Dynamic optimal execution in a mixed-market-impact Hawkes price model

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What is price impact ?

It is well known that executing market orders modifies the price of the traded assets. A market buy order increases the price while a market sell order decreases the price.
The price impact of a market order is usually decomposed in three parts :

- the immediate impact only affects the trader itself,
- the transient impact decays and vanishes with time,
- the permanent impact. It is often taken as the fundamental information of the trade on the price.

Price impact modeling : a matter of time scale

- At a very low frequency, the price impact is usually ignored.
- At a very high frequency, the price impact is built in the Limit Order Book dynamics.
- At a mesoscopic time scale, one has to model it : trade-off between tractability and realism that takes into account the trade frequency. One typically wants to solve the optimal execution problem : how to buy/sell optimally a given amount of assets within the deadline $T > 0$?

Almgren and Chriss model (1999)

It considers one large trader that starts trading at time 0. Let $X_t \in \mathbb{R}$ be the quantity of assets hold by the trader at time t.

- Asset price : $P_t = S_t^0 + \lambda(X_t - X_0) + \mu \frac{dX_t}{dt}$, with S_t^0 martingale, $\lambda, \mu \geq 0$.
- Cost of the strategy :

$$\int_0^T P_t dX_t = \int_0^T S_t^0 dX_t + \frac{\lambda}{2}(X_T - X_0)^2 + \mu \int_0^T \left(\frac{dX_t}{dt}\right)^2 dt$$

- Liquidation strategy ($X_T = 0$) that minimizes the expected costs $X_t = (1 - t/T)X_0$.

Comments : No transient impact. Other market orders $\to S_t^0$.

Obizhaeva and Wang model (2005,2013)

One large trader that trades on $[0, T]$. Other market orders create noise $\rightarrow S_t^0$ martingale.

- Let $q > 0$ and $\epsilon \in [0, 1]$. Asset price : $P_t = S_t^0 + \frac{\epsilon}{q}(X_t - X_0) + D_t$, with

$$dD_t = -\rho D_t dt + \frac{1-\epsilon}{q} dX_t,$$

  and $D_0 = 0$ (steady state).
- Cost of trading dX_t :

$$dX_t \times (P_t + \frac{1}{2q} dX_t).$$

- Optimal strategy :

$$dX_t = -\frac{X_0}{2 + \rho T}[\delta_0(dt) + \rho dt + \delta_T(dt)].$$

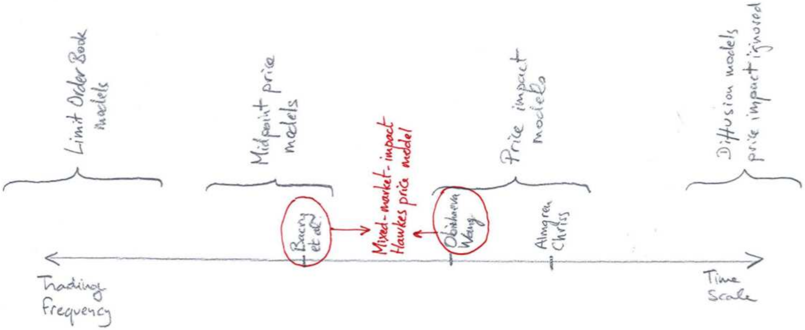
Comments on the OW model

- Possible extensions to nonlinear price impact or non exponential decay Kernel.
- Almgren and Chriss model is a limit of OW model when $\rho \to \infty$ or equivalently when the trading frequency decreases to zero.
- The price resilience ρ can be seen as the feedback of market makers.
- Impact of other market orders modeled through S_t^0. No resilience for them.

High frequency price models

- Limit Order Book models : Abergel and Jedidi, Cont and De Larrard, Huang, Lehalle and Rosenbaum,... the LOB dynamics encompasses price impact.
- There are many more LOB events than price changes
$\rightarrow$ Midpoint price models : Bacry, Delattre, Hoffmann and Muzy, Bacry and Muzy, Robert and Rosenbaum,... These models are usually meant to reproduce statistical properties of the price. The price impact is generally not directly modelled.

Position of our work



The price model : OW with a flow of orders

N_t : sum of the signed volumes ($dN_t > 0$ if buy order) of past market orders on the book between time 0 and time t.
Assumption : N is a càdlàg process, $(\mathcal{F}_t)$-adapted
$\forall t > 0, \ \sup_{s\in[0,t]} \mathbb{E}[N_s^2] < \infty$.

$$P_t = \underbrace{S_t}_{\text{fundamental price}} + \underbrace{D_t}_{\text{mesoscopic price deviation}},$$

$$\mathrm{d}S_t = \frac{\nu}{q} \underbrace{\mathrm{d}N_t}_{\text{market orders}}$$

$$\mathrm{d}D_t = \underbrace{-\rho\, D_t\, \mathrm{d}t}_{\text{market resilience}} + \frac{1-\nu}{q} \underbrace{\mathrm{d}N_t}_{\text{market orders}}.$$

Rk : Adding a martingale S_t^0 to P_t does not change what follows.

The liquidation strategy

A "strategic" trader with X_t assets at time t. We assume that

- X is a càglàd, i.e. he can react instantly to other market orders,
- X is $(\mathcal{F}_t)$-adapted, sq. integrable with bounded variation,
- $X_0 \in \mathbb{R}$, $X_{T+} = 0$.

Price : $P_t = S_t + D_t$ with $\epsilon \in [0,1]$,

$$dS_t = \frac{1}{q}\left(\nu \mathrm{d}N_t + \epsilon \mathrm{d}X_t\right),\ dD_t = -\rho\, D_t\, \mathrm{d}t + \frac{1}{q}\left((1-\nu)\mathrm{d}N_t + (1-\epsilon)\mathrm{d}X_t\right).$$

The processes P, S, D are làdlàg :

$$S_t - S_{t-} = \frac{\nu}{q}(N_t - N_{t-}),\ S_{t+} - S_t = \frac{\epsilon}{q}(X_{t+} - X_t),$$

$$D_t - D_{t-} = \frac{1-\nu}{q}(N_t - N_{t-}),\ D_{t+} - D_t = \frac{1-\epsilon}{q}(X_{t+} - X_t).$$

The liquidation cost

- Cost of the trade dX_t : $dX_t \times (P_t + \frac{1}{2q} dX_t) = dX_t \times \frac{P_t + P_{t+}}{2}$.
- Cost of the strategy :

$$C(X) = \int_{[0,T)} P_u \, \mathrm{d}X_u \; + \; \frac{1}{2q} \sum_{0 \le \tau < T} (\Delta X_\tau)^2 \; - \; P_T X_T \; + \; \frac{1}{2q} \, X_T^2$$

Optimal execution problem : find the liquidation strategy X that minimizes $\mathbb{E}[C(X)]$.

Price Manipulation Strategies

Definition 1

A Price Manipulation Strategy (PMS) in the sense of Huberman and Stanzl is a X such that $X_0 = X_{T+} = 0$ a.s. and $\mathbb{E}[C(X)] < 0$.

Theorem 2

The model does not admit PMS if, and only if the process P is a $(\mathcal{F}_t)$-martingale when $X \equiv 0$. Then, the optimal strategy X^* is

$$\Delta X_0^* = -\frac{x_0}{2+\rho T},\ \Delta X_T^* = -\frac{x_0}{2+\rho T},\ \mathrm{d}X_t^* = -\rho\frac{x_0}{2+\rho T}\mathrm{d}t \textit{ for } t \in (0,T),$$

and has the expected cost $\mathbb{E}[C(X^*)] = -P_0 x_0 + \left[\frac{1-\epsilon}{2+\rho(T-t)} + \frac{\epsilon}{2}\right] x_0^2/q.$

This is the same strategy as in the OW model.
Question : which flows of orders satisfy this condition ?

Modeling of the order flow

The Mixed-market-Impact Poisson (MIP) model : $N_t = N_t^+ - N_t^-$, where $(N_t^+)_{t\in[0,T]}$ and $(N_t^-)_{t\in[0,T]}$ are two independent compound Poisson processes (intensities κ_0^+ and κ_0^-, jump law μ on $\mathbb{R}_+$). We define

$$m_k = \int_{\mathbb{R}^+} v^k \mu(\mathrm{d}v),\ k \in \mathbb{N},\ \delta_0 = \kappa_0^+ - \kappa_0^- \text{ and } \Sigma_0 = \kappa_0^+ + \kappa_0^-,$$

and assume $m_2 < \infty$.

The optimal liquidation strategy I

Let $\epsilon \in [0,1)$. The optimal strategy to liquidate x_0 is

$$(1-\epsilon)\Delta X_0^* = -\frac{(1-\epsilon)x_0 \; + \; [1+\rho T]\left(qD_0 - \frac{m_1}{\rho}\,\delta_0\right) \; - \; \frac{\nu m_1}{4}\rho T^2 \delta_0}{2+\rho T}$$

$$(1-\epsilon)\Delta X_T^* = \frac{qD_0 - (1-\epsilon)x_0}{2+\rho T} \; - \; \frac{m_1}{\rho} \times \left[\frac{1-\frac{\nu}{4}\rho^2 T^2}{2+\rho T} \; + \; (1-\nu)\ln\left(1+\frac{\rho T}{2}\right) + \frac{\nu}{2}\,\rho T\right]\; \delta_0$$
$$+ \sum_{0<\tau<T} \frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}$$

$$(1-\epsilon)\mathrm{d}X_t^* = \left[\frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T} - (1-\nu)\ln\left(\frac{2+\rho T}{2+\rho(T-t)}\right) - \frac{\nu}{2}\,\rho t \; - \; \frac{1+\frac{\nu}{2}\rho(T-t)}{2+\rho(T-t)}\right] m_1 \delta_0 \,\mathrm{d}t$$
$$+ \left[\frac{qD_0 - (1-\epsilon)x_0}{2+\rho T} \; + \sum_{0<\tau<t} \frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}\right]\rho \mathrm{d}t - \frac{1+\rho(T-t)}{2+\rho(T-t)} \times (1-\nu)\mathrm{d}N_t$$

The optimal liquidation strategy II

The corresponding cost function is given by $\mathcal{C}(x_0, D_0, S_0)$, where

$$
\begin{aligned}
q \times \mathcal{C}(x,d,z) \quad = \quad & -q(z+d)x + \left[\frac{1-\epsilon}{2+\rho T} + \frac{\epsilon}{2}\right] x^2 + \frac{\rho T}{2+\rho T}\left[qd - \mathcal{G}_0(T)\,\frac{\delta_0 m_1}{\rho}\right] x \\
& - \frac{1}{1-\epsilon} \times \frac{\rho T/2}{2+\rho T}\left[qd - \mathcal{G}_0(T)\,\frac{\delta_0 m_1}{\rho}\right]^2 - \frac{1}{1-\epsilon} \times \frac{\nu^2}{48}\,\rho^3 T^3 \left(\frac{\delta_0 m_1}{\rho}\right)^2 \\
& - \frac{(1-\nu)^2}{1-\epsilon} \times m_2 \times \left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1 + \frac{\rho T}{2}\right)\right] \Sigma_0 .
\end{aligned}
$$

where $\mathcal{G}_0(T) = 1 + \frac{\nu}{2}\rho T$. The model admits price manipulation strategies (i.e. $\mathcal{C}(0,d,z) < 0$) unless $m_1 = m_2 = 0$ (i.e. $N \equiv 0$) and $D_0 = 0$.
Proof : guess on the quadratic cost function + verification argument

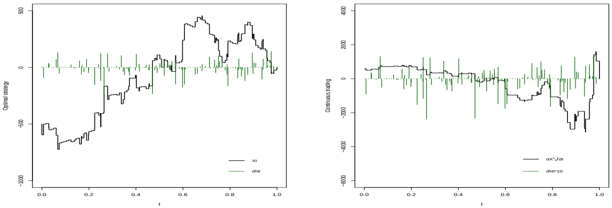
Comments

- Optimal strategy affine with respect to x_0, δ_0, D_0, N.
- Reaction to other trades in the opposite direction. This is compensated in the continuous trading rate.
- Obvious PMS when $\delta_0 \neq 0$.
- When $\delta_0 = 0$:

$$q \times \mathcal{C}(0,d,z) = -\frac{1}{1-\epsilon} \times \frac{\rho T/2}{2+\rho T} q^2 d^2 - \frac{(1-\nu)^2}{1-\epsilon} \times m_2 \times \left[\frac{T}{2} - \frac{1}{\rho} \ln\left(1 + \frac{\rho T}{2}\right)\right] \Sigma_0.$$

  $\rightarrow$ PMS take advantage of the knowledge of D_0 and the price resilience.
- Can we still have PMS if D_0 and ρ are unknown/misestimated ?

Numerical example



(a) $(X_t)_{0 \leq t \leq T}$ (b) $\mathrm{d}X_t^c/\mathrm{d}t$

FIGURE: Optimal strategy in the Poisson model for $q = 100,\ T = 1,\ \rho = 50,\ \epsilon = 0.1,\ \nu = 0.1,\ D_0 = 1,\ \kappa_0^+ = \kappa_0^- = 60,\ m_1 = 50,\ X_0 = -500$ and $\mu = \mathrm{Exp}(1/m_1)$, with the corresponding trajectory of (N_t).

Uncertainty on D_0

$D_0 = d_0$. Cost of the strategy $\tilde{X}$ obtained by using the optimal strategy with $\tilde{d}_0$

$$\mathbb{E}\left[C(\tilde{X})\right] = \mathcal{C}_0(d_0) + \frac{1}{(1-\epsilon)q} \times \frac{\rho T/2}{2+\rho T} \times q^2(\tilde{d}_0 - d_0)^2;$$

It is nonpositive iff $|\tilde{d}_0 - d_0| \;\leq\; \Delta d_0(T)$, where

$$\Delta d_0(T) \;=\; \sqrt{d_0^2 + \frac{(1-\nu)^2}{q^2} \times \frac{4\kappa_0 m_2(2+\rho T)}{\rho T}\left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]} \geq |d_0|.$$

$\rightarrow$ PMS by taking $\tilde{d}_0 = 0$.

Uncertainty on ρ

$D_0 = 0$. Cost of the strategy $\tilde{X}$ with the optimal strategy with $\tilde{\rho} > 0$ instead of ρ :

$$\mathbb{E}[C(\tilde{X})] = \frac{(1-\nu)^2}{(1-\epsilon)q} \times \frac{2\kappa_0 m_2}{\rho} \times f(r),$$

with $r = \frac{\tilde{\rho}}{\rho}$. The function f is negative (nonincreasing on $(0,1]$, nondecreasing on $[1,\infty)$ with $f(0+) < 0$ and $f(\infty) < 0$.
$\rightarrow \tilde{X}$ is thus a PMS.
For $\tilde{\rho} = 0$, the strategy $\tilde{X}$ is simply

$$\Delta\tilde{X}_0 = 0,\ (1-\epsilon)\Delta\tilde{X}_T = \sum_{0<\tau<T} \frac{(1-\nu)\Delta N_\tau}{2},\ (1-\epsilon)\mathrm{d}\tilde{X}_t = -\frac{1}{2}(1-\nu)\mathrm{d}N_t.$$

A very robust PMS

$\lambda \in (0,1)$. The following round-trip strategy $X_0^\lambda = X_{T+}^\lambda = 0$ defined by

$$X_{\tau+}^\lambda - X_\tau^\lambda = -\frac{1-\nu}{1-\epsilon} \times \lambda(N_\tau - N_{\tau-})$$

at each jump of N is a PMS :

$$\mathbb{E}[C(X^\lambda)] = 2\lambda(1-\lambda)\frac{\kappa_0 m_2 (1-\nu)^2}{q(1-\epsilon)}\left[\frac{1-\exp(-\rho T)}{\rho} - T\right] < 0,$$

and the best choice is to take $\lambda = 1/2$.

A Poissonian market order flow leads to easy PMS

Modeling of the order flow

The Mixed-market-Impact Hawkes (MIH) model. $N_t = N_t^+ - N_t^-$.
Jump law μ, respective intensities κ_t^+ and κ_t^- : càdlàg processes that follow the Markovian marked Hawkes dynamics

$$\begin{aligned} d\kappa_t^+ &= -\beta\,(\kappa_t^+ - \kappa_\infty)\,dt \,+\, \varphi_s(dN_t^+/m_1) + \varphi_c(dN_t^-/m_1) \quad, \\ d\kappa_t^- &= -\beta\,(\kappa_t^- - \kappa_\infty)\,dt \,+\, \varphi_c(dN_t^+/m_1) + \varphi_s(dN_t^-/m_1), \end{aligned}$$

where $\varphi_s, \varphi_c : \mathbb{R}^+ \to \mathbb{R}^+$ are measurable positive functions that satisfy $\iota_s = \int_{\mathbb{R}+} \varphi_s(v/m_1)\mu(dv) < \infty$, $\iota_c = \int_{\mathbb{R}+} \varphi_c(v/m_1)\mu(dv) < \infty$, $\int_{\mathbb{R}+} \varphi_s^2(v/m_1)\mu(dv) < \infty$ and $\int_{\mathbb{R}+} \varphi_c^2(v/m_1)\mu(dv) < \infty$.
We define $\alpha = \iota_s - \iota_c$, $\delta_t = \kappa_t^+ - \kappa_t^-$, $\Sigma_t = \kappa_t^+ + \kappa_t^-$ and $I_t = \int_0^t [(\varphi_s - \varphi_c)(dN_u^+/m_1) - (\varphi_s - \varphi_c)(dN_u^-/m_1)]$.

- Stationary iff $\iota_s + \iota_c < \beta$.
- $\iota_c = \iota_s = \beta = 0 \;\rightarrow$ Poisson model.
- $\nu = 1$, $\varphi_s(x) = \iota_s$, $\iota_c = 0$ and $\mu(dx) = \delta_1(dx) \;\rightarrow$ price model proposed in Bacry, Delattre, Hoffmann and Muzy.

The optimal liquidation strategy I

Let $\epsilon \in [0, 1)$. The optimal strategy to liquidate x_0 is explicit. It is a linear combination of $(x_0, D_0, \delta_0, I, N)$ and can be written as

$$X^* = X^{\text{OW}} + X^{\text{trend}} + X^{\text{dyn}},$$

where

$$\Delta X_0^{\text{OW}} = -\frac{x_0}{2 + \rho T}, \ \Delta X_T^{\text{OW}} = -\frac{x_0}{2 + \rho T}, \ \mathrm{d}X_t^{\text{OW}} = -\rho \frac{x_0}{2 + \rho T} \mathrm{d}t \text{ for } t \in (0, T),$$

is the part that is proportional to x_0 and is the Obizhaeva and Wang strategy,

The optimal liquidation strategy II

$$
(1-\epsilon)\Delta X_0^{\text{trend}} = \frac{\frac{\delta_0 m_1}{2\rho} \times [2 + \rho T \times \{1 + \zeta(\eta T) + \nu\rho[1-\zeta(\eta T)]/\eta\}] - [1+\rho T]qD_0}{2+\rho T},
$$

$$
(1-\epsilon)\Delta X_T^{\text{trend}} = \frac{\delta_0 m_1}{2\rho} \times \left[\frac{2 + \rho T \times \{1 + \zeta(\eta T) + \nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T} - 2\rho\,\Phi_\eta(0,T) - 2\exp(-\beta T)\right] + \frac{qD_0}{2+\rho T},
$$

and, on $(0,T)$,

$$
(1-\epsilon)\mathrm{d}X_t^{\text{trend}} = \frac{\delta_0 m_1}{2\rho} \times \left[\frac{2 + \rho T \times \{1 + \zeta(\eta T) + \nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T} - 2\rho\,\Phi_\eta(0,t) - 2\phi_\eta(t)\exp(-\beta t)\right]\rho\mathrm{d}t + \frac{qD_0}{2+\rho T}\,\rho\mathrm{d}t.
$$

This is the part that is proportional to (D_0, δ_0) and takes advantage of the initial trend.

The optimal liquidation strategy III

$$
\begin{aligned}
(1-\epsilon)\Delta X_0^{\text{dyn}} =& 0, \\
(1-\epsilon)\Delta X_T^{\text{dyn}} =& -\ m_1 \left[\Theta_{\chi_T}\ \Phi_\eta\ \left(\tau_{\chi_T}, T\right)\ +\ \sum_{i=1}^{\chi_T - 1} \Theta_i\ \Phi_\eta(\tau_i, \tau_{i+1}) \right]\ +\ \sum_{0<\tau\leq T} \frac{(1-\nu)\ \Delta N_\tau}{2+\rho(T-\tau)} \\
& + \frac{m_1}{2\rho} \times \sum_{0<\tau\leq T} \frac{2+\rho(T-\tau) \times \{1+\zeta(\eta(T-\tau)) + \nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)} \Delta I_\tau \\
& - \ \frac{m_1}{\rho}\ \Theta_{\chi_T} \exp(-\beta T),
\end{aligned}
$$

The optimal liquidation strategy IV

and, on $(0, T)$,

$$
\begin{aligned}
(1-\epsilon)\mathrm{d}X_t^{\mathrm{dyn}} = & -m_1\, \phi_\eta(t)\, \Theta_{\chi_t} \exp(-\beta t)\, \mathrm{d}t \; + \left[\sum_{0<\tau\leq t} \frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)} \right] \rho \mathrm{d}t \\
& + \left[\sum_{0<\tau\leq t} \frac{2+\rho(T-\tau) \times \{1+\zeta(\eta(T-\tau)) + \nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)} \, \Delta I_\tau \right] \frac{m_1}{2}\, \mathrm{d}t \\
& - \left[\Theta_{\chi_t}\, \Phi_\eta\left(\tau_{\chi_t}, t\right) \; + \sum_{i=1}^{\chi_t -1} \Theta_i\, \Phi_\eta(\tau_i, \tau_{i+1}) \right] \rho m_1\, \mathrm{d}t
\end{aligned}
$$

$$
+ \; \frac{1+\rho(T-t)}{2+\rho(T-t)} \left\{ \frac{m_1}{\rho}\, \mathrm{d}I_t - (1-\nu)\, \mathrm{d}N_t \right\} \; + \; \frac{m_1}{2\rho}(\nu\rho - \eta) \times \frac{\rho(T-t) \times [1-\zeta(\eta(T-t))]/\eta}{2+\rho(T-t)}\, \mathrm{d}I_t .
$$

This is the part that is proportional to the processes N and I and gives the optimal reaction to other trades.

The optimal liquidation strategy V

The value function of the problem is then :

$$q \times \mathcal{C}(t, x, d, z, \delta, \Sigma) = -q(z+d)x + \left[\frac{1-\epsilon}{2+\rho(T-t)} + \frac{\epsilon}{2}\right] x^2 + \frac{\rho(T-t)}{2+\rho(T-t)} \left[qd - \mathcal{G}_\eta(T-t)\frac{\delta m_1}{\rho}\right] x$$

$$- \frac{1}{1-\epsilon} \times \frac{\rho(T-t)/2}{2+\rho(T-t)} \left[qd - \mathcal{G}_\eta(T-t)\frac{\delta m_1}{\rho}\right]^2 + \hat{c}_\eta(T-t)\left(\frac{\delta m_1}{\rho}\right)^2$$

$$+ e(T-t)\Sigma + g(T-t),$$

where for $u \in [0, T]$, $\mathcal{G}_\eta(u) = \zeta(\eta u) + \nu\rho[1-\zeta(\eta u)]/\eta$,
$\hat{c}_\eta(u) = -\frac{1}{1-\epsilon} \times \left(1 - \frac{\nu\rho}{\eta}\right)^2 \times \frac{\rho u \zeta(\eta u)}{8} \times [1 + \exp(-\eta u) - 2\zeta(\eta u)]$.

The optimal liquidation strategy VI

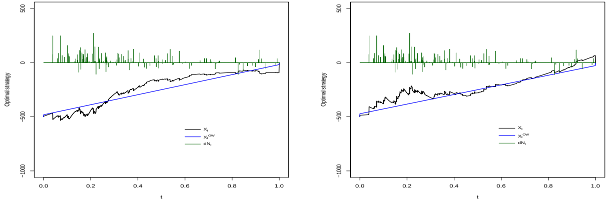
$$
\begin{aligned}
\zeta(y) &= \frac{1-\exp(-y)}{y}, \\
L(r,\lambda,t) &= r\int_0^t \frac{\exp(\lambda s)}{2+rs}\,\mathrm{d}s \;=\; \exp(-2\lambda/r)\;\left[\mathcal{E}\left(\frac{\lambda}{r}(2+rt)\right)-\mathcal{E}\left(\frac{2\lambda}{r}\right)\right] \\
\phi_\eta(t) &= \frac{1}{2(2+\rho(T-t))}\times\Big[1+\exp(-\eta(T-t))+\nu\rho(T-t)\zeta(\eta(T-t)) \\
&\quad+\frac{\beta}{\rho}\left[2+\rho(T-t)\times\{1+\zeta(\eta(T-t))+\nu\rho[1-\zeta(\eta(T-t))]/\eta\}\right]\Big],\;\eta\neq 0, \\
\Phi_\eta(s,t) &= \frac{1}{2}\left(\frac{1}{\rho}+\frac{\nu}{\eta}\right)\times[\exp(-\beta s)-\exp(-\beta t)] \\
&\quad+\;\frac{\exp(-\beta T)}{2\rho}\times\left[1+\frac{\nu(\rho-2\beta)}{\eta}+\frac{\beta}{\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right]\times[L(\rho,\beta,T-s)-L(\rho,\beta,T-t)] \\
&\quad+\;\frac{\exp(-\beta T)}{2\rho}\times\left[1-\frac{\nu\rho}{\eta}-\frac{\beta}{\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right]\times[L(\rho,\alpha,T-s)-L(\rho,\alpha,T-t)],\;\eta\neq 0.
\end{aligned}
$$

Comments

- Only depends on (φ_s, φ_c) through $\varphi_s - \varphi_c$.
- Contrary to the Poisson case, the reaction to other market orders is not always in the opposite direction and depend on the order size.
- The quantity to liquidate that maximizes $\mathbb{E}[C(X)] + P_0 \times x_0$, i.e. the expected liquidation cost with respect to the mark-to-market value is

$$x_0^* = \frac{\rho T[qD_0 - \mathcal{G}_\eta(T)\frac{\delta_0 m_1}{\rho}]}{2\left(1 + \frac{\epsilon}{2}\rho T\right)}.$$

Numerical example



(a) $(X_t)_{0 \leq t \leq T}$ (b) $\mathrm{d}X_t^{\mathrm{c}}/\mathrm{d}t$

FIGURE: Optimal strategy in the Hawkes model, in black, for $q = 100,\ T = 1,\ \beta = 20,\ \iota_{\mathrm{s}} = 16,\ \iota_{\mathrm{c}} = 2,\ \kappa_\infty = 12,\ \epsilon = 0.3,\ \nu = 0.3,\ D_0 = 0.1,\ \kappa_0^+ = \kappa_0^- = 60,\ m_1 = 50,\ X_0 = -500,\ \mu = \mathrm{Exp}(1/m_1),\ \varphi_{\mathrm{s}}(y) = 1.2 \times y^{0.2} + 0.5 \times y^{0.7} + 14.4 \times y,\ \varphi_{\mathrm{c}}(y) = 1.2 \times y^{0.2} + 0.5 \times y^{0.7} + 0.4 \times y$ for all $y > 0$, with the corresponding trajectory of (N_t).

The Mixed-Impact Hawkes Martingale (MIHM) model

Let $\mathcal{S}(\mu) = \{y \geq 0, m_1 \times y \text{ is in the support of } \mu\}$.

Proposition 3

The MIH model does not admit PMS if, and only if

$$\beta = \rho,\ \alpha = (1-\nu)\rho,\ \varphi_s(x) - \varphi_c(x) = \alpha x \text{ for } x \in \mathcal{S}(\mu), \text{ and } qD_0 = \frac{m_1}{\rho}\delta_0$$

or $\mu = Dirac(0)$ with $D_0 = 0$. The optimal execution strategy is then the same as in the OW model.

Proof $\delta_t = \kappa_t^+ - \kappa_t^-$ satisfies $\mathrm{d}\delta_t = -\beta\, \delta_t\, \mathrm{d}t + \mathrm{d}I_t$.
$\mathrm{d}P_t = -\rho D_t dt + \frac{1}{q}\mathrm{d}N_t = \frac{1}{q}(\mathrm{d}N_t - \delta_t m_1 \mathrm{d}t) + \left(\frac{m_1}{q}\delta_t - \rho D_t\right)\mathrm{d}t$.
Thus, P is a mg $\iff \frac{m_1}{\rho}\delta_t = qD_t$, and we use $dD_t = -\rho D_t dt + \frac{1-\nu}{q} dN_t$.

Comments

- When fitted to market data, one may expect to find parameters different but not "too far" from to the MIHM case.
- $\beta = \rho$: the autocorrelation of trade signs is compensated by liquidity providers (same conclusion as in Bouchaud, Gefen, Potters and Wyart).
- $\alpha = (1-\nu)\beta$: when $\iota_c = 0$, α/β is the average number of child orders. $1-\nu$: proportion of transient (vanishing) impact.
- When $\varphi_s(x) = \iota_s$ and $\varphi_c(x) = \iota_c$, $\mu = \text{Dirac}(m_1)$ comes from the fact that all market orders have the same impact regardless of their size. $\rightarrow$ clustering of size orders around typical values.
- MIHM stationary iff $\iota_c < \nu\rho/2$.

Market stability

Definition 4

We say that a market admits weak Price Manipulation Strategies (wPMS) if the cost of a liquidation strategy can be reduced by trading immediately after other market orders.

Corollary 5

In the MIH model, the market does not admit wPMS if, and only if,

$$\beta = \rho,\ \alpha = (1-\nu)\rho \text{ and } \varphi_s(x) - \varphi_c(x) = \alpha x \text{ for } x \in \mathcal{S}(\mu),$$

or $\mu = Dirac(0)$.

In this case, $(\frac{m_1}{q}\delta_t - \rho D_t) = (\frac{m_1}{q}\delta_0 - \rho D_0)e^{-\rho t} \underset{t\to\infty}{\to} 0$.
Therefore excluding PMS or wPMS is quite equivalent in the MIH model.

Low frequency asymptotics

We take $X \equiv 0$ and define $P_t^{(n)} = P_{nt}/\sqrt{n}$. By using an asymptotic result of Bacry, Delattre, Hoffmann and Muzy on Hawkes processes we get when $\varphi_s = \iota_s$ and $\varphi_c = \iota_c$

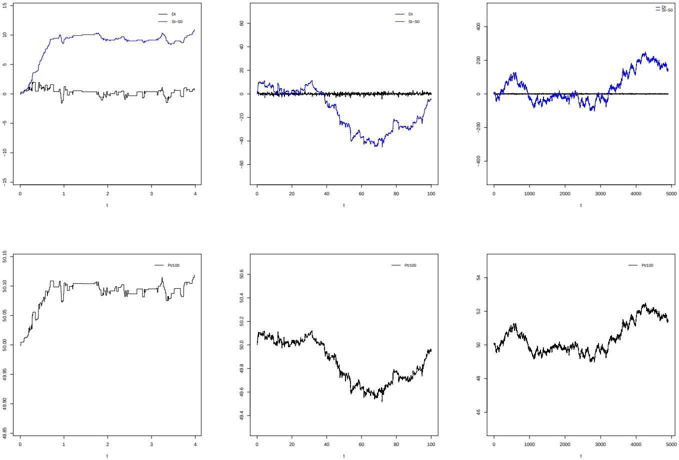
Proposition 6

When $\alpha + 2\iota_c < \beta$ *and* $\mu([0, K]) = 1$, $(P_t^{(n)}, t \in [0, T])$ *converges in law to* $(\tilde{P}_t, t \in [0, T])$ *with*

$$\tilde{P}_t = \frac{\nu}{q} \frac{\sqrt{2\kappa_\infty m_2}}{\sqrt{1 - \frac{\alpha}{\beta}} \left(1 - \frac{\alpha + 2\iota_c}{\beta}\right)} W_t.$$

MIHM model : $\tilde{P}_t = \frac{1}{q} \frac{\sqrt{2\nu\kappa_\infty m_2}}{\nu - 2\frac{\iota_c}{\rho}} W_t$.

Numerical illustration (MIHM)



Conclusion

- The Mixed-market-impact Hawkes price model makes a bridge between OW model and higher frequency models.
- The optimal execution problem can be solved explicitly in this model.
- Modeling the market order flow by a Poisson process leads to quite robust PMS.
- Instead, a particular parametrization of the Hawkes process allows to exclude PMS.

Ongoing work : fit this model to market data

Data : time stamps of midprice changes, and at each time, the order type (market order, cancellation, etc.) and the queue size at the best bid and ask price.

Year	2012	2013	2012 – 2013
Average price	32.4	44.9	38.7
Tick size	0.005	0.005	0.005
First queue average size	1398	1136	1260
m_1	776	636	714
m_2/m_1^2	3.38	4.69	3.87
Midpoint changes per hour	1909	1699	1798
Prop. triggered by trades	10.0%	7.9%	9.0%

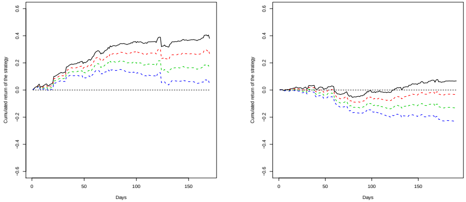
TABLE: Statistics for the stock BNP Paribas for the periods February-September 2012 and January-September 2013, between 11 a.m. and 1 p.m.

Some estimated parameters

Year	2012			2013			$2012-2013$
Hour	11-12	12-13	11-13	11-12	12-13	11-13	$11-13$
ν	0.5428	0.5774	0.4190	0.2861	0.6873	0.3014	0.3475
ρ	42.9700	25.0647	47.4424	49.9664	50.1413	52.8205	57.8499
κ_∞	25.5	19.0	23.3	19.1	12.7	18.6	20.4
β	70.4	52.7	76.1	70.1	41.6	75.4	78.1
ι_s	41.6	28.8	42.2	41.5	22.9	41.3	43.2
ι_c	13.1	10.3	13.8	11.9	9.6	12.7	14.0
$\frac{\beta-\rho}{\beta\vee\rho}$	39%	52%	38%	29%	17%	30%	26%
$\frac{\alpha-(1-\nu)\rho}{\alpha\vee((1-\nu)\rho)}$	30%	43%	1%	18%	14%	24%	22%

TABLE: Estimated resilience and Hawkes parameters for the stock BNP Paribas, Feb.-Sept. 2012 and Jan.-Sept. 2013, between 11 a.m. and 1 p.m., and deviation in % to MIHM parameters

Backtest of the (capped) optimal strategy



(g) 2012 (h) 2013

FIGURE: P&L in Millions of euros. The volume of each transaction is bounded to 1/10 of the average size of the first queue. The black line is the performance when we trade on average at the midpoint price. The dashed red, green and blue lines correspond respectively to an additional bid-ask cost of one, two and three half-ticks.