Optimal Execution and Stochastic Approximation: Learning by Trading

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Outline

1. Introduction to Stochastic Approximation Algorithms

2. Applications to Optimal Execution
   - Optimal split of orders across liquidity pools
   - Optimal posting price of limit orders: learning by trading
   - Optimal split and posting price of limit orders across lit pools

Outline

Motivations

In many fields, we often are faced with optimization or zero search problems.

Examples: In Finance,

- extraction of implicit parameter (volatility),
- calibration, optimization of an exogenous parameter for variance reduction (regression, importance sampling, etc).

Common point: all the concerned functions have a representation as an expectation, *i.e.*

$$h(\theta) = \mathbb{E}\left[H(\theta, Y)\right], \quad \text{where } Y \text{ is a } q\text{-dimensional random vector.}$$

The stochastic approximation is a tool based on simulation to solve optimization or zero search problems.

Measurement : Robbins-Monro procedure

A dose θ of a chemical product creates a **random effect** measured by $F(\theta, Y)$, Y being a random variable with distribution μ and $F : \mathbb{R}^2 \mapsto \mathbb{R}$.

We assume that the **mean effect**

$$f(\theta) = \mathbb{E}\left[F(\theta, Y)\right] \quad \text{is non-decreasing.}$$

We want to determine the dose θ^* which creates a mean effect of a given threshold α, *i.e.* to solve

$$f(\theta^*) = \alpha.$$

- **Naive idea**: $f(\theta_n) \approx \frac{1}{N_n} \sum_{k=1}^{N_n} F(\theta_n, Y_k)$, Y_k i.i.d. with distribution μ, $N_n \underset{n\to\infty}{\longrightarrow} \infty$.
- **Other idea**: use $F(\theta_n, Y_{n+1})$ instead of $f(\theta_n)$ with $\gamma_n \searrow 0$ ("local" randomization).

The **Robbins-Monro procedure** is the following

- choose arbitrarily θ_0 and administer it to a subject which reacts with the effect $F(\theta_0, Y_1)$.
- *Recurrence*: at instant n, choose a dose θ_n administered to a subject (independent of the previous ones), the effect is $F(\theta_n, Y_{n+1})$.

As $(Y_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with distribution μ, then

$$f(\theta_n) = \mathbb{E}\left[F(\theta_n, Y_{n+1}) \,|\, F(\theta_0, Y_1), \dots, F(\theta_{n-1}, Y_n)\right].$$

The **Robbins-Monro algorithm** for the choice of θ_n then reads

$$\theta_{n+1} = \theta_n - \gamma_n \left(F(\theta_n, Y_{n+1}) - \alpha\right), \quad (\gamma_n) \text{ nonnegative tending to 0.}$$

By setting $H(\theta, y) := F(\theta, y) - \alpha$, this procedure is then a zero search of the function

$$h(\theta) := f(\theta) - \alpha = \mathbb{E}\left[F(\theta, Y) - \alpha\right] = \mathbb{E}\left[H(\theta, Y)\right].$$

General Framework

Consider the following recursive procedure defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$

$$\forall\, n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1} \left(\Delta M_{n+1} + r_{n+1}\right), \tag{1}$$

where

- $h : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous locally Lipschitz function,
- θ_0 is a $\mathcal{F}_0$-measurable finite random vector,
- for every $n \geq 0$, ΔM_{n+1} is a $\mathcal{F}_n$-martingale increment,
- r_n is a $\mathcal{F}_n$-adapted reminder term.

Convergence: Martingale approach or ODE method

- **Martingale approach**: Existence of a Lyapunov function and control of both martingale increment and remainder term.
- **ODE method**: Study of the SA as a discretization of

$$ODE_h \equiv \dot{\theta} = -h(\theta).$$

Assume that $(\gamma_n)_{n\geq 1}$ is a nonnegative sequence such that

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

Then,

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^*.$$

Rate of convergence: CLT (SDE method)

Let θ^* be an equilibrium point of $\{h = 0\}$. Assume that the function h is differentiable at θ^* and that all the eigenvalues of $Dh(\theta^*)$ have a nonnegative real part. Specify the step sequence as follows

$$\forall n \geq 1, \quad \gamma_n = \frac{\alpha}{n}, \quad \alpha > \frac{1}{2\mathcal{R}e(\lambda_{min})} \tag{2}$$

where λ_{min} denotes the eigenvalue of $Dh(\theta^*)$ with the lowest real part.

Then, the previous convergence $a.s.$ is ruled on $\{\theta_n \xrightarrow{a.s.} \theta^*\}$ by the following CLT

$$\sqrt{n}\,(\theta_n - \theta^*) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \alpha\Sigma)$$

with

$$\Sigma := \int_0^{+\infty} \left(e^{-\left(Dh(\theta^*) - \frac{I_d}{2\alpha}\right)u}\right)^t \Gamma e^{-\left(Dh(\theta^*) - \frac{I_d}{2\alpha}\right)u} du.$$

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Static modelling

The principle of a *Dark pool* is the following:

- It proposes a bid price with no guarantee of executed quantity at the occasion of an OTC transaction.
- Usually this price is lower than the bid price offered on the regular market.

So one can model the impact of the existence of N dark pools ($N \geq 2$) on a given transaction as follows:

- Let $V > 0$ be the random volume to be executed,
- Let $\theta_i \in (0, 1)$ be the *discount factor* proposed by the dark pool i.
- Let r_i denote the percentage of V sent to the dark pool i for execution.
- Let $D_i \geq 0$ be the quantity of securities that can be delivered (or mase available) by the dark pool i at price $\theta_i S$.

Cost of the executed order

The reminder of the order is to be executed on the regular market, at price S.

Then the **cost** C of the whole executed order is given by

$$
\begin{aligned}
C &= S \sum_{i=1}^{N} \theta_i \min(r_i V, D_i) + S \left(V - \sum_{i=1}^{N} \min(r_i V, D_i) \right) \\
&= S \left(V - \sum_{i=1}^{N} \rho_i \min(r_i V, D_i) \right)
\end{aligned}
$$

where

$$\rho_i = 1 - \theta_i \in (0, 1), i = 1, \ldots, N.$$

Mean Execution Cost and dynamical aspect

Minimizing the mean execution cost, *given the price* S, amounts to solving the following **maximization problem**

$$\max\left\{\sum_{i=1}^{N}\rho_i\mathbb{E}\left(S\min\left(r_iV,D_i\right)\right),r\in\mathcal{P}_N\right\} \tag{3}$$

where $\mathcal{P}_N := \left\{r=(r_i)_{1\le i\le N}\in\mathbb{R}_+^N \mid \sum_{i=1}^{N}r_i=1\right\}$.

We consider the sequence $Y^n := \left(V^n, D_1^n,\ldots,D_N^n\right)_{n\ge1}$. We will make stationary assumptions on the sequence

- either $\left(Y^n\right)_{n\ge1}$ is i.i.d.,
- or $\left(Y^n\right)_{n\ge1}$ is ergodic with some rate ($e.g.$ α-mixing).

Design of the stochastic algorithm

We us a Lagrangian approach to solve this maximisation problem under constraints. We obtain that

$$r^* \in \arg\max_{\mathcal{P}_N} \Phi$$
$$\Updownarrow$$
$$\forall i \in I_N, \quad \mathbb{E}\left[V\left(\rho_i \mathbb{1}_{\{r_i^* V < D_i\}} - \tfrac{1}{N}\sum_{j=1}^{N} \rho_j \mathbb{1}_{\{r_j^* V < D_j\}}\right)\right] = 0.$$

Consequently, this leads to the following **recursive zero search procedure**

$$r_i^{n+1} = r_i^n + \gamma_{n+1} H_i(r^n, Y^{n+1}), \; r^0 \in \mathcal{P}_N, \; i \in I_N, \tag{4}$$

where for $i \in I_N$, every $r \in \mathcal{P}_N$, every $V > 0$ and every $D_1, \dots, D_N \geq 0$,

$$H_i(r, Y) = V\left(\rho_i \mathbb{1}_{\{r_i V < D_i\}} - \frac{1}{N}\sum_{j=1}^{N} \rho_j \mathbb{1}_{\{r_j V < D_j\}}\right)$$

where $(Y^n)_{n\geq 1}$ is a sequence of random vectors with non negative components such that, for every $n \geq 1$ and $i \in I_N$, $(V^n, D_i^n) \stackrel{d}{=} (V, D_i)$.

The pseudo-real data setting

V is the traded volume of a very liquid security – namely the asset BNP – during an 11 day period.

Then we selected the N most correlated assets (in terms of traded volumes) with the original asset, denoted S_i, $i = 1, \ldots, N$.

The available volumes of each dark pool i have been modeled as follows

$$\forall 1 \leq i \leq N, \quad D_i := \beta_i \left((1 - \alpha_i) V + \alpha_i S_i \frac{\mathbb{E}\, V}{\mathbb{E} S_i} \right)$$

where

- $\alpha_i \in (0, 1)$, $i = 1, \ldots, N$ are the recombining coefficients,
- β_i, $i = 1, \ldots, N$ are some scaling factors,
- $\mathbb{E}\, V$ and $\mathbb{E} S_i$ stand for the empirical mean of the data sets of V and S_i.

Parameters

The shortage situation corresponds to $\sum_{i=1}^{N} \beta_i < 1$ since it implies

$$\mathbb{E}\left[\sum_{i=1}^{N} D_i\right] < \mathbb{E}\, V.$$

We will use the following parameters

$$N = 4, \quad \beta = (0.1, 0.2, 0.3, 0.2)' \quad \text{and} \quad \alpha = (0.4, 0.6, 0.8, 0.2)'$$

and the dark pool trading (rebate) parameters are set to

$$\rho = (0.01, 0.02, 0.04, 0.06)'.$$

Long-term optimization

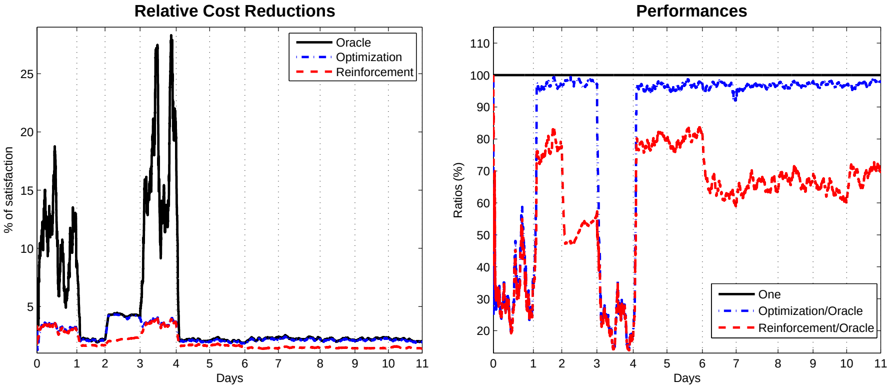

Figure: *Long term optimization*: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0.2 \leq \alpha_i \leq 0.8$ and $r_i^0 = 1/N$, $1 \leq i \leq N$.

Daily resetting of the procedure

We reset the step γ_n at the beginning of each day and the satisfaction parameters and we keep the allocation coefficients of the preceding day.

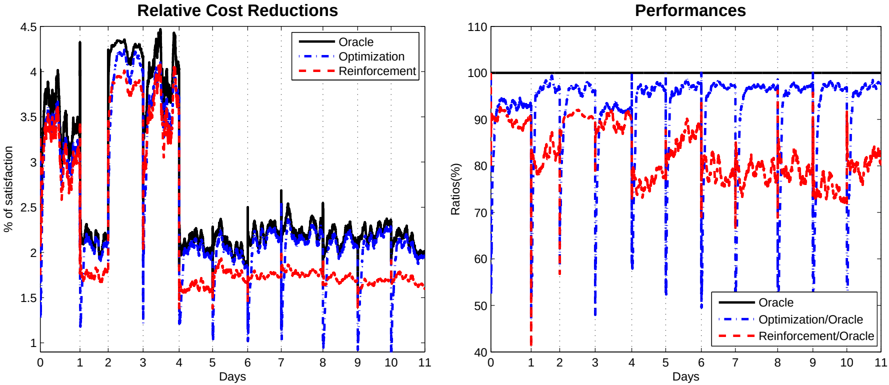

Figure: *Daily resetting of the algorithms parameters*: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0.2 \leq \alpha_i \leq 0.8$ and $r_i^0 = 1/N$ $1 \leq i \leq N$.

Outline

Modeling and design of the algorithm

We consider on a short period T a **Poisson process of "execution"** of buy orders

$$\left(N_t^{(\delta)}\right)_{0\leq t\leq T} \quad \text{with intensity} \quad \Lambda_T(\delta, S) := \int_0^T \lambda(S_t - (S_0 - \delta))dt \quad (5)$$

where

- $0 \leq \delta \leq \delta_{\max}$ with $\delta_{\max} \in (0, S_0)$ denotes the *depth of the limit order book*,
- $(S_t)_{t\geq 0}$ is a stochastic process modeling the dynamic of the *fair price of a security stock*,
- the function λ is defined on the whole real line as a finite non increasing convex function.

Optimization Problem

Then we introduce a **market impact penalization function** $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$, nondecreasing and convex, with $\Phi(0) = 0$ to model the additional cost of the execution of the remaining quantity.

Then the *resulting cost of execution* on a period $[0, T]$ reads

$$C(\delta) := \mathbb{E}\left[(S_0 - \delta)\left(Q_T \wedge N_T^{(\delta)}\right) + \kappa S_T \Phi\left(\left(Q_T - N_T^{(\delta)}\right)_+\right)\right] \tag{6}$$

where $\kappa > 0$.

Our aim is then to **minimize this cost**, namely to solve the following optimization problem

$$\min_{0 \leq \delta \leq \delta_{\max}} C(\delta). \tag{7}$$

To solve this optimization problem, we will devise a **stochastic algorithm constrained** to stay in $[0, \delta_{\max}]$.

Strategy

We will devise a **stochastic algorithm constrained** to stay in $[0, \delta_{\max}]$. To this end we have to

- prove that C, C' and C'' admit representations as expectations, *i.e.* in particular, there exists a Borel functional

$$H : [0, \delta_{\max}] \times \mathbb{D}([0, T], \mathbb{R}) \longrightarrow \mathbb{R} \quad \text{such that}$$

$$\forall \delta \in [0, \delta_{\max}], \quad C'(\delta) = \mathbb{E}\left[H\left(\delta, (S_t)_{t\in[0,T]}\right)\right]$$

- find natural assumptions on Q_T and κ such that C is twice differentiable, strictly convex on $[0, \delta_{\max}]$ with $C'(0) < 0$. Consequently

$$\operatorname{argmin}_{\delta\in[0,\delta_{\max}]} C(\delta) = \{\delta^*\}, \quad \delta^* \in (0, \delta_{\max}]$$

and

$$\delta^* = \delta_{\max} \text{ iff } C \text{ is non increasing on } [0, \delta_{\max}].$$

Design of the algorithm

Once the two points are checked, we can devise the algorithm following the **standard stochastic approximation with projection** (see [2]), namely

$$\delta_{n+1} = \mathrm{Proj}_{[0,\delta_{\max}]}\left(\delta_n - \gamma_{n+1} H\left(\delta_n, \left(\bar{S}^{(n+1)}_{t_i}\right)_{0\le i\le m}\right)\right), \; \delta_0 \in [0, \delta_{\max}], \tag{8}$$

where

- $\mathrm{Proj}_{[0,\delta_{\max}]}$ denotes the projection on $[0, \delta_{\max}]$,
- the positive step sequence $(\gamma_n)_{n\ge1}$ satisfies at least the minimal *decreasing step* assumption

$$\sum_{n\ge1} \gamma_n = +\infty \quad \text{and} \quad \gamma_n \to 0, \tag{9}$$

- $\left\{\left(\bar{S}^{(n)}_{t_i}\right)_{0\le i\le m}, n \ge 0\right\}$ is either a sequence of i.i.d. copies of the true underlying dynamics of $(S_{t_i})_{0\le i\le m}$ or at least of its Euler scheme or a sequence sharing some averaging properties in the sense of [4] (*e.g.* stationary α-mixing).

Numerical experiments: Market data

As market data, we use the bid prices of Accor SA (ACCP.PA) of 11/11/2010 for the fair price process $(S_t)_{t\in[0,T]}$. We divide the day into periods of 15 trades which will denote steps of the stochastic procedure. Let N_{cycles} be the number of these periods. For every $n \in N_{\mathrm{cycles}}$, we have a sequence of bid prices $(S^{(n)}_{t_i})_{1\le i\le 15}$ and we approximate the jump intensity of the Poisson process $\Lambda_{T^n}(\delta, S)$, where $T^n = \sum_{i=1}^{15} t_i$, by

$$\forall n \in N_{\mathrm{cycles}}, \quad \Lambda_{T^n}(\delta, S) = A \sum_{i=2}^{15} e^{-k(S^{(n)}_{t_i} - S_{t_1} + \delta)}(t_i - t_{i-1}).$$

The empirical mean of the intensity function

$$\bar{\Lambda}(\delta, S) = \frac{1}{N_{\mathrm{cycles}}} \sum_{n=1}^{N_{\mathrm{cycles}}} \Lambda_{T^n}(\delta, S)$$

is plotted on the following figure.

Intensity function

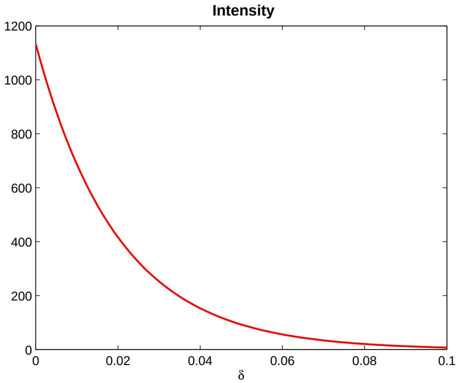

Figure: Fit of the exponential model on real data (Accor SA (ACCP.PA) 11/11/2010): $A = 1/50$, $k = 50$ and $N_{\text{cycles}} = 220$.

The penalization function is of the following form

$$\Phi(x) = (1 + \eta(x))x \quad \text{with} \quad \eta(x) = A' e^{k' x}.$$

Cost function and its derivative

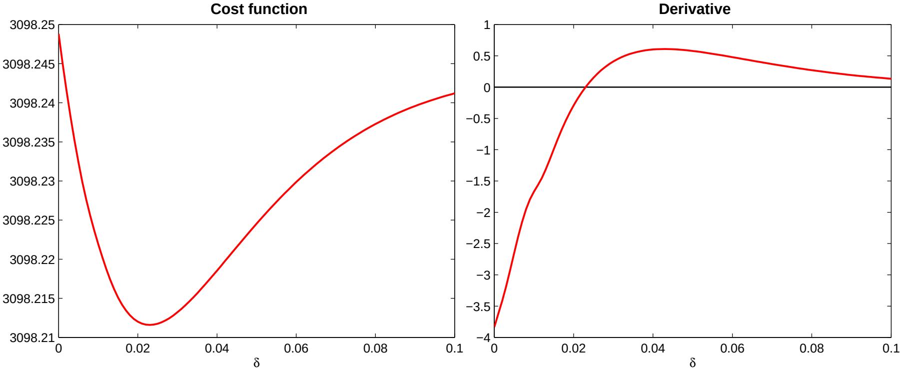

Figure: $\eta \neq 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1$, $A' = 0.001$, $k' = 0.0005$ and $N_{\text{cycles}} = 220$.

δ and posting price obtained by SA

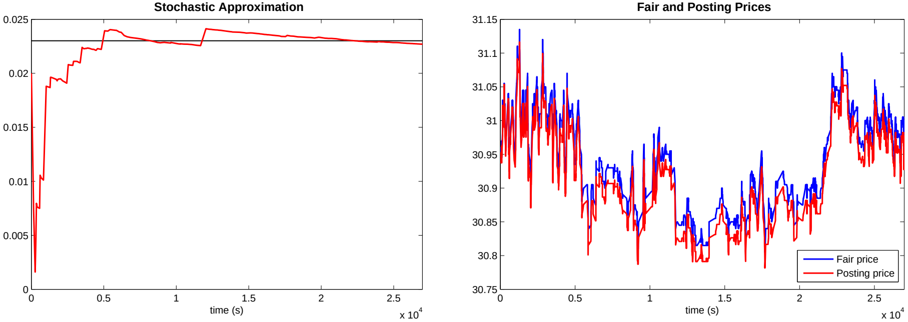

Figure: $\eta \neq 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1$, $A' = 0.001$, $k' = 0.0005$ and $N_{\text{cycles}} = 220$. Crude algorithm with $\gamma_n = \frac{1}{550n}$.

Stochastic algorithm with RP averaging

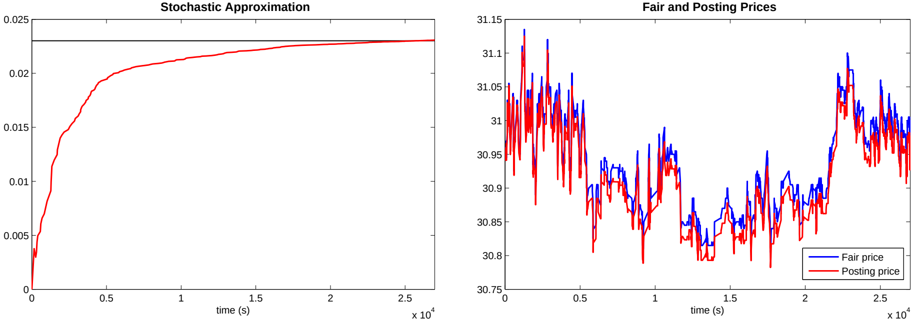

Figure: $\eta \neq 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1$, $A' = 0.001$, $k' = 0.0005$ and $N_{\text{cycles}} = 220$. Averaging algorithm (Ruppert and Poliak) with $\gamma_n = \frac{1}{550 n^{0.95}}$.

Outline

Framework and Modelling

- Assume that a trader wants to buy a volume V of an asset across N lit pools with limit orders.
- She has to determinate the proportions $r = (r^i)_{1 \leq i \leq N}$ to sent to each lit pools and the posting prices $(S - \delta) = (S - \delta^i)_{1 \leq i \leq N} \in [0, \delta_{\max}]^N$.
- The execution flow at the distance δ^i of the reference price S is modelled by a random variable $Q^i(\delta^i) = \bar{Q}^i e^{-k^i \delta^i}$ where $\bar{Q}^i$ is a positive random variable modeling the executed quantity at the first limit an $k^i > 0$.
- If she is not fully executed, she sends a market order of the remaining quantity.

Mean Execution Cost

The **mean resulting cost of execution** is the **sum of each mean execution costs** on the lit pools, namely it reads

$$C(r,\delta) := \sum_{i=1}^{N} \mathbb{E}\left[(S-\delta^i)(r^i V \wedge Q^i(\delta^i)) + \kappa S (r^i V - Q^i(\delta^i))_+\right], \quad (10)$$

where $\kappa > 0$ is a free tuning parameter.

Our aim is then to **minimize this cost** by choosing the proportions and the distances to post at, namely to solve the following optimization problem

$$\min_{r \in \mathcal{P}_N, \delta \in [0,\delta_{\max}]^N} C(r,\delta). \quad (11)$$

We take advantage of the representation of C and its first two derivatives as expectations to devise a recursive stochastic algorithm.

Design of the stochastic algorithm

Based on a **Lagrangian approach** for the optimal proportions and on the **representations as expectations** for C' and C'', we can formally devise a recursive stochastic gradient descent

$$r_{n+1} = \mathrm{Proj}_{\mathcal{P}_N}\left(r_n - \gamma_{n+1} H(r_n, \delta_n, \bar{Q}_n e^{-k\delta_n})\right), \quad n \geq 0,$$

$$\delta_{n+1} = \mathrm{Proj}_{[0,\delta_{\max}]^N}\left(\delta_n - \gamma_{n+1} G(r_n, \delta_n, \bar{Q}_n e^{-k\delta_n})\right), \quad n \geq 0,$$

where, for every $i \in \{1, \ldots, N\}$,

$$H(r_n^i, \delta_n^i, \bar{Q}_n^i e^{-k^i \delta_n^i}) = V\left((S - \delta_n^i)\mathbb{1}_{\{r_n^i V \leq \bar{Q}_n^i e^{-k^i \delta_n^i}\}} + \kappa S \mathbb{1}_{\{r_n^i V \geq \bar{Q}_n^i e^{-k^i \delta_n^i}\}}\right)$$
$$-\frac{1}{N}\sum_{j=1}^N V\left((S - \delta_n^j)\mathbb{1}_{\{r_n^j V \leq \bar{Q}_n^j e^{-k^j \delta_n^j}\}} + \kappa S \mathbb{1}_{\{r_n^j V \geq \bar{Q}_n^j e^{-k^j \delta_n^j}\}}\right),$$

and

$$G(r_n^i, \delta_n^i, \bar{Q}_n^i e^{-k^i \delta_n^i}) = -r_n^i V \wedge \bar{Q}_n^i e^{-k^i \delta_n^i} + k^i(\kappa S - (S - \delta_n^i))\mathbb{1}_{\{r_n^i V \leq \bar{Q}_n^i e^{-k^i \delta_n^i}\}}.$$

Convergence of the SA procedure

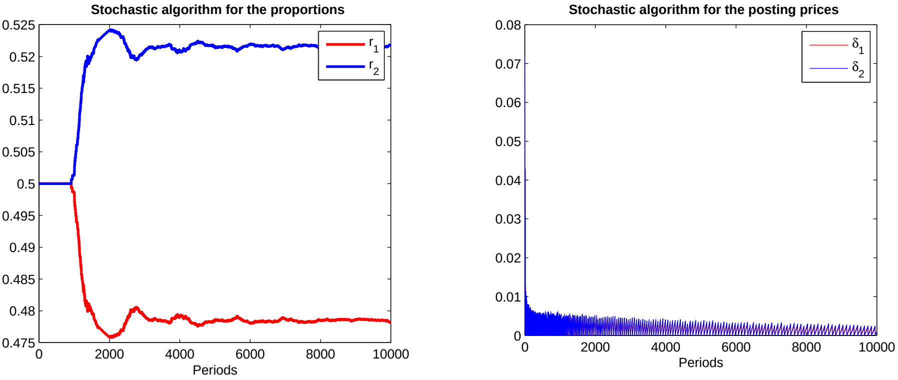

Figure: Convergence of the stochastic algorithm with $N = 2$, $V = 150$, $S = 100$, $m_Q = (8090)^t$, $v_Q = (11)^t$, $k = (2025)^t$, $\kappa = 1$, $r_0^i = 1/N$, $\delta_0^i = 0.05$ and $n = 10000$.

Convergence of the SA procedure with RP averaging

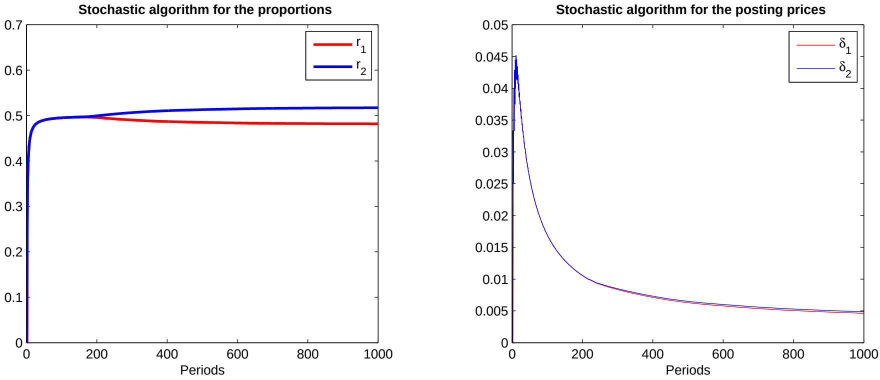

Figure: Convergence of the stochastic algorithm with $N = 2$, $V = 150$, $S = 100$, $m_Q = (80 90)^t$, $v_Q = (1 1)^t$, $k = (20 25)^t$, $\kappa = 1$, $r_0^i = 1/N$, $\delta_0^i = 0.05$ and $n = 1000$.

Perspectives

- Optimal split of large volumes across both dark and lit pools (with integrated or mid-point book for dark pools and limit orders for lit pools).
- Optimal posting price of limit order with other kind of matching mechanism : pro-rata or a mix between FIFO and pro-rata. This could be generalized whith order split too.
- Optimal execution for portfolios.

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Thank you for your attention