

# A closed-form execution strategy to target VWAP

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- ▶ Executions are delegated to brokers
- ▶ How can investors measure the performance of brokers?
  - ▶ Obtaining what the market has borne is sometimes enough,
  - ▶ A popular benchmark is

$$\text{VWAP} = \frac{\int_0^T S_u \mu_u du}{\int_0^T \mu_u du}, \quad (1)$$

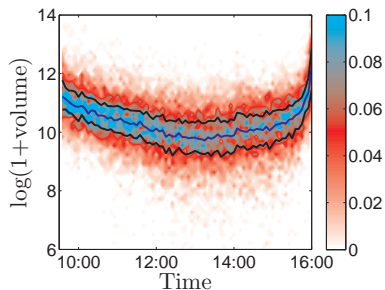
where  $S_t$  is the midprice and  $\mu_t$  is the volume traded by the entire market.

**Siblings:**  
**VWAP and other Volume-based Algorithms**

# Targeting VWAP via POV or POCV

- ▶ Brokers may guarantee VWAP and charge a fee
  - ▶ Incentive is to outperform VWAP
- ▶ Account for price impact:
  - ▶ slice parent order over a time window,
  - ▶ do 'bits' in a volume-based fashion to target VWAP.
- ▶ Formulation: set up a performance criteria where the investor/broker
  - ▶ seeks to execute a large order over a trading horizon  $T$ ,
  - ▶ maximises sales proceeds when liquidating shares,
  - ▶ accounts for permanent and temporary impact of trades,
  - ▶ tracks (penalises deviations from) POV or POCV,
  - ▶ ensures that by the end of trading horizon target quantity is achieved.

# Volume patterns



**Figure :** ORCL traded volume for orders sent to NASDAQ in all of 2013 using 5 minute buckets – heat-map of the data and 25%, 50% and 75% quantiles.

# Price Impact of Market Orders

# Permanent Impact

We assume a linear relationship between net order-flow and changes in the midprice, thus for every trading day we perform the regression

$$\Delta S_n = b \mu_n + \varepsilon_n, \quad (2)$$

where

- ▶  $\Delta S_n = S_{n\tau} - S_{(n-1)\tau}$  is the change in the midprice,
- ▶  $\mu_n$  is net order-flow defined as the difference between the volume of buy and sell MOs during the time interval  $[(n-1)\tau, n\tau]$ , and
- ▶  $\varepsilon_n$  is the error term (assumed normal). In the empirical analysis we choose  $\tau = 5$  min.

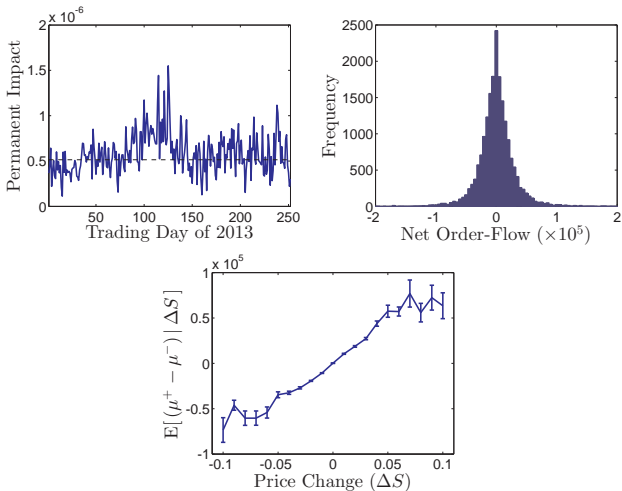
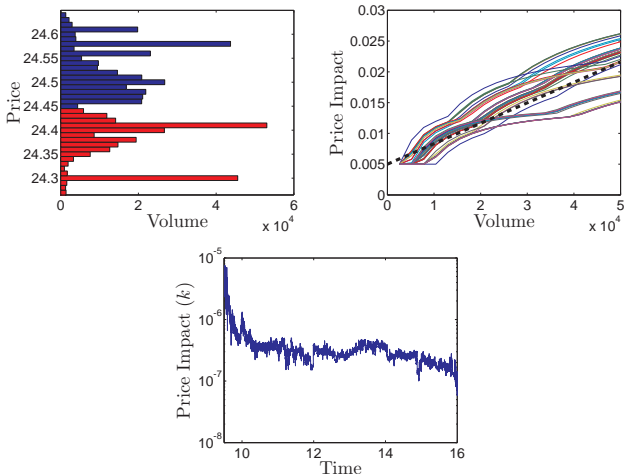


Figure : Order-Flow and effect on the drift of midprice of INTC.



# Temporary Impact

- ▶ Assume that temporary price impact is linear in the rate of trading so the difference between the execution price that the investor receives and the midprice is  $k\nu$ ,
- ▶ To do this we take a snapshot of the LOB each second,
- ▶ Determine the price per share for various volumes (by walking through the LOB),
- ▶ Compute the difference between the price per share and the best quote at that time,
- ▶ Perform a linear regression.



**Figure :** An illustration of how the temporary impact may be estimated from snapshots of the LOB using INTC on Nov 1, 2013. The first panel is at 11:00am, the second from 11:00am to 11:01am and the third contains the entire day.

# Parameters

**Table :** Permanent and temporary price impact parameters for Nasdaq stocks, ADV of MOs, average midprice,  $\sigma$  volatility (hourly) of arithmetic price changes, mean arrival (hourly) of MOs  $\lambda^\pm$ , and average volume of MOs  $\mathbb{E}[\eta^\pm]$ . Data are from Nasdaq 2013.

	FARO		SMH		NTAP	
	mean	stdev	mean	stdev	mean	stdev
ADV	23,914	14,954	233,609	148,580	1,209,628	642,376
midprice	40.55	6.71	37.90	2.44	38.33	3.20
$\sigma$	0.151	0.077	0.067	0.039	0.078	0.045
$b$	1.41E-04	9.61E-05	5.45E-06	4.20E-06	5.93E-06	2.31E-06
$k$	1.86E-04	2.56E-04	8.49E-07	8.22E-07	3.09E-06	1.75E-06
$b/k$	1.02	0.83	7.43	6.24	2.04	0.77
$\lambda^+$	16.81	9.45	47.29	28.13	300.52	144.48
$\mathbb{E}[\eta^+]$	103.56	21.16	377.05	118.05	308.45	53.09
$\lambda^-$	17.62	10.69	46.37	27.62	293.83	136.13
$\mathbb{E}[\eta^-]$	104.00	21.79	381.70	126.74	312.81	49.86

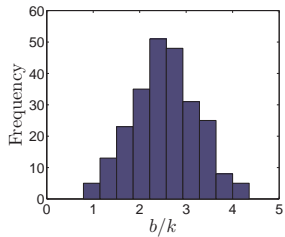
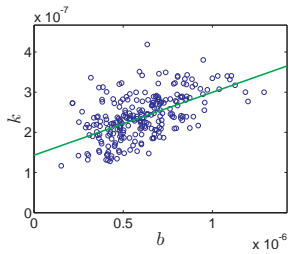


Figure : Price Impact INTC using daily observations for 2013.

## Targeting VWAP via POV

# The Model

- ▶ The investor searches for an optimal speed to liquidate  $\mathfrak{N}$  shares over a trading horizon  $T$ .
- ▶ Inventory

$$dQ_t^\nu = -\nu_t dt, \quad Q_0^\nu = \mathfrak{N}, \quad (3)$$

where  $\nu_t$  is the speed of liquidation.

- ▶ The rest of the market trades at speed  $\mu^+$  for buy side and  $\mu^-$  for sell side.
- ▶ And the idea is to target, at every instant in time, a percentage  $\rho \in [0, 1]$  of the overall traded volume.
- ▶ The overall traded volume, at every instant in time, is

$$\mu_t^+ + \mu_t^- + \nu_t.$$

# The Model, price impact

- ▶ Execution price

$$\hat{S}_t^\nu = S_t^\nu - k \nu_t, \quad (4)$$

$k > 0$  is the temporary impact parameter.

- ▶ The midprice satisfies

$$dS_t^\nu = b (\mu_t^+ - (\nu_t + \mu_t^-)) dt + dM_t, \quad S_0^\nu = S, \quad (5)$$

where  $b \geq 0$  is the permanent impact parameter, and  $M = \{M_t\}_{0 \leq t \leq T}$  is a martingale (independent of all other processes).

- ▶ Cash

$$dX_t^\nu = \hat{S}_t^\nu \nu_t dt, \quad X_0^\nu = X_0. \quad (6)$$

## The Model (cont)

Performance criteria is

$$H^\nu(t, x, S, \boldsymbol{\mu}, q) = \mathbb{E}_{t, x, S, \boldsymbol{\mu}, q} \left[ X_{\tau^\nu}^\nu + Q_{\tau^\nu}^\nu (S_{\tau^\nu}^\nu - \alpha Q_{\tau^\nu}^\nu) - \tilde{\varphi} \int_t^{\tau^\nu} (\nu_u - \chi_u^\nu)^2 du \right], \quad (7)$$

where  $\boldsymbol{\mu} = \{\mu^+, \mu^-\}$ ,

$$\chi_t^\nu := \tilde{\rho} \times (\mu_t^+ + \mu_t^- + \nu_t), \quad (8)$$

$\tau^\nu = T \wedge \inf\{t : Q_t^\nu = 0\}$ , and her value function

$$H(t, x, S, \boldsymbol{\mu}, q) = \sup_{\nu \in \mathcal{A}} H^\nu(t, x, S, \boldsymbol{\mu}, q), \quad (9)$$

where  $\mathcal{A}$  is the set of admissible strategies consisting of  $\mathcal{F}$ -predictable processes such that  $\int_0^T |\nu_u| du < +\infty$ ,  $\mathbb{P}$ -a.s.



# DPE

The value function satisfies

$$0 = (\partial_t + \mathcal{L}^M + \mathcal{L}^\mu) H + \sup_{\nu} \left\{ (S - k\nu) \nu \partial_x H - \nu \partial_q H + b((\mu^+ - \mu^-) - \nu) \partial_S H - \varphi (\nu - \rho \mu)^2 \right\}, \quad (10)$$

for  $q > 0$ , with terminal condition

$$H(t, x, S, \mu, q) = x + q(S - \alpha q), \quad (11)$$

where  $\mu = \mu^+ + \mu^-$ ,  $\mathcal{L}^\mu$  represents the infinitesimal generator of  $\mu$ ,  $\mathcal{L}^M$  the infinitesimal generator of  $M$ , and we have introduced the re-scaled parameters

$$\varphi = \tilde{\varphi}(1 - \tilde{\rho})^2 \quad \text{and} \quad \rho = \tilde{\rho}/(1 - \tilde{\rho}).$$

## Solving the DPE

The DPE (10) admits the solution

$$H(t, x, S, \mu, q) = x + qS + h_0(t, \mu) + h_1(t, \mu)q + h_2(t)q^2, \quad q > 0$$

where

$$h_2(t) = - \left( \frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b} \right)^{-1} - \frac{1}{2}b, \quad (12a)$$

$$\begin{aligned} h_1(t, \mu) &= \frac{\varphi\rho}{(T-t) + \zeta} \int_t^T \mathbb{E}_{t, \mu} [\mu_s^+ + \mu_s^-] ds \\ &+ \frac{b}{(T-t) + \zeta} \int_t^T ((T-s) + \zeta) \mathbb{E}_{t, \mu} [\mu_s^+ - \mu_s^-] ds, \end{aligned} \quad (12b)$$

$$h_0(t, \mu) = \int_t^T \mathbb{E}_{t, \mu} \left[ \frac{(h_1(t, \mu_s) - 2\varphi\rho(\mu_s^+ + \mu_s^-))^2}{4(k+\varphi)} - \varphi\rho^2(\mu_s^+ + \mu_s^-)^2 \right] ds, \quad (12c)$$

and the constant

$$\zeta = \frac{k+\varphi}{\alpha - \frac{1}{2}b}.$$

## Optimal liquidation speed (Target POV!)

$$\nu_t^* = \frac{1}{(T-t) + \zeta} Q_t^{\nu^*} \quad (13a)$$

$$+ \frac{\varphi}{k + \varphi} \rho \left\{ (\mu_t^+ + \mu_t^-) - \frac{1}{(T-t) + \zeta} \int_t^T \mathbb{E}[(\mu_s^+ + \mu_s^-) | \mathcal{F}_t^\mu] ds \right\} \quad (13b)$$

$$- \frac{b}{k + \varphi} \int_t^T \frac{(T-s) + \zeta}{(T-t) + \zeta} \mathbb{E}[(\mu_s^+ - \mu_s^-) | \mathcal{F}_t^\mu] ds, \quad \forall t < \tau^* \quad (13c)$$

where  $\mathcal{F}_t^\mu$  denotes the natural filtration generated by  $\mu$ , and  $\nu_t^* = 0$  for  $t \geq \tau$ , is the admissible optimal control we seek.

# Optimal liquidation speed (targeting POV)

- ▶ The first component is TWAP-like.
- ▶ The second component consists of targeting instantaneous total order-flow plus a weighted average of future expected total order-flow. Although the POV target is  $\rho \mu_t$ , the strategy targets a lower amount since  $\frac{\varphi \rho}{k + \varphi} \leq \rho$ , where equality is achieved if the costs of missing the target are  $\varphi \rightarrow \infty$  and  $k$  remains finite, or there is no temporary impact  $k \downarrow 0$ .
- ▶ The last component acts to correct her trading based on her expectations of the net order-flow from that point in time until the end of the trading horizon. When there is a current surplus of buy trades she slows down her trading rate to allow the midprice to appreciate before liquidating the rest of her order.

## From POV to VWAP

Investor must liquidate all shares. Do this by letting  $\alpha \rightarrow \infty$

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \nu_t^* &= \frac{1}{T-t} Q_t^{\nu^*} \\ &+ \frac{\varphi}{k+\varphi} \left( \rho \mu_t - \frac{1}{T-t} \int_t^T \mathbb{E}[\mu_s^+ + \mu_s^- | \mathcal{F}_t^\mu] ds \right) \\ &- \frac{b}{k+\varphi} \frac{\int_t^T (T-s) \mathbb{E}[\mu_s^+ - \mu_s^- | \mathcal{F}_t^\mu] ds}{T-t}, \end{aligned}$$

and follow this by taking the limit  $\varphi \rightarrow \infty$ , resulting in

$$\lim_{\varphi \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \nu_t^* = \frac{1}{T-t} Q_t^{\nu^*} + \rho \left( (\mu_t^+ + \mu_t^-) \right) \quad (15)$$

$$- \frac{1}{T-t} \int_t^T \mathbb{E} [ (\mu_s^+ + \mu_s^-) | \mathcal{F}_t^\mu ] ds \Big). \quad (16)$$

## Targeting VWAP via POCV

# Accumulated Volume

- ▶ Accumulated volume  $V$  of orders, excluding the agent's, is

$$V_t = \int_0^t (\mu_u^+ + \mu_u^-) du.$$

- ▶ The investor's performance criteria is now modified to

$$H^\nu(t, x, S, \mu, V, q) = \mathbb{E}_{t,x,S,\mu,V,q} \left[ X_{\tau^\nu}^\nu + Q_{\tau^\nu}^\nu (S_{\tau^\nu}^\nu - \alpha Q_{\tau^\nu}^\nu) - \tilde{\varphi} \int_t^{\tau^\nu} \left( (\mathfrak{N} - Q_u^\nu) - \tilde{\rho} (V_u + (\mathfrak{N} - Q_u^\nu)) \right)^2 du \right]$$

and  $\tau^\nu = T \wedge \inf\{t : Q_t^\nu = 0\}$ .

- ▶ The investor's value function is

$$H(t, x, S, \mu, V, q) = \sup_{\nu \in \mathcal{A}} H^\nu(t, x, S, \mu, V, q). \quad (17)$$

# DPE

The value function should satisfy the DPE

$$0 = (\partial_t + \mathcal{L}^M + \mathcal{L}^{\mu, V}) H - \varphi ((\mathfrak{N} - q) - \rho V)^2 + \sup_{\nu} \{ b((\mu^+ - \mu^-) - \nu) \partial_S H + (S - k\nu) \nu \partial_x H - \nu \partial_q H \}, \quad (18)$$

for  $q > 0$ , subject to  $H(T, x, S, y, q) = x + q(S - \alpha q)$  and

$$\varphi = \tilde{\varphi}(1 - \tilde{\rho})^2 \quad \text{and} \quad \rho = \tilde{\rho}/(1 - \tilde{\rho}).$$

Here,  $\mathcal{L}^{\mu, V}$  denotes the infinitesimal generator of the joint process  $(\mu_t, V_t)_{0 \leq t \leq T}$ .



The DPE (18) admits the solution

$$H(t, x, S, \mu, V, q) = x + qS + h_0(t, \mu, V) + h_1(t, \mu, V)q + h_2(t)q^2, \quad q > 0 \quad (19)$$

where

$$h_2(t) = -\sqrt{k\varphi} \frac{\gamma e^{\xi(T-t)} + e^{\xi(T-t)}}{\gamma e^{\xi(T-t)} - e^{-\xi(T-t)}} - \frac{1}{2}b, \quad (20a)$$

$$\begin{aligned} h_1(t, \mu, V) &= 2\varphi \int_t^T \ell(u, t) (\mathfrak{N} - \rho \mathbb{E}_{t, \mu, V} [V_u]) du \\ &\quad + b \int_t^T \ell(u, t) \mathbb{E}_{t, \mu, V} [(\mu_u^+ - \mu_u^-)] du, \end{aligned} \quad (20b)$$

$$h_0(t, \mu, V) = \int_t^T \mathbb{E}_{t, \mu} \left[ \frac{1}{4k} (h_1(u, \mu_u, V_u))^2 - \varphi (\mathfrak{N} - \rho V_u)^2 \right] du, \quad (20c)$$

the function

$$\ell(u, t) := \frac{\gamma e^{\xi(T-u)} - e^{-\xi(T-u)}}{\gamma e^{\xi(T-t)} - e^{-\xi(T-t)}}, \quad (20d)$$

and the constants

$$\xi = \sqrt{\frac{\varphi}{k}}, \quad \text{and} \quad \gamma = \frac{\alpha - \frac{1}{2}b + \sqrt{k\varphi}}{\alpha - \frac{1}{2}b - \sqrt{k\varphi}}.$$

# Optimal control for POCV

Moreover, the trading speed, for  $q > 0$ , is given by

$$\nu_t^* = \xi \frac{1 + \gamma}{\gamma e^{\xi(T-t)} - e^{-\xi(T-t)}} Q_t^{\nu^*} \quad (21a)$$

$$- \xi^2 \int_t^T \ell(u, t) \left\{ \left( \mathfrak{N} - Q_t^{\nu^*} \right) - \rho \mathbb{E} \left[ V_u \mid \mathcal{F}_t^{\mu, V} \right] \right\} du \quad (21b)$$

$$- \frac{b}{2k} \int_t^T \ell(u, t) \mathbb{E} \left[ (\mu_u^+ - \mu_u^-) \mid \mathcal{F}_t^{\mu, V} \right] du, \quad \forall t < \tau^{\nu^*}. \quad (21c)$$

And the optimal inventory path

$$Q_t^{\nu^*} = \ell(t, 0) \mathfrak{N} + \int_0^t \int_u^T \ell(u, s) \left\{ \frac{\varphi}{k} \left( \mathfrak{N} - \rho \mathbb{E} \left[ V_s \mid \mathcal{F}_u^{\mu, V} \right] \right) + \frac{b}{2k} \mathbb{E} \left[ (\mu_s^+ - \mu_s^-) \mid \mathcal{F}_u^{\mu, V} \right] \right\} ds du. \quad (22)$$

## Interpreting liquidation speed

- ▶ (21a) is an Almgren-Chriss like strategy which approaches TWAP near maturity.
- ▶ (21b) corrects for the weighted average of the difference between what the investor has liquidated so far ( $\mathfrak{N} - Q_t^{\nu^*}$ ) and her future targeted volume  $\rho \mathbb{E} \left[ V_u \mid \mathcal{F}_t^{\mu, \nu} \right]$ .
- ▶ (21c) contributes a weighted average of future net order-flow and accounts for the permanent impact on the midprice. If future net order-flow is expected to be buy heavy, then investor slows down
- ▶ Finally, as  $t \rightarrow T$  the second and third terms become negligible and the investor ignores order-flow entirely and instead focuses on completing her trades.

# Targeting VWAP

Let  $\alpha \rightarrow \infty$ , so that the investor ensures that she completely liquidates her position by the terminal time,  $\gamma \rightarrow 1$ , therefore  $\ell(u, t) \rightarrow \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-t))}$  and so we have

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \nu_t^* &= \xi \frac{Q_t^{\nu^*}}{\sinh(\xi(T-t))} \\ &- \xi^2 \int_t^T \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-t))} \left\{ (\mathfrak{N} - Q_t^{\nu^*}) - \rho \mathbb{E} \left[ V_u \mid \mathcal{F}_t^{\mu, V} \right] \right\} du \\ &- \frac{b}{2k} \int_t^T \frac{\sinh(\xi(T-u))}{\sinh(\xi(T-t))} \mathbb{E} \left[ (\mu_u^+ - \mu_u^-) \mid \mathcal{F}_t^{\mu, V} \right] du. \end{aligned}$$

# Strategy Performance

Assume that other market participants trades follow

$$d\mu_t^+ = -\kappa^+ \mu_t^+ dt + \eta_{1+N_t^+}^+ dN_t^+, \quad (23a)$$

$$d\mu_t^- = -\kappa^- \mu_t^- dt + \eta_{1+N_t^-}^- dN_t^-, \quad (23b)$$

where

- ▶  $\kappa^\pm \geq 0$  are the mean-reversion rates,
- ▶  $N_t^+$  and  $N_t^-$  are independent homogeneous Poisson processes with intensities  $\lambda^+$  and  $\lambda^-$ , respectively,
- ▶  $\{\eta_1^\pm, \eta_2^\pm, \dots\}$  are non-negative i.i.d. random variables with distribution function  $F$ , with finite first moment, independent from all processes.
- ▶ In addition, we require  $\kappa^\pm > \lambda^\pm \mathbb{E}[\eta_1^\pm]$  to ensure that  $\mu^\pm$  remain bounded  $\mathbb{P}$ -a.s..

## VWAP as target

Compute VWAP as

$$\text{VWAP} = \frac{\int_0^T S_u^\nu (\mu_u^+ + \mu_u^- + \nu_u) du}{\int_0^T (\mu_u^+ + \mu_u^- + \nu_u) du}, \quad (24a)$$

while the execution price is computed as

$$\text{Exec. Price} = \frac{X_T^\nu}{\mathfrak{N}} = \frac{\int_0^T \hat{S}_u^\nu \nu_u du}{\mathfrak{N}}. \quad (24b)$$

## POV-VWAP performance

**Table :** The statistics of the execution price, VWAP, and relative error (computed as  $(\text{Exec. Price} - \text{VWAP})/\text{VWAP}$  for each simulation) and reported in basis points (i.e.,  $\times 10^4$ ).

		FARO		SMH		NTAP	
		VWAP	Rel.Error	VWAP	Rel.Error	VWAP	Rel.Error
quantile	mean	\$ 40.54	8.9	\$ 37.90	2.98	\$ 38.30	0.19
	stdev	\$0.11	16.9	\$0.04	6.10	\$0.06	0.87
	5%	\$40.35	-4.2	\$37.83	-1.03	\$38.20	-0.78
	25%	\$40.46	-0.4	\$37.87	-0.16	\$38.26	-0.20
	50%	\$40.54	2.5	\$37.90	0.53	\$38.30	0.03
	75%	\$40.61	12.0	\$37.92	3.60	\$38.34	0.36
	95%	\$40.72	42.6	\$37.96	15.29	\$38.40	1.66
%t : $\nu_t^* < 0$		27.3%		18.4%		0.6%	

# POCV-VWAP performance

**Table :** The statistics of the execution price, VWAP, and relative error (computed as (Exec. Price – VWAP)/VWAP for each simulation) and reported in basis points (i.e.,  $\times 10^4$ ). **For POCV we set  $\varphi = 10^5 \times k$**

		FARO		SMH		NTAP	
		VWAP	Rel.Error	VWAP	Rel.Error	VWAP	Rel.Error
quantile	mean	\$40.54	5.0	\$37.89	2.75	\$38.30	1.88
	stdev	\$0.11	14.2	\$0.04	6.02	\$0.06	2.35
	5%	\$40.35	-13.6	\$37.83	-5.43	\$38.20	-1.65
	25%	\$40.46	-3.0	\$37.87	-0.78	\$38.26	0.28
	50%	\$40.54	1.9	\$37.89	1.79	\$38.30	1.72
	75%	\$40.61	11.0	\$37.92	5.51	\$38.34	3.27
	95%	\$40.73	31.9	\$37.96	13.76	\$38.40	6.00
%t : $\nu_t^* < 0$		0.83%		0.013%		0.55%	