Optimal Order Display in Limit Order Markets with Liquidity Competition

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[1]Based on joint work with Gökhan Cebiroğlu; to appear in JEDC

Outline

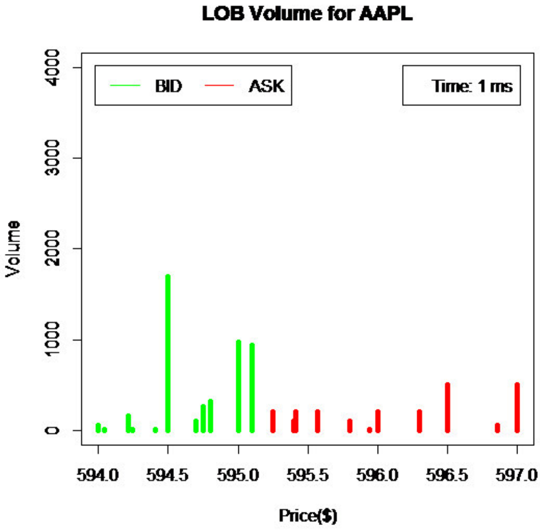
- Limit Order Books (LOBs) and hidden liquidity
- A sequential trade model with
  - a *hidden liquidity trader* that needs to buy a (large) position
  - *liquidity competition* at submission and more competitive price level
  - random *market order flow* executing standing liquidity
- Optimal dispaly strategies under market impact
  - openly displayed orders impact market dynamics
  - exchanges allow traders to shield orders from public view
  - tradeoff between loss in execution priority and reduced impact
- Closed form solutions under pure liquidity competition
  - model performs well for liquid stocks and medium order sizes
  - show expected shortfalls for selected (liquid) stocks

Hidden Liquidity in LOBs

Hidden Liquidity in Limit Order Books

- Electronic exchanges organize trading through *limit order books*
- Market participants can submit
  - market orders for immediate execution
  - limit orders for future execution
  - cancellations of standing orders
- Orders are executed according to a set of priority rules:
  - price priority
  - time priority

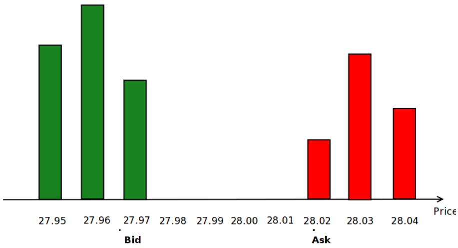
188 ms in the life of the Apple stock



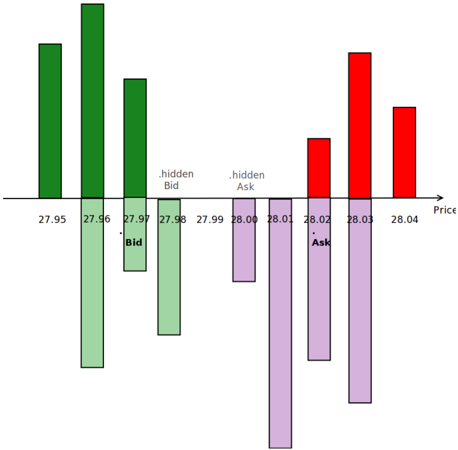
Hidden Liquidity in Limit Order Books

- Electronic exchanges organize trading through *limit order books*
- Market participants can submit
  - market orders for immediate execution
  - limit orders for future execution
  - cancellations of standing orders
- Orders are executed according to a set of priority rules:
  - price priority
  - time priority
  - *display priority*
- Markets allow traders to hide some/all of their orders
  - Hidden orders account for 20%-30% of liquidity
  - If orders can be fully hidden, there may be liquidity in the spread

Hidden orders



Hidden orders



Why Hiding?

- Large limit orders have market impact
  - private information ("scare away other traders")
  - large displayed orders encourage price undercutting ("impatience")
- Hiding orders helps to reduce market impact
  - tradeoff between reduced market impact and loss in time priority
  - we study optimal display decisions under liquidity competition
- Market impact of market orders has been studied extensively
  - plenty of models of portfolio liquidation with market orders
  - no model of portfolio liquidation using limit orders (with impact)
  - market impact: assume that volume imbalances drive order flow

The Model

The Model

We consider a benchmark model with a single risk neutral trader who:

- needs to buy N shares by some time T
- submits a *limit order* at the best bid price B_0
- can chose to openly display $\Delta \in [0, N]$ shares
- faces same side liquidity competition
    - at the submission price level ("loss in time priority")
    - at more competitive price levels ("undercutting")
- cancels all unexecuted orders at time T
    - resubmits cancelled orders as market orders
    - faces "opposite side liquidity competition"

Trading Costs

Order flows (limit and market) depend on the visible state of the book.

- A random number Z^Δ is executed before time T
- The *relative* execution price is given by

$$P^\Delta := \left(1 - \frac{Z^\Delta}{N}\right) S_T^\Delta$$

  where $S_T^\Delta := \frac{A_T^\Delta - B_0}{B_0}$ denotes the *effective spread*.
- Both same and opposite side order flow (limit and market) matters
- The two sides of the market are assumed conditionally independent given Δ; the expected relative execution price is:

$$W(\Delta) := \left(1 - \frac{\mathbb{E}Z^\Delta}{N}\right) \mu(\Delta).$$

Flow Dynamics

- We consolidate order arrivals into single submissions with laws:
  - aggregate limit order volume ($y \geq 0$) at *submission price level:*

$$f_y(u) = (1-q)\mathbf{1}_{\{u=0\}} + \frac{q}{\beta} e^{-\frac{u}{\beta}} \mathbf{1}_{\{u>0\}}.$$

  - aggregate limit order volume ($\hat{y} \geq 0$) at *more competitive levels*:

$$f_{\hat{y}}(u) = (1-\hat{q})\mathbf{1}_{\{u=0\}} + \frac{\hat{q}}{\hat{\beta}} e^{-\frac{u}{\hat{\beta}}} \mathbf{1}_{\{u>0\}}.$$

  - aggregate *market order volume* ($x \geq 0$):

$$f_x(u) = (1-p)\mathbf{1}_{\{u=0\}} + \frac{p}{\alpha} e^{-\frac{u}{\alpha}} \mathbf{1}_{\{u>0\}}.$$

- The volume y has priority over the hidden order $N - \Delta$ ("display priority"); $\hat{y}$ has priority over the full order N ("price priority")

Theorem (Expected Execution Volume)

The expected execution volume is given by

$$V = \alpha p(1-\hat{\beta}_r)e^{-\frac{D(1-c)}{\alpha}}\left\{(1-\beta_r)\left(e^{-\frac{\Delta}{\alpha}} - e^{-\frac{N}{\alpha}}\right) + \left(1 - e^{-\frac{\Delta}{\alpha}}\right)\right\}$$

where D is the volume at submission level, c cancellation ratio and

$$\hat{\beta}_r = \hat{q}\frac{\hat{\beta}}{\alpha+\hat{\beta}}; \quad \beta_r = q\frac{\beta}{\alpha+\beta}.$$

Market Impact of Limit Orders

Assumption (Market impact of limit order)

The variables $p, \alpha, \beta, \hat{\beta}$ and μ depend the order imbalance

$$I := I(\Delta) = D_b - D_a + \Delta.$$

In particular, the expected relative execution price is of the form:

$$W^{\Delta} = W(\Delta, p(\Delta), \alpha(\Delta), \beta(\Delta), \hat{\beta}(\Delta), \mu(\Delta))$$

and

$$\frac{d}{d\Delta} W = M_{\text{Impact}} + M_{\text{Priority}}$$

Remark

We calibrated this model (see below) but there is typically no closed-form solution for optimal display sizes.

A Model of Pure Liquidity Competition

The Model

Assumption (Pure liquidity competition)

Only order flow at more competitive prices depends on imbalances, and

$$\hat{\beta} = \hat{\beta}_0 + \hat{\beta}_1 \Delta$$

We define the intensity of liquidity competition ξ as

$$\xi = \frac{\hat{\beta}_1}{1 + \hat{\beta}_0 / \alpha}$$

and recall that the *Lambert Function* is any function that solves:

$$w = \Phi(w) e^{\Phi(w)}, w \in \mathbb{C}.$$

Theorem (Optimal display sizes)

Assume all model parameters are independent of the display size, except

$$\hat{\beta}(\Delta) = \hat{\beta}_0 + \hat{\beta}_1 \Delta.$$

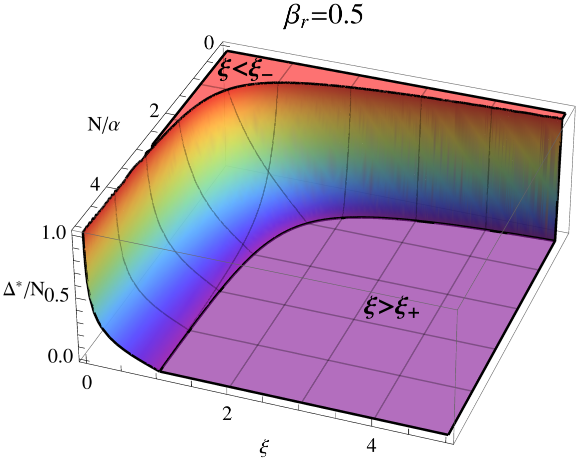
Assume moreover that $\hat{q} = 1$*. Then*

$$\frac{\Delta^*}{N} = \begin{cases} 1 & \textit{if} \quad \xi \leq \xi_- \\ -\frac{\alpha}{N}(1 + \xi^{-1} + \Phi_{-1}(\overline{w})) & \textit{if} \quad \xi_- < \xi < \xi_+ \ , \\ 0 & \textit{if} \quad \xi \geq \xi_+ \end{cases}$$

with $\quad \overline{w} := -\gamma e^{-\xi^{-1}-1}, \qquad \gamma := \dfrac{1 - e^{-\frac{N}{\alpha}}(1-\beta_r)}{\beta_r},$

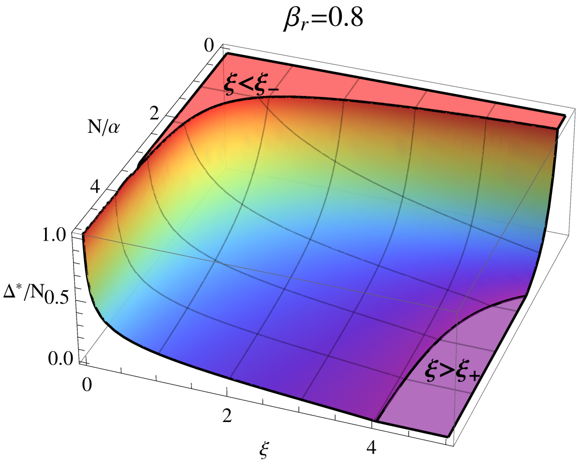
and $\quad \xi_- := \left(\gamma e^{\frac{N}{\alpha}} - 1 - \dfrac{N}{\alpha}\right)^{-1}, \quad \xi_+ := \Big(\gamma - 1\Big)^{-1}.$

Optimal Display Ratios



Little competition at submission price level: hide more

Optimal Display Ratios



More competition at submission price level: hide less

Estimating Display Ratios

The Data

- NASDAQ ITCH order-message data; 01/11 - 04/11
- random selection of 31 stocks from the S&P500 index
- aggregation of cancellation, submission, ... into 1 min snapshots

	Average Stock Properties				Average Model Parameters			
	MQ (\$)	S (bps)	$TrVol$ (1000\$)	Var	D_{bid} ($shares$)	α ($shares$)	p	μ (bps)
q_1	32	37.27	2.55	25.07	1,705	647	0.23	8.83
q_2	23	9.29	19.14	13.01	6,101	1,940	0.62	1.90
q_3	64	3.96	425.03	8.15	18,391	11,488	0.90	2.24
All	41	15.35	171.44	14.63	9,582	5,261	0.61	4.04

Model Parameter

- For the full model we used a linear regression model for $\beta_r, \hat{\beta}_r, \mu$

$$\gamma[I] = b_0 + b_1 I + \varepsilon_\gamma$$

and

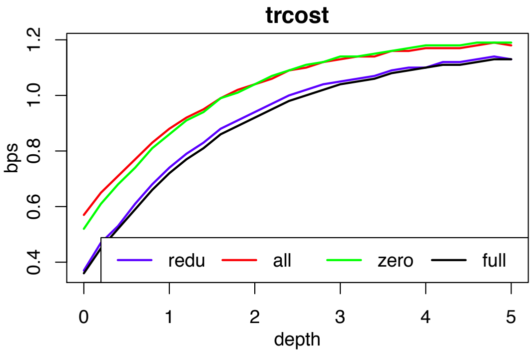
$$\log \alpha[I] = a_0 + a_1 I + \varepsilon_a; \quad p[I] = \frac{1}{1 + e^{\kappa_0 + \kappa_1 I + \varepsilon_p}}$$

- For the reduced model we used an inverse linear model for

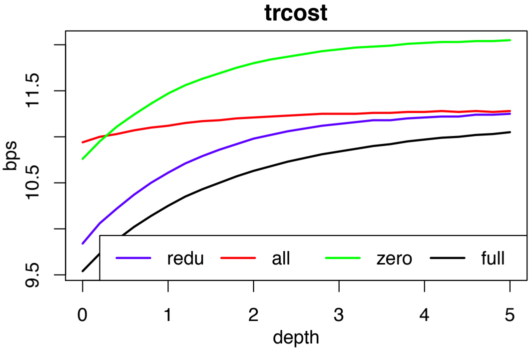
$$\hat{\beta}_r[I] = 1 - \frac{1}{\zeta}; \quad \zeta = \zeta_0 + \zeta_1 I + \varepsilon_\zeta.$$

- Average r^2 goodness-of-fit ranges from 40% to 63% and is stable across all stocks.

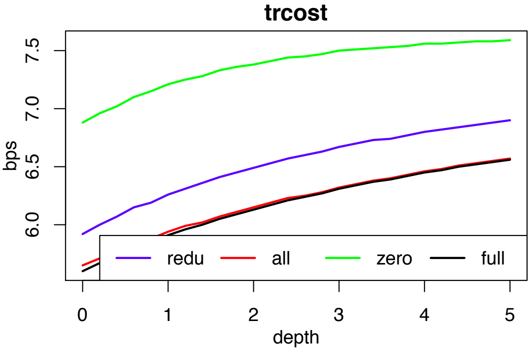
Transaction Costs: Liquid Stocks: N=αp



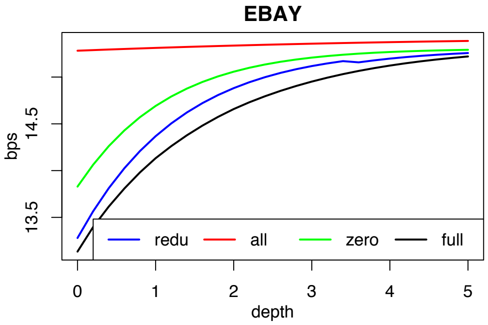
Transaction Costs: Liquid Stocks: N=$10\alpha p$



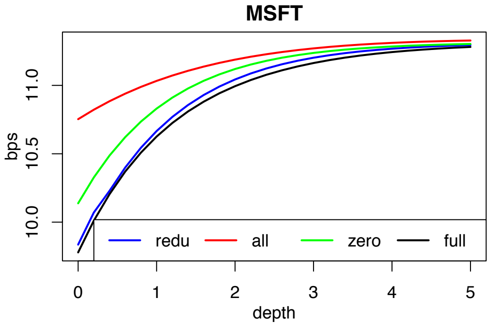
Transaction Stocks: Less Liquid Stocks: N=$10\alpha p$



Implementation Shortfall: EBAY



Implementation Shortfall: MSFT



Conclusion

- We studied a model of optimal order display under market impact
- Closed form solution for pure liquidity competition
- Reduced model performs well for liquid stocks
- Open problems:
  - more general flow dynamics
  - general analysis of limit order impact
  - dynamic model of optimal order placement
  - ...

Thank You!