Carnets d'ordres pilotés par des processus de Hawkes

workshop sur les *Mathématiques des marchés financiers en haute fréquence*

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What is a limit order book ?

- The **limit order book** is the list, at a given time, of all buy and sell limit orders, with their corresponding prices and volumes
- The order book evolves over time according to the arrival of new orders

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  - **limit order**: specify a price at which one is willing to buy (sell) a certain number of shares
  - **market order**: immediately buy (sell) a certain number of shares at the best available opposite quote(s)
  - **cancellation order**: cancel an existing limit order

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Price dynamics

The price dynamics becomes a by-product of the order book dynamics

Plan

1. Stylized facts of limit order books
   - A word on data
   - Arrival times of orders
   - Cancellation of orders
   - Intraday seasonality
   - Competitive liquidity
2. Mathematical modelling of limit order books
   - Zero-intelligence models
     - Stability and long-time dynamics
     - Large-scale limit of the price process
   - Modelling dependencies
     - Hawkes processes
     - Application to order book modelling
     - Stability and long-time dynamics
3. Simulation with Hawkes processes

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Limit order book events

Timestamp	Side	Level	Price	Quantity
33480.158	B	1	121.1	480
33480.476	B	2	121.05	1636
33481.517	B	5	120.9	1318
33483.218	*B*	*1*	*121.1*	*420*
33484.254	B	1	121.1	556
33486.832	A	1	121.15	187
33489.014	B	2	121.05	1397
33490.473	*B*	*1*	*121.1*	*342*
33490.473	*B*	*1*	*121.1*	*304*
33490.473	*B*	*1*	*121.1*	*256*
33490.473	A	1	121.15	237

Table: Tick by tick data file sample

Trades

Timestamp	Last	Last quantity
33483.097	121.1	60
33490.380	121.1	214
33490.380	121.1	38
33490.380	121.1	48

Table: Trades data file sample.

These data are made available (for free) to the scientific community
`http://fiquant.mas.ecp.fr/liquidity-watch`

Trades and order book data processing

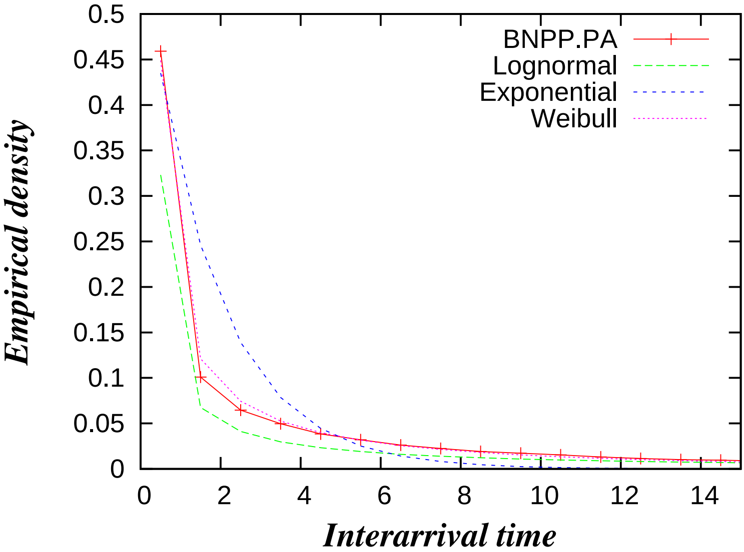
For each stock and each trading day:

1. Parse the tick by tick data file to compute order book state variations;
2. Parse the trades file and for each trade:
    1. Compare the trade price and volume to likely market orders whose timestamps are in $[t^{Tr} - \Delta t, t^{Tr} + \Delta t]$, where t^{Tr} is the trade timestamp and Δt is a predefined time window;
    2. Match the trade to the first likely market order with the same price and volume and label the corresponding event as a market order;
    3. Remaining negative variations are labeled as cancellations.

Doing so, we have an average matching rate of around 85% for CAC 40 stocks. As a byproduct, one gets the sign of each matched trade.

Plan

Arrival time of orders



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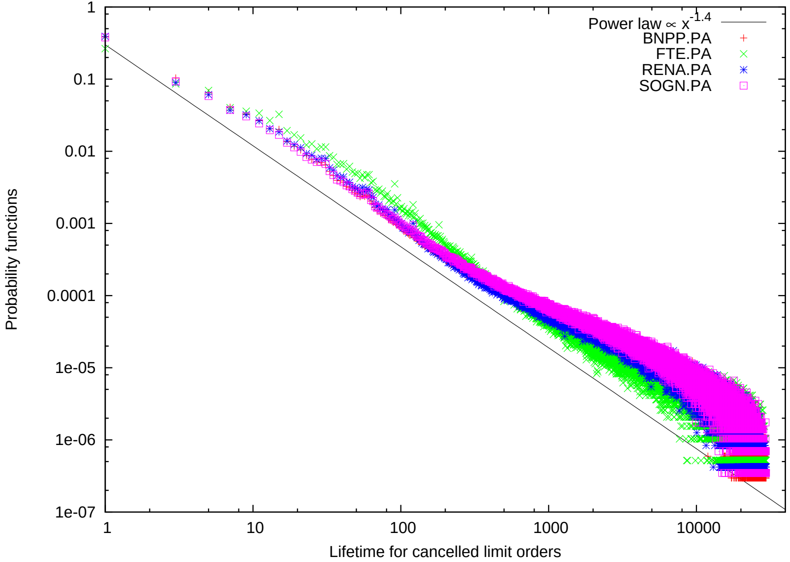

Figure: Distribution of estimated lifetime of cancelled limit orders.

Plan

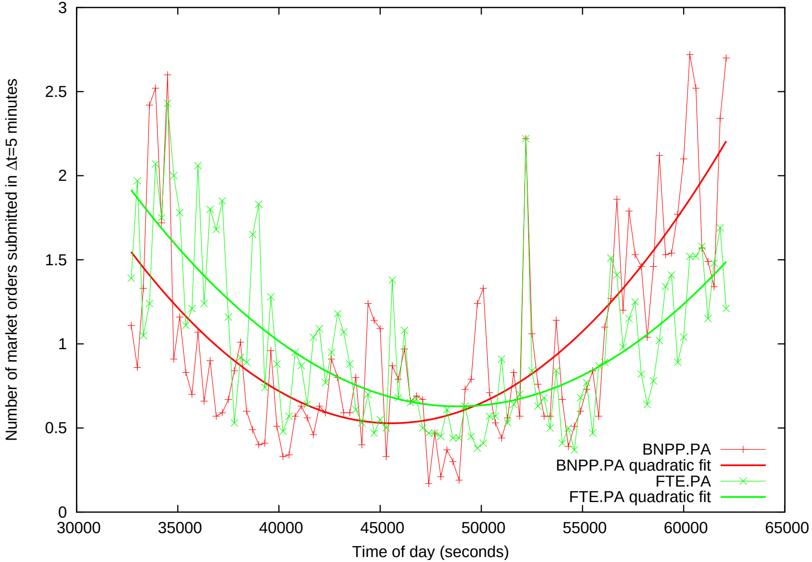

Figure: Normalized average number of market orders in a 5-minute interval.

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Muni Toke (2011)

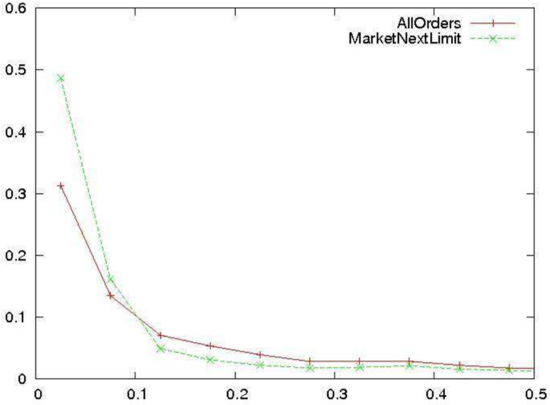

Figure: Evidence of liquidity providing

Competitive liquidity

Muni Toke (2011)Anane et al. (2015)

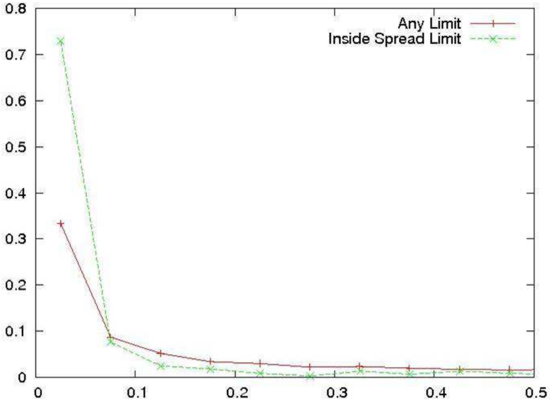

Figure: Evidence of liquidity taking

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Mathematical modelling

A set of reasonable assumptions:

- the limit order book is described as a point process;
- several types of events can happen;
- two events cannot occur simultaneously (simple point process).

The main questions of interest are

- the stationarity and ergodicity of the order book;
- the dynamics of the induced price...

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- the limit order book is described as a point process;
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The main questions of interest are

- the stationarity and ergodicity of the order book;
- the dynamics of the induced price...
- ... particularly, its behaviour at larger time scales.

$\Rightarrow$ Linking microstructure modelling and continuous-time finance

Some notations

- The order book is represented by a finite-size vector of quantities

$$\mathbf{X}(t) := (\mathbf{a}(t); \mathbf{b}(t)) := (a_1(t), \ldots, a_K(t); b_1(t), \ldots, b_K(t));$$

- $\mathbf{a}(t)$: ask side of the order book
- $\mathbf{b}(t)$: bid side of the order book
- ΔP: tick size
- q: unit volume
- $P = \frac{P^A + P^B}{2}$: mid-price
- $A(p), B(p)$: cumulative number of sell (buy) orders up to price level p

Notations

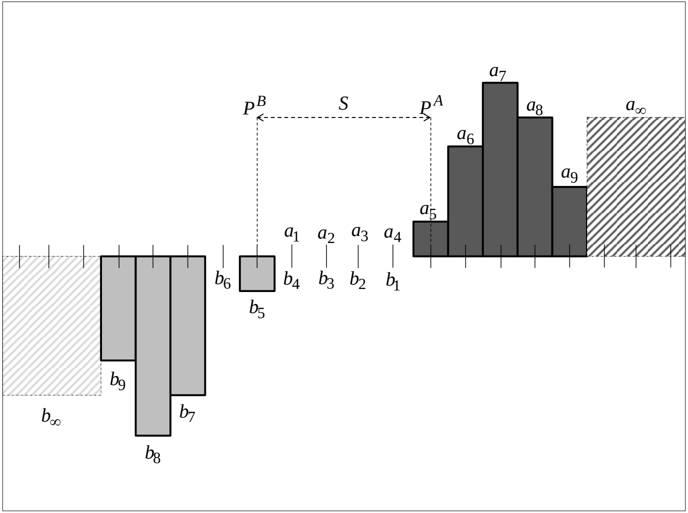

Figure: Order book notations

Plan

Zero-intelligence model

Inspired by Smith et al. (2003), Abergel and Jedidi (2013) analyzes an elementary LOB model where the events affecting the order book are described by **independent Poisson processes**:

- $M^{\pm}$: arrival of new **market order**, with intensity $\dfrac{\lambda^{M^{\pm}}}{q}$;
- $L_i^{\pm}$: arrival of a **limit order** at level i, with intensity $\dfrac{\lambda_i^{L^{\pm}} \Delta P}{q}$;
- $C_i^{\pm}$: **cancellation** of a limit order at level i, with intensity $\lambda_i^{C^+} \dfrac{a_i}{q}$ and $\lambda_i^{C^-} \dfrac{|b_i|}{q}$
  $\Rightarrow$ Cancellation rate is **proportional** to the outstanding quantity at each level.

Under these assumptions, $(\mathbf{X}(t))_{t \geq 0}$ is a **Markov process** with state space $\mathcal{S} \subset \mathbb{Z}^{2K}$.

Order book dynamics

$$
\begin{cases}
da_i(t) = -\left(q - \sum_{k=1}^{i-1} a_k\right)_+ dM^+(t) + qdL_i^+(t) - qdC_i^+(t) \\
\qquad + (J^{M^-}(\mathbf{a}) - \mathbf{a})_i dM^-(t) + \sum_{i=1}^{K} (J^{L_i^-}(\mathbf{a}) - \mathbf{a})_i dL_i^-(t) \\
\qquad + \sum_{i=1}^{K} (J^{C_i^-}(\mathbf{a}) - \mathbf{a})_i dC_i^-(t), \\
db_i(t) = \text{similar expression},
\end{cases}
$$

where $J^{M^\pm}$, $J^{L_i^\pm}$, and $J^{C_i^\pm}$ are shift operators corresponding to the effect of order arrivals on the reference frame

The shift operator

For instance, the shift operator corresponding to the arrival of a sell market order is

$$J^{M^-}(\mathbf{a}) = \left(\underbrace{0, 0, \ldots, 0}_{\text{k times}}, a_1, a_2, \ldots, a_{K-k} \right),$$

with

$$k := \inf\{p : \sum_{j=1}^{p} |b_j| > q\} - \inf\{p : |b_p| > 0\}.$$

Infinitesimal generator

$$
\begin{aligned}
\mathcal{L}f(\mathbf{a};\mathbf{b}) = {} & \lambda^{M^+}\left(f\left([a_i-(q-A(i-1))_+]_+;J^{M^+}(\mathbf{b})\right)-f\right) \\
& + \sum_{i=1}^{K}\lambda_i^{L^+}\left(f\left(a_i+q;J^{L_i^+}(\mathbf{b})\right)-f\right) \\
& + \sum_{i=1}^{K}\lambda_i^{C^+}a_i\left(f\left(a_i-q;J^{C_i^+}(\mathbf{b})\right)-f\right) \\
& + \lambda^{M^-}\left(f\left(J^{M^-}(\mathbf{a});[b_i+(q-B(i-1))_+]_-\right)-f\right) \\
& + \sum_{i=1}^{K}\lambda_i^{L^-}\left(f\left(J^{L_i^-}(\mathbf{a});b_i-q\right)-f\right) \\
& + \sum_{i=1}^{K}\lambda_i^{C^-}|b_i|\left(f\left(J^{C_i^-}(\mathbf{a});b_i+q\right)-f\right)
\end{aligned}
$$

Stability of the order book

Stationary order book distribution

Abergel and Jedidi (2013) If $\underline{\lambda_C} = \min_{1 \le i \le N}\{\lambda_i^{C^\pm}\} > 0$, then $(\mathbf{X}(t))_{t\ge0} = (\mathbf{a}(t); \mathbf{b}(t))_{t\ge0}$ is an ergodic Markov process. In particular $(\mathbf{X}(t))$ has a unique stationary distribution Π. Moreover, the rate of convergence of the order book to its stationary state is exponential.

The proof relies on the use of a Lyapunov function

The proportional cancellation rate helps a lot... and so do the boundary conditions!

A general approach to study price asymptotics

Combining ergodic theory and martingale convergence

The ergodicity of the order book allows for a direct study of the asymptotic behaviour of the price, based on Foster-Lyapunov-type criteria Meyn and Tweedie (1993)Glynn and Meyn (1996) and the convergence of martingales Ethier and Kurtz (2005):

- the evolution of the price is $dP_t = \sum_{i=1}^{K'} F_i(X_t)dN_t^i$;
- the rescaled, centered price is $\tilde{P}_t^n \equiv \frac{P_{nt} - \int_0^{nt} \sum_{i=1}^{K'} F_i(X_s)\lambda_i ds}{\sqrt{n}}$
- its predictable quadratic variation is $< \tilde{P}^n, \tilde{P}^n >_t = \frac{\int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{n}$
- ergodicity ensures the convergence of $\frac{\int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{nt}$ as $n \to \infty$

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This approach is easily extended to state-dependent intensities

A similar approach - in a somewhat different modelling context - is used in recent papers by Horst and co-authors Horst and Paulsen (2015).

A general approach made precise I

Write as above

$$P_t = P_0 + \sum_i \int_0^t F_i(X_s) dN_s^i$$

and its compensator

$$Q_t = \sum_i \int_0^t \lambda^i F_i(X_s) ds.$$

Define

$$h = \sum_i \lambda_i F_i(X)$$

and let

$$\alpha = \overset{a.s.}{\lim_{t \to +\infty}} \frac{1}{t} \sum_i \int_0^t \lambda^i (F_i(X_s)) ds = \int h(X) \Pi(dX).$$

Finally, introduce the solution g to the Poisson equation

$$\mathcal{L}g = h - \alpha$$

A general approach made precise II

and the associated martingale

$$Z_t = g(X_t) - g(X_0) - \int_0^t \mathcal{L}g(X_s)ds \equiv g(X_t) - g(X_0) - Q_t + \alpha t.$$

Then, the deterministically centered, rescaled price

$$\bar{P}_t^n \equiv \frac{P_{nt} - \alpha nt}{\sqrt{n}}$$

converges in distribution to a Wiener process $\bar{\sigma}W$. The asymptotic volatility $\bar{\sigma}$ satisfies the identity

$$\bar{\sigma}^2 = \lim_{t \to +\infty} \frac{1}{t} \sum_i \int_0^t \lambda^i((F_i - \Delta^i(g))(X_s))^2 ds \equiv \sum_i \int \lambda_i((F_i - \Delta^i(g))(X))^2 \Pi(dx).$$

where $\Delta^i(g)(X)$ denotes the jump of the process $g(X)$ when the process N^i jumps and the limit order book is in the state X.

Plan

Hawkes processes

Zero-intelligence models fail to capture the dependencies between various types of orders:

- clustering of market orders;
- interplay between liquidity taking and providing;
- leverage effect,

as has been demonstrated in many contributions, see e.g. Muni Toke (2011)Muni Toke and Pomponio (2012).

Hawkes processes

Hawkes processes provide an *ad hoc* tool to describe the mutual excitations of the arrivals of different types of orders.
In D dimensions, the process N_s^j has a stochastic intensity λ_t^j such that

$$\lambda_t^j = \lambda_0^j + \sum_{p=1}^{D} \int_0^t \phi_{jp}(t-s)dN_s^p. \tag{2.1}$$

A simplifying choice is the exponential kernel

$$\phi_{jp}(s) = \alpha_{jp} \exp(-\beta_{jp} s) \tag{2.2}$$

leading to markovian processes. A classical result states that the process is stationary *iff* the spectral radius of the matrix

$$[\frac{\alpha_{jp}}{\beta_{jp}}] \tag{2.3}$$

is < 1, see Brémaud and Massoulié (1996).

As an example, Muni Toke (2011) study variations of the model below

A better order book model

$$
\begin{aligned}
\lambda^M(t) &= \lambda_0^M + \int_0^t \alpha_{MM} e^{-\beta_{MM}(t-s)} dN^M(s), \\
\lambda^L(t) &= \lambda_0^L + \int_0^t \alpha_{LL} e^{-\beta_{LL}(t-s)} dN^L(s) \\
&+ \int_0^t \alpha_{ML} e^{-\beta_{ML}(t-s)} dN^M(s).
\end{aligned}
$$

- MM and LL effect for clustering of orders
- ML effect as observed on data
- No global LM effect observed on data

Infinitesimal generator

$$
\begin{aligned}
\mathcal{L}F(\vec{a};\vec{b};\vec{\mu}) = {} & \lambda^{M^+}\left(F\left([a_i-(q-A(i-1))_+]_+;J^{M^+}(\vec{b});\vec{\mu}+\Delta^{M^+}(\vec{\mu})\right)-F\right) \\
& + \sum_{i=1}^{K}\lambda_i^{L^+}\left(F\left(a_i+q;J^{L_i^+}(\vec{b});\vec{\mu}+\Delta^{L_i^+}(\vec{\mu})\right)-F\right) \\
& + \sum_{i=1}^{K}\lambda_i^{C^+}a_i\left(F\left(a_i-q;J^{C_i^+}(\vec{b})\right)-F\right) \\
& + \lambda^{M^-}\left(F\left(J^{M^-}(\vec{a});[b_i+(q-B(i-1))_+]_-;\vec{\mu}+\Delta^{M^-}(\vec{\mu})\right)-F\right) \\
& + \sum_{i=1}^{K}\lambda_i^{L^-}\left(F\left(J^{L_i^-}(\vec{a});b_i-q;\vec{\mu}+\Delta^{L_i^-}(\vec{\mu})\right)-F\right) \\
& + \sum_{i=1}^{K}\lambda_i^{C^-}|b_i|\left(f\left(J^{C_i^-}(\vec{a});b_i+q\right)-f\right) \\
& - \sum_{i,j=1}^{D}\beta_{ij}\mu_{ij}\frac{\partial F}{\partial \mu_{ij}}. \qquad (2.4)
\end{aligned}
$$

Large-time behaviour

Results similar to those obtained in the zero-intelligence case hold Abergel and Jedidi (2015)

Large-time behaviour for Hawkes-process driven LOB

- Under the usual stationarity conditions for the intensities, there exists a Lyapunov function $V = \sum |a_i| + \sum |b_i| + \sum U_k \lambda_k$ and the LOB converges exponentially to its stationary distribution Π
- The rescaled, (deterministically) centered price converges to a Wiener process

The proofs rely on the same decomposition as in the Poisson arrival case, thanks to the fact that the proportional cancellation rate remains bounded away from zero
Some extra care is required to prove that the solution to the Poisson equation is in $\mathbf{L}^2(\Pi(dx))$

Plan

Hawkes model

In Muni Toke (2011), the flow of limit and market orders are modelled by Hawkes processes N^L and N^M with intensities λ and μ defined as follows:

$$\begin{cases} \mu^M(t) = \mu_0^M + \int_0^t \alpha_{MM} e^{-\beta_{MM}(t-s)} dN_s^M \quad , \\ \lambda^L(t) = \lambda_0^L + \int_0^t \alpha_{LM} e^{-\beta_{LM}(t-s)} dN_s^M + \int_0^t \alpha_{LL} e^{-\beta_{LL}(t-s)} dN_s^L \end{cases}$$

Numerical results on the order book: fitting

Fitting and simulation

Model	μ_0	α_{MM}	β_{MM}	λ_0	α_{LM}	β_{LM}	α_{LL}	β_{LL}
HP	0.22	-	-	1.69	-	-	-	-
LM	0.22	-	-	0.79	5.8	1.8	-	-
MM	0.09	1.7	6.0	1.69	-	-	-	-
MM LL	0.09	1.7	6.0	0.60	-	-	1.7	6.0
MM LM	0.12	1.7	6.0	0.82	5.8	1.8	-	-
MM LL LM	0.12	1.7	5.8	0.02	5.8	1.8	1.7	6.0

Common parameters: $m_1^P = 2.7, \nu_1^P = 2.0, s_1^P = 0.9$
$_1^V = 275, m_2^V = 380$
$\lambda^C = 1.35, \delta = 0.015$

Table: Estimated values of parameters used for simulations.

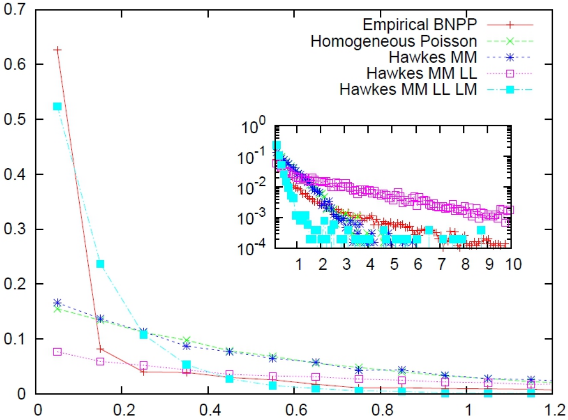

Figure: Empirical density function of the distribution of the interval times between a market order and the following limit order for three simulations, namely HP, MM+LL, MM+LL+LM, compared to empirical measures. In inset, same data using a semi-log scale

Impact on the bid-ask spread

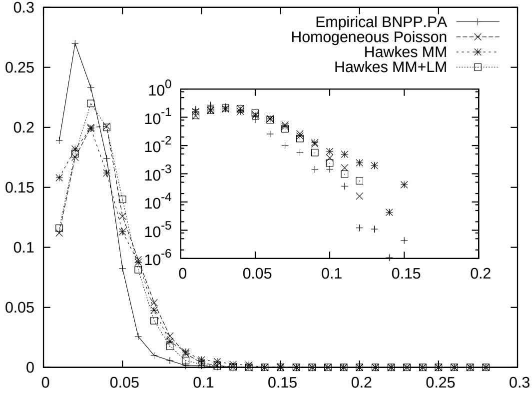

Figure: Empirical density function of the distribution of the bid-ask spread for three simulations, namely HP, MM, MM+LM, compared to empirical measures. In inset, same data using a semi-log scale. X-axis is scaled in euro (1 tick is 0.01 euro).

References I

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