Optimal Order Display in Limit Order Markets with Liquidity Competition

ULRICH HORST

Humboldt-Universität zu Berlin
Department of Mathematics
and
School of Business and Economics

1Based on joint work with Gökhan Cebiroğlu; to appear in JEDC
Outline

- Limit Order Books (LOBs) and hidden liquidity
- A sequential trade model with
  - a hidden liquidity trader that needs to buy a (large) position
  - liquidity competition at submission and more competitive price level
  - random market order flow executing standing liquidity
- Optimal display strategies under market impact
  - openly displayed orders impact market dynamics
  - exchanges allow traders to shield orders from public view
  - tradeoff between loss in execution priority and reduced impact
- Closed form solutions under pure liquidity competition
  - model performs well for liquid stocks and medium order sizes
  - show expected shortfalls for selected (liquid) stocks
Hidden Liquidity in LOBs
Electronic exchanges organize trading through *limit order books*. Market participants can submit:
- market orders for immediate execution
- limit orders for future execution
- cancellations of standing orders

Orders are executed according to a set of priority rules:
- price priority
- time priority
188 ms in the life of the Apple stock
Hidden Liquidity in Limit Order Books

- Electronic exchanges organize trading through *limit order books*
- Market participants can submit
  - market orders for immediate execution
  - limit orders for future execution
  - cancellations of standing orders
- Orders are executed according to a set of priority rules:
  - price priority
  - time priority
  - *display priority*
- Markets allow traders to hide some/all of their orders
  - Hidden orders account for 20%-30% of liquidity
  - If orders can be fully hidden, there may be liquidity in the spread
Hidden orders
Hidden orders
Why Hiding?

- Large limit orders have market impact
  - private information ("scare away other traders")
  - large displayed orders encourage price undercutting ("impatience")
- Hiding orders helps to reduce market impact
  - tradeoff between reduced market impact and loss in time priority
  - we study optimal display decisions under liquidity competition
- Market impact of market orders has been studied extensively
  - plenty of models of portfolio liquidation with market orders
  - no model of portfolio liquidation using limit orders (with impact)
  - market impact: assume that volume imbalances drive order flow
The Model
The Model

We consider a benchmark model with a single risk neutral trader who:

- needs to buy $N$ shares by some time $T$
- submits a limit order at the best bid price $B_0$
- can chose to openly display $\Delta \in [0, N]$ shares
- faces same side liquidity competition
  - at the submission price level ("loss in time priority")
  - at more competitive price levels ("undercutting")
- cancels all unexecuted orders at time $T$
  - resubmits cancelled orders as market orders
  - faces “opposite side liquidity competition”
Trading Costs

Order flows (limit and market) depend on the visible state of the book.

- A random number $Z^\Delta$ is executed before time $T$
- The relative execution price is given by
  \[ P^\Delta := \left(1 - \frac{Z^\Delta}{N}\right) S^\Delta_T \]

where $S^\Delta_T := \frac{A^\Delta_T - B_0}{B_0}$ denotes the effective spread.

- Both same and opposite side order flow (limit and market) matters
- The two sides of the market are assumed conditionally independent given $\Delta$; the expected relative execution price is:
  \[ W(\Delta) := \left(1 - \frac{\mathbb{E}Z^\Delta}{N}\right) \mu(\Delta). \]
Flow Dynamics

- We consolidate order arrivals into single submissions with laws:
  - aggregate limit order volume ($y \geq 0$) at submission price level:
    \[ f_y(u) = (1 - q)1_{\{u=0\}} + \frac{q}{\beta} e^{-\frac{u}{\beta}} 1_{\{u>0\}}. \]
  - aggregate limit order volume ($\hat{y} \geq 0$) at more competitive levels:
    \[ f_{\hat{y}}(u) = (1 - \hat{q})1_{\{u=0\}} + \frac{\hat{q}}{\beta} e^{-\frac{u}{\beta}} 1_{\{u>0\}}. \]
  - aggregate market order volume ($x \geq 0$):
    \[ f_x(u) = (1 - p)1_{\{u=0\}} + \frac{p}{\alpha} e^{-\frac{u}{\alpha}} 1_{\{u>0\}}. \]
  - The volume $y$ has priority over the hidden order $N - \Delta$ ("display priority"); $\hat{y}$ has priority over the full order $N$ ("price priority")
Theorem (Expected Execution Volume)

The expected execution volume is given by

\[ V = \alpha p (1 - \hat{\beta}_r) e^{-\frac{D(1-c)}{\alpha}} \left\{ (1 - \beta_r) \left( e^{-\frac{\Delta}{\alpha}} - e^{-\frac{N}{\alpha}} \right) + \left( 1 - e^{-\frac{\Delta}{\alpha}} \right) \right\} \]

where \( D \) is the volume at submission level, \( c \) cancellation ratio and

\[ \hat{\beta}_r = \hat{q} \frac{\hat{\beta}}{\alpha + \hat{\beta}}; \quad \beta_r = q \frac{\beta}{\alpha + \beta}. \]
Assumption (Market impact of limit order)

The variables $p$, $\alpha$, $\beta$, $\hat{\beta}$ and $\mu$ depend the order imbalance

$$I := I(\Delta) = D_b - D_a + \Delta.$$ 

In particular, the expected relative execution price is of the form:

$$W^\Delta = W(\Delta, p(\Delta), \alpha(\Delta), \beta(\Delta), \hat{\beta}(\Delta), \mu(\Delta))$$

and

$$\frac{d}{d\Delta} W = M_{\text{Impact}} + M_{\text{Priority}}$$

Remark

We calibrated this model (see below) but there is typically no closed-form solution for optimal display sizes.
A Model of Pure Liquidity Competition
The Model

Assumption (Pure liquidity competition)

*Only order flow at more competitive prices depends on imbalances, and*

\[ \hat{\beta} = \hat{\beta}_0 + \hat{\beta}_1 \Delta \]

We define the intensity of liquidity competition \( \xi \) as

\[ \xi = \frac{\hat{\beta}_1}{1 + \hat{\beta}_0 / \alpha} \]

and recall that the *Lambert Function* is any function that solves:

\[ w = \Phi(w) e^{\Phi(w)}, w \in \mathbb{C}. \]
Theorem (Optimal display sizes)

Assume all model parameters are independent of the display size, except

$$\hat{\beta}(\Delta) = \hat{\beta}_0 + \hat{\beta}_1 \Delta.$$ 

Assume moreover that $\hat{q} = 1$. Then

$$\frac{\Delta^*}{N} = \begin{cases} 
1 & \text{if } \xi \leq \xi_- \\
-\frac{\alpha}{N}(1 + \xi^{-1} + \Phi_{-1}(\overline{w})) & \text{if } \xi_- < \xi < \xi_+ , \\
0 & \text{if } \xi \geq \xi_+ 
\end{cases}$$

with $\overline{w} := -\gamma e^{-\xi^{-1} - 1}$, $\gamma := \frac{1 - e^{-\frac{N}{\alpha}(1 - \beta_r)}}{\beta_r}$,

and $\xi_- := \left(\gamma e^{\frac{N}{\alpha}} - 1 - \frac{N}{\alpha}\right)^{-1}$, $\xi_+ := \left(\gamma - 1\right)^{-1}$. 
Optimal Display Ratios

Little competition at submission price level: hide more
Optimal Display Ratios

More competition at submission price level: hide less
Estimating Display Ratios
## The Data

- NASDAQ ITCH order-message data; 01/11 - 04/11
- random selection of 31 stocks from the S&P500 index
- aggregation of cancellation, submission, ... into 1 min snapshots

<table>
<thead>
<tr>
<th></th>
<th>Average Stock Properties</th>
<th>Average Model Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$MQ$ ($)</td>
<td>$S$ (bps)</td>
</tr>
<tr>
<td>$q_1$</td>
<td>32</td>
<td>37.27</td>
</tr>
<tr>
<td>$q_2$</td>
<td>23</td>
<td>9.29</td>
</tr>
<tr>
<td>$q_3$</td>
<td>64</td>
<td>3.96</td>
</tr>
<tr>
<td>All</td>
<td>41</td>
<td>15.35</td>
</tr>
</tbody>
</table>
For the full model we used a linear regression model for $\beta_r, \hat{\beta}_r, \mu$

$$\gamma[l] = b_0 + b_1 l + \varepsilon_\gamma$$

and

$$\log \alpha[l] = a_0 + a_1 l + \varepsilon_a; \quad p[l] = \frac{1}{1 + e^{\kappa_0 + \kappa_1 l + \varepsilon_p}}$$

For the reduced model we used an inverse linear model for

$$\hat{\beta}_r[l] = 1 - \frac{1}{\zeta}; \quad \zeta = \zeta_0 + \zeta_1 l + \varepsilon_\zeta.$$

Average $r^2$ goodness-of-fit ranges from 40% to 63% and is stable across all stocks.
Transaction Costs: Liquid Stocks: $N=\alpha p$
Transaction Costs: Liquid Stocks: N=10αρ
Transaction Stocks: Less Liquid Stocks: $N=10\alpha p$
Implementation Shortfall: EBAY

![Graph showing the relationship between depth and bps across different conditions: redu, all, zero, and full. The x-axis represents depth, ranging from 0 to 5, and the y-axis represents bps, ranging from 13.5 to 14.5. Different lines correspond to different conditions, with 'redu' in blue, 'all' in red, 'zero' in green, and 'full' in black.](image)
Implementation Shortfall: MSFT
Conclusion

• We studied a model of optimal order display under market impact
• Closed form solution for pure liquidity competition
• Reduced model performs well for liquid stocks
• Open problems:
  • more general flow dynamics
  • general analysis of limit order impact
  • dynamic model of optimal order placement
  • ...
Thank You!