{3N_i}\boldsymbol{\nabla}_\parallel(\kappa^c_{\parallel i}\partial_\parallel T_i)$$
$$+\frac{2S^E_i}{3N_i}-\frac{2T_i}{3N_i}\left(N_i\boldsymbol{\nabla}_\parallel V_{\parallel i}-\frac{1}{e}\boldsymbol{\nabla}_\parallel j_\parallel\right)+\frac{2}{3N_i}\left(\frac{20}{3}\mu_{ii}\right)\nabla_\perp^2T_i, \tag{4}$$

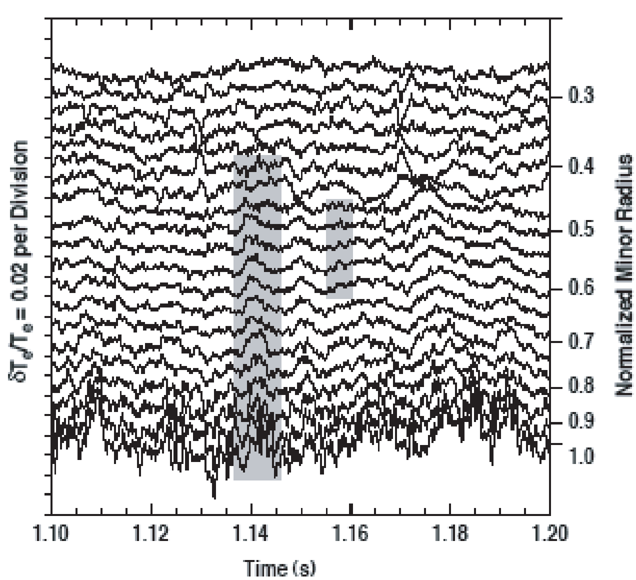
$$\frac{\partial T_e}{\partial t}+(\mathbf{V_E}+V_{\parallel e}\mathbf{b}_0)\cdot\boldsymbol{\nabla}T_e=\frac{4}{3}\left(\frac{cT_e}{N_ieB}\right)\mathbf{b}_0\times\boldsymbol{\kappa}\cdot\left(\boldsymbol{\nabla}P_e-N_ie\boldsymbol{\nabla}\phi+\frac{5}{2}N_i\boldsymbol{\nabla}T_e\right)+\frac{2}{3N_i}\boldsymbol{\nabla}_\parallel(\kappa^c_{\parallel e}\partial_\parallel T_e)$$
$$+\frac{2\eta_\parallel}{3N_i}j_\parallel^2-\frac{2T_e}{3}\boldsymbol{\nabla}_\parallel V_{\parallel e}+0.71\frac{2T_e}{3N_ie}\boldsymbol{\nabla}_\parallel j_\parallel+\frac{2S^E_e}{3N_i}, \tag{5}$$

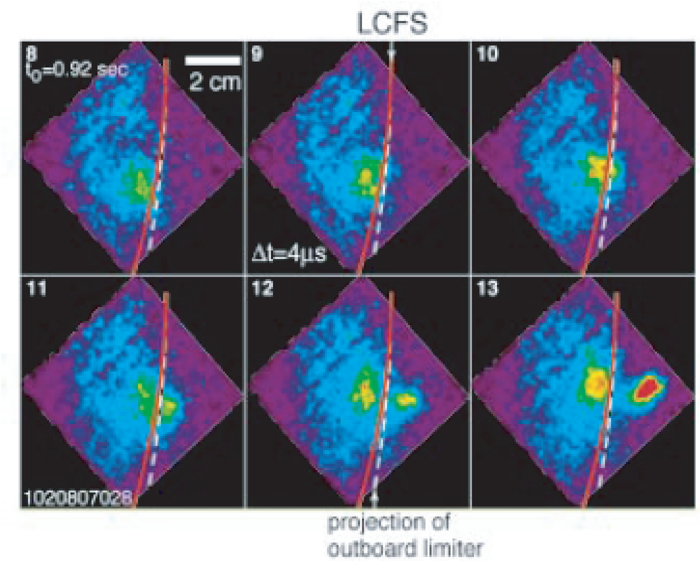
$$\frac{\partial V_{\parallel i}}{\partial t}+(\mathbf{V_E}+V_{\parallel i}\mathbf{b}_0)\cdot\boldsymbol{\nabla}V_{\parallel i}=-\frac{1}{N_iM_i}\partial_\parallel P-\frac{1}{N_iM_i}P_i\boldsymbol{\nabla}\times(\mathbf{b}/\omega_{ci})\cdot\boldsymbol{\nabla}V_{\parallel i}$$
$$-\frac{2}{3}\frac{1}{N_iM_i}B^{3/2}\partial_\parallel(B^{-3/2}(P_\parallel-P_\perp)_i)+\frac{S^m_{\parallel i}}{N_iM_i}-\frac{S^p_i}{N_i}V_{\parallel i}, \tag{6}$$

Is this Hamiltonian? How badly do we want our models to be?

Importance of coherent structures

- Zonal flows and avalanches are examples of CS in 3D. Other CS are more elusive but may nevertheless affect transport properties.





We explore role of the Hamiltonian property with a simple model

Consider standard FLR fluid equations for p, φ, v with two models for the electron response:

- **full Boltzmann response** This generalizes the Boltzmann model $n = \phi$, so as to ensure Galilean invariance,

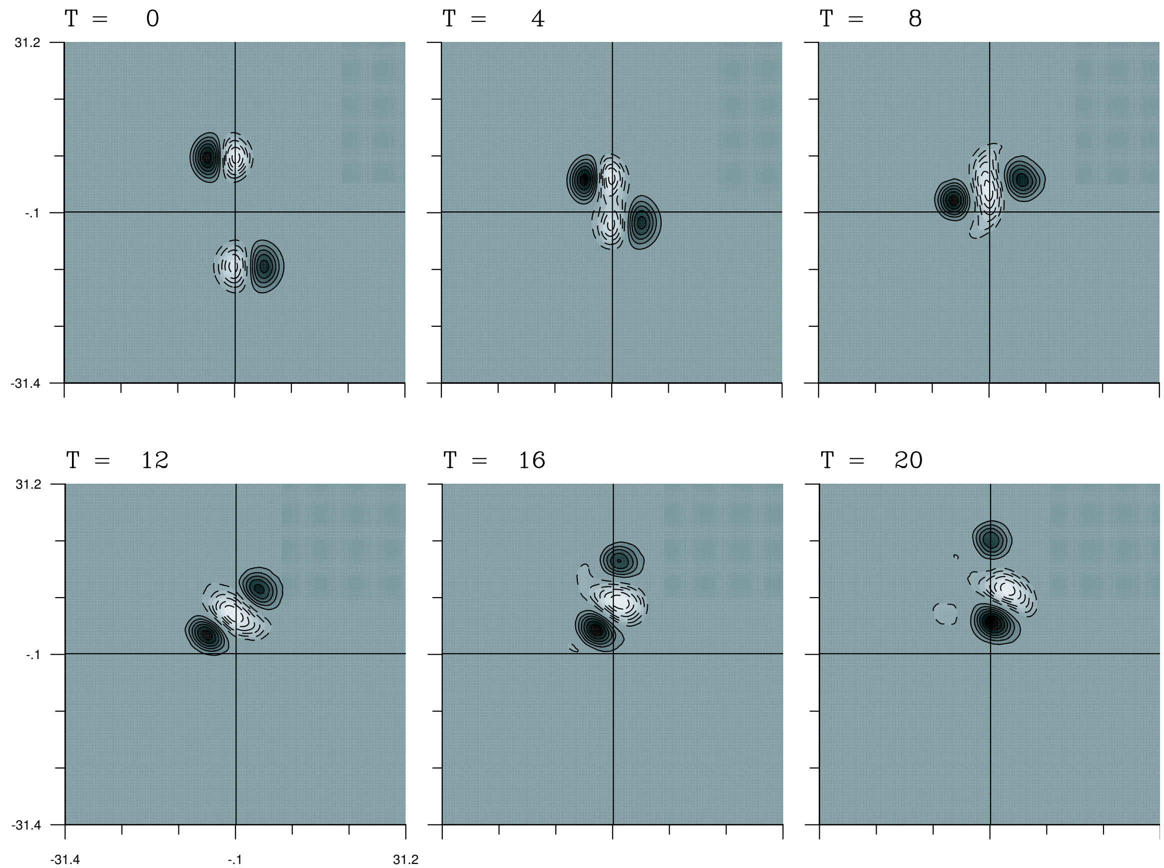
$$n = \phi - ux, \tag{11}$$

  where u is a constant background velocity in the y direction.

- **parallel Boltzmann response**

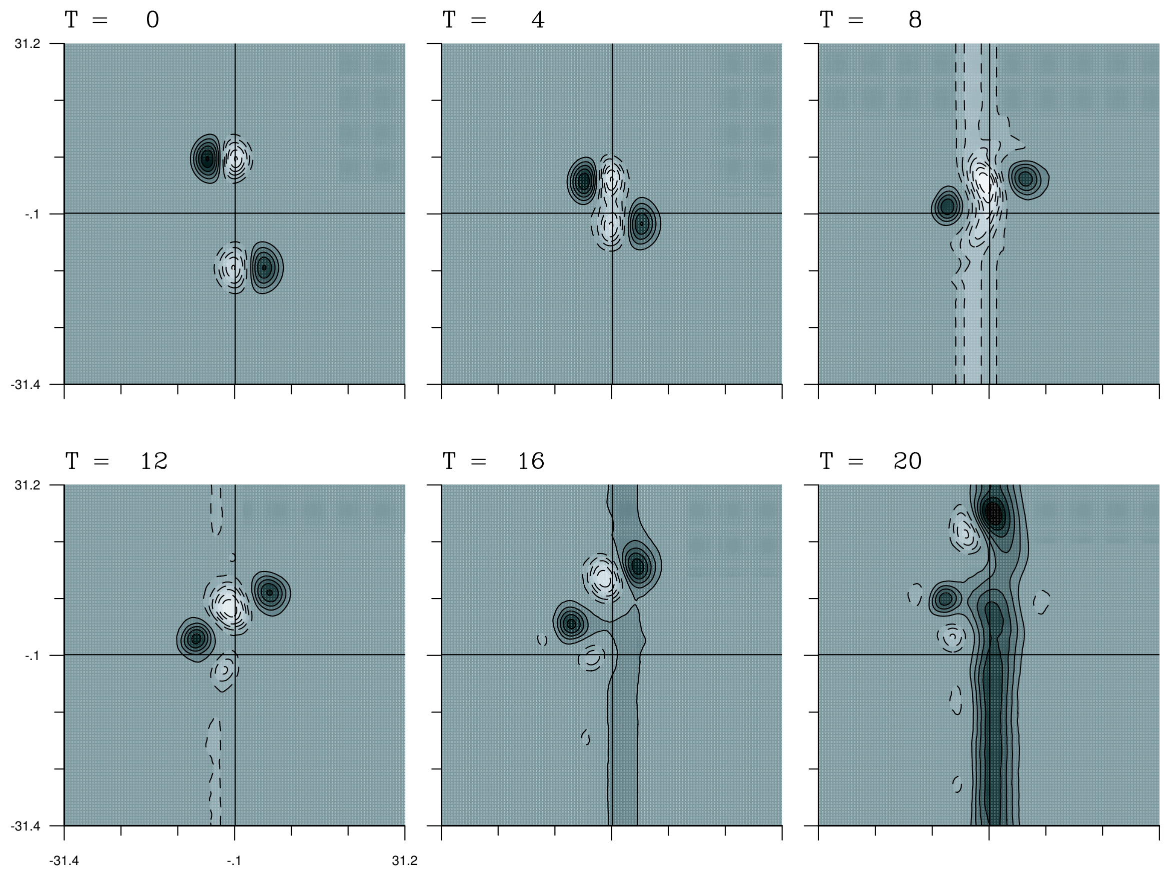
$$n = \tilde{\phi} := \phi - \bar{\phi}, \tag{12}$$

Modon collision under Boltzmann model



Bipolar modons retain coherence with Boltzmann electron response.

Modon collision under parallel Boltzmann model



The parallel Boltzmann response leads to zonal flow generation

Results of the Poisson bracket construction

- The Fully Boltzmann model is Hamiltonian

- There are three families of detailed conservation laws (Casimirs) constraining the motion.

- For $\Gamma > 0$, we were unable to write the parallel Boltzmann model in Hamiltonian form.

- Does this mean that there are no CS for parallel Boltzmann electrons and $\Gamma > 0$?

Existence of equilibria for $\Gamma > 0$

- We solve the equilibrium equation in powers of $\Gamma \ll 1$ for the Parallel Boltzmann model. The first-order equation is

$$\mathbf{v}_E \cdot \nabla \,(\text{first order corrections}) = \frac{\partial^2 \bar{\phi}_0}{\partial x^2} \frac{\partial^2 \phi_0}{\partial x \partial y}.$$

- Near the center of a convection cell, the solubility condition for the above equation is

$$\left[\frac{2\pi \partial_x^2 \bar{\phi}_0 \partial_{xy} \phi_0}{\sqrt{\partial_x^2 \bar{\phi}_0 \partial_y^2 \bar{\phi}_0 - (\partial_{xy}\phi_0)^2}} \right]_{\mathbf{x}=\mathbf{x}_{\mathrm{max}}} = 0.$$

  This is satisfied only if the major axes of the streamlines are aligned with the coordinate axes.

  Conclusion: There are no elliptical CS for $\Gamma > 0$.

Discussion

- We conjecture that the existence of Casimirs is responsible for the robustness of coherent structures.

- The parallel Boltzmann model appears to be non-Hamiltonian.

- The parallel Boltzmann model lacks non-trivial coherent structures.

- We need Hamiltonian FLR fluid models.