Carnets d’ordres pilotés par des processus de Hawkes

workshop sur les Mathématiques des marchés financiers en haute fréquence

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fiquant.mas.ecp.fr/limit-order-books

14 Avril 2015
1 Stylized facts of limit order books
   - A word on data
   - Arrival times of orders
   - Cancellation of orders
   - Intraday seasonality
   - Competitive liquidity

2 Mathematical modelling of limit order books
   - Zero-intelligence models
     - Stability and long-time dynamics
     - Large-scale limit of the price process
   - Modelling dependencies
     - Hawkes processes
     - Application to order book modelling
     - Stability and long-time dynamics

3 Simulation with Hawkes processes
What is a limit order book?

- The **limit order book** is the list, at a given time, of all buy and sell limit orders, with their corresponding prices and volumes.
- The order book evolves over time according to the arrival of new orders.
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The order book evolves over time according to the arrival of new orders.

3 main types of orders:

- **limit order**: specify a price at which one is willing to buy (sell) a certain number of shares.
- **market order**: immediately buy (sell) a certain number of shares at the best available opposite quote(s).
- **cancellation order**: cancel an existing limit order.
What is a limit order book?

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Price dynamics

The price dynamics becomes a by-product of the order book dynamics.
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3. Simulation with Hawkes processes
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3 Simulation with Hawkes processes
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**Table:** Tick by tick data file sample
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**Table:** Trades data file sample.

These data are made available (for free) to the scientific community [http://fiquant.mas.ecp.fr/liquidity-watch](http://fiquant.mas.ecp.fr/liquidity-watch)
For each stock and each trading day:

1. Parse the tick by tick data file to compute order book state variations;
2. Parse the trades file and for each trade:
   1. Compare the trade price and volume to likely market orders whose timestamps are in $[t^{Tr} - \Delta t, t^{Tr} + \Delta t]$, where $t^{Tr}$ is the trade timestamp and $\Delta t$ is a predefined time window;
   2. Match the trade to the first likely market order with the same price and volume and label the corresponding event as a market order;
   3. Remaining negative variations are labeled as cancellations.

Doing so, we have an average matching rate of around 85% for CAC 40 stocks. As a byproduct, one gets the sign of each matched trade.
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3 Simulation with Hawkes processes
Arrival time of orders

Empirical density vs Interarrival time for different distributions: BNPP.PA, Lognormal, Exponential, Weibull.

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3 Simulation with Hawkes processes
**Figure:** Distribution of estimated lifetime of cancelled limit orders.
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3. Simulation with Hawkes processes
Figure: Normalized average number of market orders in a 5-minute interval.
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3. **Simulation with Hawkes processes**
Competitive liquidity

Muni Toke (2011)

Figure: Evidence of liquidity providing

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Competitive liquidity


Figure: Evidence of liquidity taking
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3. Simulation with Hawkes processes
A set of reasonable assumptions:
- the limit order book is described as a point process;
- several types of events can happen;
- two events cannot occur simultaneously (simple point process).

The main questions of interest are
- the stationarity and ergodicity of the order book;
- the dynamics of the induced price...
A set of reasonable assumptions:

- the limit order book is described as a point process;
- several types of events can happen;
- two events cannot occur simultaneously (simple point process).

The main questions of interest are

- the stationarity and ergodicity of the order book;
- the dynamics of the induced price...
- ... particularly, its behaviour at larger time scales.

⇒ Linking microstructure modelling and continuous-time finance
Some notations

- The order book is represented by a finite-size vector of quantities

  \[ \mathbf{X}(t) := (\mathbf{a}(t); \mathbf{b}(t)) := (a_1(t), \ldots, a_K(t); b_1(t), \ldots, b_K(t)); \]

- \( \mathbf{a}(t) \): ask side of the order book
- \( \mathbf{b}(t) \): bid side of the order book
- \( \Delta P \): tick size
- \( q \): unit volume
- \( P = \frac{P^A + P^B}{2} \): mid-price
- \( A(p), B(p) \): cumulative number of sell (buy) orders up to price level \( p \)
Figure: Order book notations
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3. Simulation with Hawkes processes
Inspired by Smith et al. (2003), Abergel and Jedidi (2013) analyzes an elementary LOB model where the events affecting the order book are described by independent Poisson processes:

- $M^\pm$: arrival of new market order, with intensity $\lambda^M^\pm$;

- $L^\pm_i$: arrival of a limit order at level $i$, with intensity $\lambda^L^\pm_i \Delta P$;

- $C^\pm_i$: cancellation of a limit order at level $i$, with intensity $\lambda^C^\pm_i$ and $\lambda^C^-_i \frac{|b_i|}{q}$

$\Rightarrow$ Cancellation rate is proportional to the outstanding quantity at each level.

Under these assumptions, $(X(t))_{t \geq 0}$ is a Markov process with state space $S \subset \mathbb{Z}^{2K}$. 
Order book dynamics

\[
\begin{align*}
\left. \begin{array}{l}
da_i(t) = & - \left( q - \sum_{k=1}^{i-1} a_k \right) dM^+(t) + qdL_i^+(t) - qdC_i^+(t) \\
& + (J^M^- (a) - a)_i dM^-(t) + \sum_{i=1}^{K} (J^{L^-}_i (a) - a)_i dL_i^-(t) \\
& + \sum_{i=1}^{K} (J^{C^-}_i (a) - a)_i dC_i^-(t),
\end{array} \right\}
db_i(t) = \text{similar expression},
\end{align*}
\]

where \( J^{M^\pm}_i, J^{L^\pm}_i, \) and \( J^{C^\pm}_i \) are shift operators corresponding to the effect of order arrivals on the reference frame.
The shift operator

For instance, the shift operator corresponding to the arrival of a sell market order is

\[ J^M_-(a) = \left(0, 0, \ldots, 0, a_1, a_2, \ldots, a_{K-k}\right), \]

with

\[ k := \inf\{p : \sum_{j=1}^{p} |b_j| > q\} - \inf\{p : |b_p| > 0\}. \]
\[ \mathcal{L} f (a; b) = \lambda^M f (a_i - (q - A(i - 1))_+; J^M(b)) - f \]
\[ + \sum_{i=1}^{K} \lambda_i^{L+} f (a_i + q; J^L_i(b)) - f \]
\[ + \sum_{i=1}^{K} \lambda_i^{C+} a_i f (a_i - q; J^C_i(b)) - f \]
\[ + \lambda^M f (J^M(a); [b_i + (q - B(i - 1))_+] - f) \]
\[ + \sum_{i=1}^{K} \lambda_i^{L-} f (J^L_i(a); b_i - q) - f \]
\[ + \sum_{i=1}^{K} \lambda_i^{C-} |b_i| f (J^C_i(a); b_i + q) - f \]
Stability of the order book

Stationary order book distribution

Abergel and Jedidi (2013) If \( \lambda_C = \min_{1 \leq i \leq N} \{ \lambda_i^{C_{\pm}} \} > 0 \), then 
\( (X(t))_{t \geq 0} = (a(t); b(t))_{t \geq 0} \) is an ergodic Markov process. In particular 
\( (X(t)) \) has a unique stationary distribution \( \Pi \). Moreover, the rate of 
convergence of the order book to its stationary state is exponential.

The proof relies on the use of a Lyapunov function
The proportional cancellation rate helps a lot... and so do the boundary 
conditions!
Combining ergodic theory and martingale convergence


- The evolution of the price is
  \[ dP_t = \sum_{i=1}^{K'} F_i(X_t) dN^i_t; \]

- The rescaled, centered price is
  \[ \tilde{P}_t^n = \frac{P_n t - \int_0^n \sum_{i=1}^{K'} F_i(X_s) \lambda_i ds}{\sqrt{n}} \]

- Its predictable quadratic variation is
  \[ < \tilde{P}_t^n, \tilde{P}_t^n >_t = \frac{\int_0^n \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{n} \]

- Ergodicity ensures the convergence of
  \[ \frac{\int_0^n \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{nt} \] as \( n \to \infty \)
A general approach to study price asymptotics

Combining ergodic theory and martingale convergence


- the evolution of the price is \( dP_t = \sum_{i=1}^{K'} F_i(X_t) dN^i_t \);
- the rescaled, centered price is \( \tilde{P}^n_t = \frac{P_{nt} - \int_0^{nt} \sum_{i=1}^{K'} F_i(X_s) \lambda_i ds}{\sqrt{n}} \);
- its predictable quadratic variation is \( <\tilde{P}^n, \tilde{P}^n>_t = \frac{\int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{n} \);
- ergodicity ensures the convergence of \( \int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds / nt \) as \( n \to \infty \).

This approach is easily extended to state-dependent intensities. A similar approach - in a somewhat different modelling context - is used in recent papers by Horst and co-authors Horst and Paulsen (2015).
Write as above

\[ P_t = P_0 + \sum_i \int_0^t F_i(X_s) dN^i_s \]

and its compensator

\[ Q_t = \sum_i \int_0^t \lambda_i F_i(X_s) ds. \]

Define

\[ h = \sum_i \lambda_i F_i(X) \]

and let

\[ \alpha = \lim_{t \to +\infty} \frac{1}{t} \sum_i \int_0^t \lambda_i(F_i(X_s)) ds = \int h(X) \Pi(dX). \]

Finally, introduce the solution \( g \) to the Poisson equation

\[ \mathcal{L}g = h - \alpha \]
and the associated martingale

\[ Z_t = g(X_t) - g(X_0) - \int_0^t \mathcal{L}g(X_s)ds \equiv g(X_t) - g(X_0) - Q_t + \alpha t. \]

Then, the deterministically centered, rescaled price

\[ \bar{P}_n^t \equiv \frac{P_{nt} - \alpha nt}{\sqrt{n}} \]

converges in distribution to a Wiener process \( \bar{\sigma}W \). The asymptotic volatility \( \bar{\sigma} \) satisfies the identity

\[ \bar{\sigma}^2 = \lim_{t \to +\infty} \frac{1}{t} \sum_i \int_0^t \lambda_i((F_i - \Delta^i(g))(X_s))^2 ds \equiv \sum_i \int \lambda_i((F_i - \Delta^i(g))(X))^2 \Pi(dx). \]

where \( \Delta^i(g)(X) \) denotes the jump of the process \( g(X) \) when the process \( N^i \) jumps and the limit order book is in the state \( X \).
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3. **Simulation with Hawkes processes**
Hawkes processes

Zero-intelligence models fail to capture the dependencies between various types of orders:
- clustering of market orders;
- interplay between liquidity taking and providing;
- leverage effect,

as has been demonstrated in many contributions, see e.g. Muni Toke (2011) Muni Toke and Pomponio (2012).
Hawkes processes provide an *ad hoc* tool to describe the mutual excitations of the arrivals of different types of orders. In $D$ dimensions, the process $N^j_s$ has a stochastic intensity $\lambda^j_t$ such that

$$\lambda^j_t = \lambda^j_0 + \sum_{p=1}^{D} \int_0^t \phi_{jp}(t-s) dN^p_s.$$  \hfill (2.1)

A simplifying choice is the exponential kernel

$$\phi_{jp}(s) = \alpha_{jp} \exp(-\beta_{jp} s)$$  \hfill (2.2)

leading to markovian processes. A classical result states that the process is stationary *iff* the spectral radius of the matrix

$$\begin{bmatrix} \alpha_{jp} \\ -\beta_{jp} \end{bmatrix}$$  \hfill (2.3)

is $< 1$, see Brémaud and Massoulié (1996).
Introducing dependence in the order flow

As an example, Muni Toke (2011) study variations of the model below

A better order book model

\[ \lambda^M(t) = \lambda_0^M + \int_0^t \alpha_{MM} e^{-\beta_{MM}(t-s)} dN^M(s), \]
\[ \lambda^L(t) = \lambda_0^L + \int_0^t \alpha_{LL} e^{-\beta_{LL}(t-s)} dN^L(s) + \int_0^t \alpha_{ML} e^{-\beta_{ML}(t-s)} dN^M(s). \]

- MM and LL effect for clustering of orders
- ML effect as observed on data
- No global LM effect observed on data
\[ \mathcal{L}F(\vec{a}; \vec{b}; \vec{\mu}) = \lambda^{M+}(F([a_i - (q - A(i - 1))_+]_+; J^{M+}(\vec{b}); \vec{\mu} + \Delta^{M+}(\vec{\mu})) - F) \]

\[
+ \sum_{i=1}^{K} \lambda^{L+}_i (F(a_i + q; J^{L+}_i(\vec{b}); \vec{\mu} + \Delta^{L+}_i(\vec{\mu})) - F) \\
+ \sum_{i=1}^{K} \lambda^{C+}_i a_i(F(a_i - q; J^{C+}_i(\vec{b})) - F) \\
+ \lambda^{M-}(F(J^{M-}(\vec{a}); [b_i + (q - B(i - 1))_+]_-; \vec{\mu} + \Delta^{M-}(\vec{\mu})) - F) \\
+ \sum_{i=1}^{K} \lambda^{L-}_i (F(J^{L-}_i(\vec{a}); b_i - q; \vec{\mu} + \Delta^{L-}_i(\vec{\mu})) - F) \\
+ \sum_{i=1}^{K} \lambda^{C-}_i |b_i|(f(J^{C-}_i(\vec{a}); b_i + q) - f) \\
- \sum_{i,j=1}^{D} \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}}. \]

(2.4)
Large-time behaviour

Results similar to those obtained in the zero-intelligence case hold
Abergel and Jedidi (2015)

Large-time behaviour for Hawkes-process driven LOB

- Under the usual stationarity conditions for the intensities, there exists a Lyapunov function
  \[ V = \sum |a_i| + \sum |b_i| + \sum U_k \lambda_k \]
  and the LOB converges exponentially to its stationary distribution \( \Pi \)
- The rescaled, (deterministically) centered price converges to a Wiener process

The proofs rely on the same decomposition as in the Poisson arrival case, thanks to the fact that the proportional cancellation rate remains bounded away from zero
Some extra care is required to prove that the solution to the Poisson equation is in \( L^2(\Pi(dx)) \)
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3 Simulation with Hawkes processes
In Muni Toke (2011), the flow of limit and market orders are modelled by Hawkes processes $N^L$ and $N^M$ with intensities $\lambda$ and $\mu$ defined as follows:

\[
\begin{align*}
\mu^M(t) &= \mu^M_0 + \int_0^t \alpha_{MM} e^{-\beta_{MM}(t-s)} dN^M_s , \\
\lambda^L(t) &= \lambda^L_0 + \int_0^t \alpha_{LM} e^{-\beta_{LM}(t-s)} dN^M_s + \int_0^t \alpha_{LL} e^{-\beta_{LL}(t-s)} dN^L_s
\end{align*}
\]
Numerical results on the order book: fitting

Fitting and simulation

<table>
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<th>( \alpha_{LL} )</th>
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<td>5.8</td>
<td>1.8</td>
<td>1.7</td>
<td>6.0</td>
</tr>
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</table>

Common parameters: 

\[
\begin{align*}
m_1^P &= 2.7, \quad \nu_1^P = 2.0, \quad s_1^P = 0.9 \\
\nu_1 &= 275, \quad m_2^V = 380 \\
\lambda^C &= 1.35, \quad \delta = 0.015
\end{align*}
\]

**Table:** Estimated values of parameters used for simulations.
Impact on arrival times

**Figure:** Empirical density function of the distribution of the interval times between a market order and the following limit order for three simulations, namely HP, MM+LL, MM+LL+LM, compared to empirical measures. In inset, same data using a semi-log scale.
Impact on the bid-ask spread

Figure: Empirical density function of the distribution of the bid-ask spread for three simulations, namely HP, MM, MM+LM, compared to empirical measures. In inset, same data using a semi-log scale. X-axis is scaled in euro (1 tick is 0.01 euro).


