Risk-Consistent Conditional Systemic Risk Measures

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Workshop on Systemic Risk and Financial Networks
IPAM, UCLA

March 24, 2015

Motivation

- Let

$$X = (X_1, \dots, X_d) \in L_d^\infty(\mathcal{F})$$

represent (monetary) risk factors associated to a system of d interacting financial institutions.

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represent (monetary) risk factors associated to a system of d interacting financial institutions.

- *Traditional approach to risk management*: Measuring stand-alone risk of each institution

$$\eta(X_i)$$

for some univariate risk measure $\eta : L^\infty(\mathcal{F}) \to \mathbb{R}$.

Motivation

Axiomatic characterization of (univariate) risk measures:

A map $\eta : L^\infty(\mathcal{F}) \to \mathbb{R}$ is a monetary risk measure, if it is

- *Monotone:* $X_1 \geq X_2 \Rightarrow \eta(X_1) \leq \eta(X_2)$.
- *Cash-invariant:*] $\eta(X + m) = \eta(X) - m$ for all $m \in \mathbb{R}$.
  (*Constant-on-constants:* $\eta(m) = -m$ for all $m \in \mathbb{R}$.)

A monetary risk measure is called convex, if it is

- *Convex:* $\eta(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\eta(X_1) + (1-\lambda)\eta(X_2)$ for $\lambda \in [0,1]$.
  (*Quasi-convex:* $\eta(\lambda X_1 + (1-\lambda)X_2) \leq \max\{\eta(X_1), \eta(X_2)\}$.)

A convex risk measure is called coherent, if it is

- *Pos. homogeneity:* $\eta(\lambda X) = \lambda\eta(X)$ for $\lambda \geq 0$.

Motivation

However,

- Financial crisis: traditional approach to regulation and risk management insufficiently captures

  Systemic risk: risk that in case of an adverse (local) shock substantial parts of the system default.

- *Question*: Given the system $X = (X_1, \dots, X_d)$, what is an appropriate risk measure

$$\rho : L_d^\infty(\mathcal{F}) \to \mathbb{R}$$

  of systemic risk?

Motivation

- Most proposals of systemic risk measures in the post-crisis literature are of the form

$$\rho(X) := \eta\left(\Lambda(X)\right)$$

for some univariate risk measure

$$\eta : L^\infty(\mathcal{F}) \to \mathbb{R}$$

and some aggregation function

$$\Lambda : \mathbb{R}^d \to \mathbb{R}.$$

Motivation

Some examples:

- Appropriate aggregation to reflect systemic risk?

Systemic Expected Shortfall

[Acharya et al., 2011]

$$\rho(X) := ES_q\left(\sum_{i=1}^{d} X_i\right)$$

where ES_q is the univariate Expected Shortfall at level $q \in [0, 1]$.

Motivation

Deposit Insurance [Lehar, 2005], [Huang, Zhou & Zhu, 2011]

$$\rho(X) := E\left(\sum_{i=1}^{d} -X_i^-\right)$$

SystRisk [Brunnermeier & Cheridito, 2013]

$$\rho(X) := \eta_{SystRisk}\left(\sum_{i=1}^{d} -\alpha_i X_i^- + \beta_i (X_i^+ - v_i)\right)$$

where $\eta_{SystRisk}$ is some utility-based univariate risk measure.

Motivation

Contagion model

- Π_{ji} denotes the proportion of the total liabilities L_j of bank j which it owes to bank i.
- X_i represents the capital endowment of bank i.
- If a bank i defaults, i.e. $X_i < 0$, this generates further losses in the system by contagion.
- Systemic risk concerns the total loss in the network generated by a profit/loss profile $X = (X_1, \ldots, X_d)$.

Motivation

[Chen, Iyengar & Moallemi, 2013], [Eisenberg, Noe, 2001]

Total loss in the system induced by some initial loss profile $x \in \mathbb{R}^d$:

$$\Lambda(x) := \min_{y,b \in \mathbb{R}^d_-} \quad \sum_{i=1}^{d} -y_i - \gamma b_i$$

$$\text{subject to} \quad y_i = x_i - b_i + \sum_{j=1}^{d} \Pi_{ji} y_j \quad \forall i$$

- Bank i decreases its liabilities to the remaining banks by y_i.
- This decreases the equity values of its creditors, which can result in a further default.
- A regulator injects b_i (weighted by $\gamma > 1$) into bank i.

Motivation

- Various extensions of the Eisenberg & Noe framework to include further channels of contagion, see e.g.
  - [Amini, Filipovic & Minca, 2013]
  - [Awiszus & Weber, 2015]
  - [Cifuentes, Ferrucci & Shin, 2005]
  - [Gai & Kapadia, 2010]
  - [Rogers & Veraart, 2013]

Motivation

- Conditional risk measuring interesting in systemic risk
  $\rightarrow$ identification of systemic relevant structures

CoVar and **CoES** [Adrian, Brunnermeier, 2011]

$$\rho(X) := VaR_q\left(\sum_{i=1}^{d} X_i \mid X_j \leq -VaR_q(X_j)\right)$$

[Acharya et al., 2011]

$$ES_q\left(X_j \mid \sum_{i=1}^{d} X_i \leq -VaR_q(\sum_{i=1}^{d} X_i)\right)$$

Motivation

$\rightarrow$ conditional risk measure

$$\eta : L^{\infty}(\mathcal{F}) \rightarrow L^{\infty}(\mathcal{G})$$

for some sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$

- Broad literature on conditional risk measures in a dynamic setup (e.g. [Detlefsen & Scandolo, 2005], [Föllmer & Schied, 2011], [Frittelli & Maggis, 2011], [Tutsch, 2007]).
- In the context of systemic risk, see also [Föllmer, 2014] and [Föllmer & Klüppelberg, 2014]:

Motivation

But also conditional aggregation $\Lambda(x,\omega)$ appears naturally; f.ex.:

- The way of aggregation might depend on macroeconomic factors:
  - Countercyclical regulation: more severe when economy is in good shape than in times of financial distress
  - Stochastic discounting: $\Lambda(x,\omega) = \widetilde{\Lambda}(x)D(\omega)$, where D is some stochastic discount factor.
- Liability matrix $\Pi = \Pi(\omega)$ in contagion model above might be stochastic (derivative exposures between banks)

Motivation

Objective of this presentation: Structural analysis of conditional systemic risk measures $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \to L^\infty(\mathcal{G})$, $\mathcal{G} \subset \mathcal{F}$, of the form

$$\rho_{\mathcal{G}}(X) := \eta_{\mathcal{G}}\left(\Lambda_{\mathcal{G}}(X)\right)$$

where $\eta_{\mathcal{G}}$ is some univariate conditional risk measure and $\Lambda_{\mathcal{G}}$ some conditional aggregation function.

Motivation

Structure of remaining presentation:

1. *Axiomatic characterization of conditional systemic risk measures of this type*
2. *Examples and simulation study*
3. *Strong consistency and risk-consistency*

AXIOMATIC CHARACTERIZATION OF RISK-CONSISTENT CONDITIONAL SYSTEMIC RISK MEASURES

Risk-Consistent Conditional Systemic Risk Measures

Aim: Axiomatic characterization of conditional systemic risk measures of the form $\eta\left(\Lambda(X)\right)$ in terms of *properties on constants* and *risk-consistent properties*.

- Risk-consistent properties ensure a consistency between local - that is ω-wise - risk assessment and the measured global risk.

For deterministic risk measures [Chen, Iyengar & Moallemi, 2013], f. ex.:

- *Risk-monotonicity*: if for given risk vectors X and Y we have $\rho(X(\omega)) \geq \rho(Y(\omega))$ in a.a. states ω, then $\rho(X) \geq \rho(Y)$.

Risk-Consistent Conditional Systemic Risk Measures

In the deterministic case:

- [Chen, Iyengar & Moallemi, 2013] on finite probability space
- [Kromer, Overbeck & Zilch, 2013] general probability space

Our contribution:

- Conditional setting
- More comprehensive structural analysis
- More flexible aggregation and axiomatic setting

Risk-Consistent Conditional Systemic Risk Measures

Notation: Given $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$:

- A realization of a function
$$\rho_{\mathcal{G}} : L_d^{\infty}(\mathcal{F}) \to L^{\infty}(\mathcal{G})$$
is a function
$$\rho_{\mathcal{G}}(\cdot,\cdot) : L_d^{\infty}(\mathcal{F}) \times \Omega \to \mathbb{R}$$
such that $\rho_{\mathcal{G}}(X,\cdot) \in \rho_{\mathcal{G}}(X)$ for all $X \in L_d^{\infty}(\mathcal{F})$.

- A realization $\rho_{\mathcal{G}}(\cdot,\cdot)$ has *continuous path* if $\rho_{\mathcal{G}}(\cdot,\omega) : \mathbb{R}^d \to \mathbb{R}$ is continuous for all $\omega \in \Omega$.

- We denote the constant random vector $X_{\widehat{\omega}}$, $\widehat{\omega} \in \Omega$, by
$$X_{\widehat{\omega}}(\omega) := X(\widehat{\omega}) \; \forall \omega \in \Omega$$

Risk-Consistent Conditional Systemic Risk Measures

Definition: A function $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \to L^\infty(\mathcal{G})$ is called *risk-consistent conditional systemic risk measure* (CSRM), if it is

Monotone on constants: $x, y \in \mathbb{R}^d$ with $x \geq y \Rightarrow \rho_{\mathcal{G}}(x) \leq \rho_{\mathcal{G}}(y)$

and if there exists a realization $\rho_{\mathcal{G}}(\cdot, \cdot)$ with *continuous paths* such that $\rho_{\mathcal{G}}$ is

Risk-monotone: For all $X, Y \in L_d^\infty(\mathcal{F})$

$$\rho_{\mathcal{G}}(X_\omega, \omega) \geq \rho_{\mathcal{G}}(Y_\omega, \omega) \; a.s. \Rightarrow \rho_{\mathcal{G}}(X, \omega) \geq \rho_{\mathcal{G}}(Y, \omega) \; a.s.$$

(Risk-) regular: $\rho_{\mathcal{G}}(X, \omega) = \rho_{\mathcal{G}}(X_\omega, \omega)$ a.s. $\forall X \in L_d^\infty(\mathcal{G})$

Further risk-consistent properties: We say that $\rho_{\mathcal{G}}$ is

Risk-convex: If for $X, Y, Z \in L_d^\infty(\mathcal{F})$, $\alpha \in L^\infty(\mathcal{G})$ with $0 \leq \alpha \leq 1$

$$\rho_{\mathcal{G}}(Z_\omega, \omega) = \alpha(\omega)\rho_{\mathcal{G}}(X_\omega, \omega) + \big(1 - \alpha(\omega)\big)\rho_{\mathcal{G}}(Y_\omega, \omega) \text{ a.s.}$$

$$\Rightarrow \rho_{\mathcal{G}}(Z, \omega) \leq \alpha(\omega)\rho_{\mathcal{G}}(X, \omega) + (1 - \alpha(\omega))\rho_{\mathcal{G}}(Y, \omega) \text{ a.s.}$$

Risk-quasiconvex: If for $X, Y, Z \in L_d^\infty(\mathcal{F})$, $\alpha \in L^\infty(\mathcal{G})$, $0 \leq \alpha \leq 1$

$$\rho_{\mathcal{G}}(Z_\omega, \omega) = \alpha(\omega)\rho_{\mathcal{G}}(X_\omega, \omega) + \big(1 - \alpha(\omega)\big)\rho_{\mathcal{G}}(Y_\omega, \omega) \text{ a.s.}$$

$$\Rightarrow \rho_{\mathcal{G}}(Z, \omega) \leq \rho_{\mathcal{G}}(X, \omega) \vee \rho_{\mathcal{G}}(Y, \omega) \text{ a.s.}$$

Risk-pos.-homogeneous: If for $X, Y \in L_d^\infty(\mathcal{F})$, $0 \leq \alpha \in L^\infty(\mathcal{G})$

$$\rho_{\mathcal{G}}(Y_\omega, \omega) = \alpha(\omega)\rho_{\mathcal{G}}(X_\omega, \omega) \text{ a.s.}$$

$$\Rightarrow \rho_{\mathcal{G}}(Y, \omega) \leq \alpha(\omega)\rho_{\mathcal{G}}(X, \omega) \text{ a.s.}$$

Risk-Consistent Conditional Systemic Risk Measures

Further properties on constants: We say that $\rho_{\mathcal{G}}$ is

Convex on constants: If for all $x, y \in \mathbb{R}^d$ and $\lambda \in [0,1]$

$$\rho_{\mathcal{G}}\left(\lambda x + (1-\lambda)y\right) \leq \lambda \rho_{\mathcal{G}}(x) + (1-\lambda)\rho_{\mathcal{G}}(y)$$

Positively homogeneous on constants: If for all $x \in \mathbb{R}^d$ and $\lambda \geq 0$

$$\rho_{\mathcal{G}}(\lambda x) = \lambda \rho_{\mathcal{G}}(x)$$

Risk-Consistent Conditional Systemic Risk Measures

Proposition: Let $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \to L^\infty(\mathcal{G})$ be a function which has a realization with continuous paths. Further suppose that

$$\rho_{\mathcal{G}}(x) = \sum_{i=1}^{s} a_i(x)\mathbb{I}_{A_i},\ x \in \mathbb{R}^d,$$

where $a_i(x) \in \mathbb{R}$ and A_i are pairwise disjoint sets such that $\Omega = \bigcup_{i=1}^{s} A_i$ for $s \in \mathbb{N} \cup \{\infty\}$. Define

$$k : \Omega \to \mathbb{N};\ \omega \mapsto i \text{ such that } \omega \in A_i.$$

Then $\rho_{\mathcal{G}}$ is risk-monotone if and only if

$$\rho_{\mathcal{G}}(X_\omega)\mathbb{I}_{A_{k(\omega)}} \geq \rho_{\mathcal{G}}(Y_\omega)\mathbb{I}_{A_{k(\omega)}} \text{ for a.a. } \omega \Rightarrow\ \rho_{\mathcal{G}}(X) \geq \rho_{\mathcal{G}}(Y).$$

Also the remaining risk-consistent properties can be expressed in a similar way without requiring a particular realization of $\rho_{\mathcal{G}}$.

Risk-Consistent Conditional Systemic Risk Measures

Proposition: The following holds for a risk-consistent CSRM $\rho_{\mathcal{G}}$:

- If $\rho_{\mathcal{G}}$ is risk-monotone and monotone on constants, then $\rho_{\mathcal{G}}$ is monotone.
- If $\rho_{\mathcal{G}}$ is risk-quasiconvex and convex on constants, then $\rho_{\mathcal{G}}$ is quasiconvex.
- If $\rho_{\mathcal{G}}$ is risk-convex and convex on constants, then $\rho_{\mathcal{G}}$ is convex;
- $\rho_{\mathcal{G}}$ is risk positively homogeneous and positively homogeneous on constants iff $\rho_{\mathcal{G}}$ is positively homogeneous;

Risk-Consistent Conditional Systemic Risk Measures

Definition: A function $\Lambda_{\mathcal{G}} : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a *conditional aggregation function* (CAF), if

1. $\Lambda_{\mathcal{G}}(x, \cdot) \in \mathcal{L}^{\infty}(\mathcal{G})$ for all $x \in \mathbb{R}^d$.
2. $\Lambda_{\mathcal{G}}(\cdot, \omega)$ is *continuous* for all $\omega \in \Omega$.
3. $\Lambda_{\mathcal{G}}(\cdot, \omega)$ is *monotone* (increasing) for all $\omega \in \Omega$.

Furthermore, $\Lambda_{\mathcal{G}}$ is called *concave* (*positively homogeneous*) if $\Lambda_{\mathcal{G}}(\cdot, \omega)$ is *concave* (*positively homogeneous*) for all $\omega \in \Omega$.

Given a CAF $\Lambda_{\mathcal{G}}$, we extend the aggregation from deterministic to random vectors in the following way:

$$\begin{aligned} \widetilde{\Lambda}_{\mathcal{G}}: \quad & L_d^{\infty}(\mathcal{F}) \to L^{\infty}(\mathcal{F}) \\ & X(\omega) \ \mapsto \Lambda_{\mathcal{G}}\left(X(\omega), \omega\right) \end{aligned}$$

Notice that the aggregation of random vectors is ω-wise in the sense that given a certain state $\omega \in \Omega$, in that state we aggregate the sure payoff $X(\omega)$.

Definition: Let $F, G \in L^{\infty}(\mathcal{F})$. A function $\eta_{\mathcal{G}} : L^{\infty}(\mathcal{F}) \to L^{\infty}(\mathcal{G})$ is a *conditional base risk measure* (CBRM), if it is

Monotone: $F \geq G \Rightarrow \eta_{\mathcal{G}}(F) \leq \eta_{\mathcal{G}}(G)$.

Constant on $\mathcal{G}$-constants: $\eta_{\mathcal{G}}(\alpha) = -\alpha \quad \forall\, \alpha \in L^{\infty}(\mathcal{G})$.

Additional properties of CBRMs:

Convexity: For all $\alpha \in L^\infty(\mathcal{G})$ with $0 \le \alpha \le 1$

$$\eta_{\mathcal{G}}\left(\alpha F + (1-\alpha)G\right) \le \alpha\eta_{\mathcal{G}}(G) + (1-\alpha)\eta_{\mathcal{G}}(G)$$

Quasiconvexity: For all $\alpha \in L^\infty(\mathcal{G})$ with $0 \le \alpha \le 1$

$$\eta_{\mathcal{G}}\left(\alpha F + (1-\alpha)G\right) \le \eta_{\mathcal{G}}(F) \vee \eta_{\mathcal{G}}(G)$$

Positive homogeneity: For all $\alpha \in L^\infty(\mathcal{G})$ with $\alpha \ge 0$

$$\eta_{\mathcal{G}}(\alpha F) = \alpha\eta_{\mathcal{G}}(F)$$

Theorem: (*Decomposition of conditional systemic risk measures*)

A map $\rho_{\mathcal{G}} : L_d^{\infty}(\mathcal{F}) \to L^{\infty}(\mathcal{G})$ is a risk-consistent CSRM if and only if there exists a CBRM $\eta_{\mathcal{G}} : L^{\infty}(\mathcal{F}) \to L^{\infty}(\mathcal{G})$ and a CAF $\Lambda_{\mathcal{G}}$ such that

$$\rho_{\mathcal{G}}(X) = \eta_{\mathcal{G}}\left(\widetilde{\Lambda}_{\mathcal{G}}(X)\right) \quad \forall X \in L_d^{\infty}(\mathcal{F}),$$

where $\widetilde{\Lambda}_{\mathcal{G}}(X) := \Lambda_{\mathcal{G}}(X(\omega), \omega)$.

Theorem cont': Furthermore, the following equivalences hold:

- $\rho_{\mathcal{G}}$ is risk-convex iff $\eta_{\mathcal{G}}$ is convex;
- $\rho_{\mathcal{G}}$ is risk-quasiconvex iff $\eta_{\mathcal{G}}$ is quasiconvex;
- $\rho_{\mathcal{G}}$ is risk-positive homogeneous iff $\eta_{\mathcal{G}}$ is positive homogeneous;

and

- $\rho_{\mathcal{G}}$ is convex on constants iff $\Lambda_{\mathcal{G}}$ is concave;
- $\rho_{\mathcal{G}}$ is positive homogeneous on constants iff $\Lambda_{\mathcal{G}}$ is positive homogeneous.

EXAMPLES AND SIMULATION STUDY

Examples

CoVar (analogue *CoES*) [Adrian & Brunnermeier, 2011]:

- Let

$$\eta_{\mathcal{G}}(F) := \mathrm{VaR}_{\lambda}(F|\mathcal{G}) := -\operatorname*{essinf}_{G \in L^{\infty}(\mathcal{G})} \left\{ \mathbb{P}(F \leq G \mid \mathcal{G}) > \lambda \right\},$$

  where $F \in L^{\infty}(\mathcal{F})$, $\lambda \in L^{\infty}(\mathcal{G})$, and $0 < \lambda < 1$.

- For a fixed $j \in \{1, ..., d\}$ and $q \in (0, 1)$, define the event $A := \{X_j \leq -\mathrm{VaR}_q(X_j)\}$ and let $\mathcal{G} := \sigma(A)$.
- Define $\Lambda(x) := \sum x_i \,, x \in \mathbb{R}$.
- Then the CoVar is represented by $\eta_{\mathcal{G}}(\Lambda(X))$, which is a positively homogeneous CSRM.

Examples

Example: Extended contagion model

- Π_{ji} denotes the proportion of the total interbank liabilities L_j of bank j which it owes to bank i.
- X_i represents the capital endowment of bank i.
- If a bank i defaults, i.e. $X_i < 0$, this generates further losses in the system by contagion.
- Systemic risk concerns the total loss in the network generated by a profit/loss profile $X = (X_1, \ldots, X_d)$.

Examples

More realistic contagion aggregation:

- Total aggregated loss in the system for loss profile $x \in \mathbb{R}^d$:

$$\Lambda(x) := \min_{y,b\in\mathbb{R}^d} \sum_{i=1}^{d} \left(x_i + b_i + \left(\Pi^\top y\right)_i \right)^- + \gamma b_i$$

$$\text{subject to} \quad y_i = \max\left(x_i + b_i + \sum_{j=1}^{d} \Pi_{ji} y_j \,, -L_i \right) \quad \forall i$$

$$\text{and} \quad y \le 0, b \ge 0.$$

- Bank i decreases its liabilities to the remaining banks by y_i.
- This decreases the equity values of its creditors, which can result in a further default.
- A regulator injects b_i (weighted by $\gamma > 1$) into bank i.

Examples

Conditional contagion aggregation:

- Let the proportional liability matrix $\Pi_{ji}(\omega) \in L^\infty(\mathcal{G})$ be stochastic.
- Then the corresponding CAF becomes

$$\Lambda(x,\omega) := \min_{y,b \in \mathbb{R}^d} \sum_{i=1}^{d} \left(x_i + b_i + \left(\Pi^\top(\omega) y \right)_i \right)^- + \gamma b_i$$

$$\text{subject to} \quad y_i = \max\left(y_i + b_i \leq x_i + \sum_{j=1}^{d} \Pi_{ji}(\omega) y_j \, , -L_i \right) \; \forall i$$

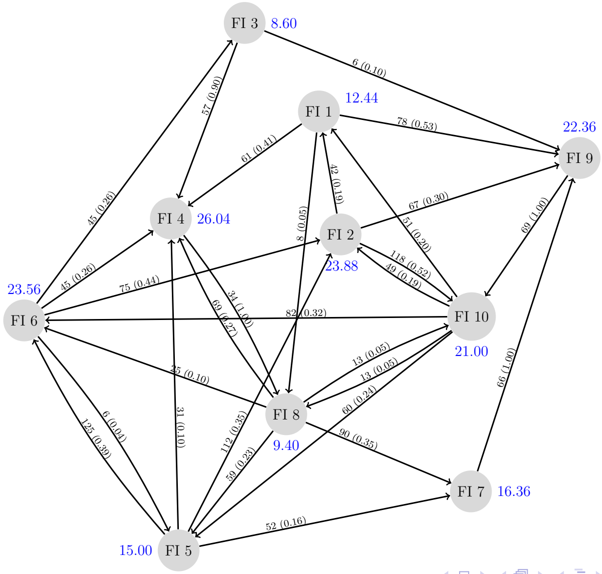
$$\text{and} \quad y \leq 0, b \geq 0.$$

Examples

Numerical case study

- $d = 10$ financial institutions;
- One realization of an Erdös-Rényi graph with success probability $p = 0.35$ and with half-normal distributed weights/liabilities;
- Initial capital endowment proportional to the total interbank assets;
- 0.5 correlated normally distributed shocks on this initial capital;

Examples


FI 3 8.60
6 (0.10)
FI 1 12.44
78 (0.53)
22.36
FI 9
57 (0.90)
61 (0.41)
45 (0.26)
42 (0.19)
67 (0.30)
69 (1.00)
FI 4 26.04
8 (0.05)
FI 2
51 (0.20)
23.88
118 (0.52)
49 (0.19)
23.56
45 (0.26)
75 (0.44)
FI 6
82 (0.32)
34 (1.00)
69 (0.27)
FI 10
21.00
13 (0.05)
13 (0.05)
25 (0.10)
66 (1.00)
60 (0.24)
FI 8
9.40
6 (0.04)
125 (0.39)
31 (0.10)
112 (0.35)
59 (0.23)
90 (0.35)
FI 7 16.36
52 (0.16)
15.00 FI 5

Examples

Statistics for Λ *for 30000 shock scenarios*:

γ	1.6	2.6	'∞'
Mean	78.26	115.69	195.27
5% Quantile	333.09	541.41	977.79
Standard Deviation	121.79	198.17	337.98
$\sum b_i$	38.54	30.45	0.00
Initially defaulted banks		2.78	
Defaulted banks without regulator		3.83	
Defaulted banks with regulator	2.93	3.33	3.83

Examples

Systemic ranking by CoVar:

$\gamma = 2.6$	FI j	2	3	6	4	7	1	10	9	5	8
	$\text{CoVaR}^j_{0.1}$	266.94	297.28	298.49	308.61	320.58	322.56	332.94	355.23	362.27	367.68
$\gamma = \infty$	FI j	2	4	3	7	9	6	1	10	8	5
	$\text{CoVaR}^j_{0.1}$	397.73	419.11	423.18	459.33	471.81	473.61	481.40	548.21	563.60	601.09
	FI j	2	6	10	3	1	7	5	9	8	4
	$-\text{VaR}_{0.1}(x_j)$	13.30	-7.67	-15.05	-17.01	-20.69	-22.98	-26.89	-30.48	-32.11	-33.41
	FI j	4	3	7	9	1	6	2	10	8	5
	L_j	34	63	66	69	147	171	227	255	256	320

Table 4.3: Systemic importance ranking based on $\text{CoVaR}^j_{0.1}$.

STRONG CONSISTENCY AND RISK-CONSISTENCY

Strong consistency and risk-consistency

We now consider CSRMs $\rho_{\mathcal{G}} : L^\infty_d(\mathcal{F}) \to L^\infty(\mathcal{G})$ **fulfilling**:

Monotonicity: $X \geq Y \quad \Longrightarrow \quad \rho_{\mathcal{G}}(X) \leq \rho_{\mathcal{G}}(Y)$

Strong Sensitivity: $X \geq Y$ and $\mathbb{P}(X > Y) > 0$

$$\Longrightarrow \quad \mathbb{P}\big(\rho_{\mathcal{G}}(X) < \rho_{\mathcal{G}}(Y)\big) > 0$$

Locality: $\rho_{\mathcal{G}}(X\mathbb{I}_A + Y\mathbb{I}_{A^c}) = \rho_{\mathcal{G}}(X)\mathbb{I}_A + \rho_{\mathcal{G}}(Y)\mathbb{I}_{A^c} \quad \forall A \in \mathcal{G}$

Lebesgue property: For any uniformly bounded sequence $(X_n)_{n\in\mathbb{N}}$ in $L^\infty_d(\mathcal{F})$ such that $X_n \to X$ $\mathbb{P}$-a.s.

$$\rho_{\mathcal{G}}(X) = \lim_{n\to\infty} \rho_{\mathcal{G}}(X_n) \quad \mathbb{P}\text{-a.s.}$$

Strong consistency and risk-consistency

- On $(\Omega, \mathcal{F}, \mathbb{P})$ let $\mathcal{E}$ be a family of sub-σ-algebras of $\mathcal{F}$ such that $\{\emptyset, \Omega\}, \mathcal{F} \in \mathcal{E}$.

Definition: A family $\left(\rho_{\mathcal{G}}\right)_{\mathcal{G}\in\mathcal{E}}$ of conditional systemic risk measures (CSRM)

$$\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \to L^\infty(\mathcal{G})$$

is (strongly) consistent if for all $\mathcal{G}, \mathcal{H} \in \mathcal{E}$ with $\mathcal{G} \subseteq \mathcal{H}$ and $X, Y \in L_d^\infty(\mathcal{F})$

$$\rho_{\mathcal{H}}(X) \leq \rho_{\mathcal{H}}(Y) \;\; a.s. \Longrightarrow \rho_{\mathcal{G}}(X) \leq \rho_{\mathcal{G}}(Y) \;\; a.s..$$

Strong consistency and risk-consistency

Remark: In case $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{H}}$ are univariate monetary risk measures that are constant-on-constants, strong consistency is equivalent to

$$\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}\big(\rho_{\mathcal{H}}(X)\big).$$

Strong consistency and risk-consistency

Definition: Given a CSRM $\rho_{\mathcal{G}}$ we define

$$f_{\rho_{\mathcal{G}}} : L^{\infty}(\mathcal{G}) \to L^{\infty}(\mathcal{G}); \alpha \mapsto \rho_{\mathcal{G}}(\alpha \mathbf{1}_d)$$

and its corresponding inverse function (which is well-defined)

$$f_{\rho_{\mathcal{G}}}^{-1} : \operatorname{Im} f_{\rho_{\mathcal{G}}} \to L^{\infty}(\mathcal{G}).$$

Strong consistency and risk-consistency

Lemma: $f_{\rho_{\mathcal{G}}}$ and $f_{\rho_{\mathcal{G}}}^{-1}$ are antitone, strongly sensitive, local, and fulfill the Lebesgue property. Further

$$\rho_{\mathcal{G}}(L_d^\infty(\mathcal{F})) = f_{\rho_{\mathcal{G}}}(L^\infty(\mathcal{G})).$$

Strong consistency and risk-consistency

Lemma: The following statements are equivalent

1. $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ is strongly consistent;
2. $\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_d\right) \quad \forall X \in L_d^\infty(\mathcal{F}), \mathcal{G} \subseteq \mathcal{H}$

Strong consistency and risk-consistency

Lemma: The following statements are equivalent

1. $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ is strongly consistent;
2. $\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}\left(f^{-1}_{\rho_{\mathcal{H}}}(\rho_{\mathcal{H}}(X))\mathbf{1}_d\right) \quad \forall X \in L^\infty_d(\mathcal{F}), \mathcal{G} \subseteq \mathcal{H}$

Remark: In case $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{H}}$ are univariate risk measures that are constant-on-constants, $f^{-1}_{\rho_{\mathcal{H}}} = -id$ and 2. reduces to

2. $\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}(\rho_{\mathcal{H}}(X))$

Strong consistency and risk-consistency

Remark: Let $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ be a family of CRMs and define the normalized family

$$(\widetilde{\rho}_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}} := (-f_{\rho_{\mathcal{G}}}^{-1} \circ \rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$$

.

- Then $(\widetilde{\rho}_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ is a family of CRMs which is consistent iff $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ is consistent.

- Further,

$$f_{\widetilde{\rho}_{\mathcal{G}}} = -id$$

which can be considered as a vector generalization of the constant-on-constants property for univariate risk measures.

Strong consistency and risk-consistency

Theorem: Let $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ be strongly consistent. Assume there exists a continuous realizations $\rho_{\mathcal{G}}(\cdot,\cdot)$ for all $\mathcal{G}\in\mathcal{E}$ and that

$$f_{\rho_{\mathcal{F}}}^{-1}\circ\rho_{\mathcal{F}}(x)\in\mathbb{R}\quad\forall x\in\mathbb{R}^d.$$

Then for all $\mathcal{G}\in\mathcal{E}$ the risk measure $\rho_{\mathcal{G}}$ is risk-consistent, i.e. there exists an CAF $\Lambda_{\mathcal{G}}$ and a CBRM $\eta_{\mathcal{G}}$ such that

$$\rho_{\mathcal{G}}=\eta_{\mathcal{G}}\circ\Lambda_{\mathcal{G}}\,.$$

Strong consistency and risk-consistency

Theorem cont': Further, the CAF are strongly consistent in the sense that for all $\mathcal{G} \subseteq \mathcal{H}$

$$\Lambda_{\mathcal{H}}(X) \leq \Lambda_{\mathcal{H}}(Y) \quad \Longrightarrow \quad \Lambda_{\mathcal{G}}(X) \leq \Lambda_{\mathcal{G}}(Y) \quad (X, Y \in L_d^{\infty}(\mathcal{F})),$$

which is equivalent to

$$f_{\Lambda_{\mathcal{G}}}^{-1}(\Lambda_{\mathcal{G}}(X)) = f_{\Lambda_{\mathcal{F}}}^{-1}(\Lambda_{\mathcal{F}}(X)), \text{ for all } X \in L_d^{\infty}(\mathcal{F}).$$

Strong consistency and risk-consistency

Law-invariant, consistent systemic risk measures:

Definition: A CSRM is called conditional law-invariant if

$$\mu_X(\cdot|\mathcal{G}) = \mu_Y(\cdot|\mathcal{G}) \Longrightarrow \rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}(Y),$$

where $\mu_X(\cdot|\mathcal{G})$ and $\mu_Y(\cdot|\mathcal{G})$ are the $\mathcal{G}$-conditional distributions of $X, Y \in L_d^\infty(\mathcal{F})$ resp.

Assumption: There exists $\mathcal{H} \in \mathcal{E}$ such that $(\Omega, \mathcal{H}, \mathbb{P})$ is atomless, and $(\Omega, \mathcal{F}, \mathbb{P})$ is conditionally atomless given $\mathcal{H}$.

Strong consistency and risk-consistency

Theorem [Föllmer, 2014]: Let $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ be a family of univariate, conditionally law-invariant, monetary risk measures. Then $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ is strongly consistent iff the $\rho_{\mathcal{G}}$ are certainty equivalents of the form

$$\rho_{\mathcal{G}}(F) = u^{-1}\big(\mathbb{E}_{\mathbb{P}}\left(u(F) \mid \mathcal{G}\right)\big), \quad \forall F \in L^{\infty}_{,}(\mathcal{F})$$

where $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous.

Strong consistency and risk-consistency

Theorem: Let $(\rho_\mathcal{G})_{\mathcal{G}\in\mathcal{E}}$ be a family of conditionally law-invariant CSRMs. Then $(\rho_\mathcal{G})_{\mathcal{G}\in\mathcal{E}}$ is strongly consistent iff each $\rho_\mathcal{G}$ is of the form

$$\rho_\mathcal{G}(X) = g_\mathcal{G}\left(f_u^{-1}\left(\mathbb{E}_\mathbb{P}\left(u(X) \mid \mathcal{G}\right)\right)\right), \quad \forall X \in L_d^\infty(\mathcal{F}),$$

where

- $u : \mathbb{R}^d \to \mathbb{R}$ is strictly increasing and continuous
- $f_u^{-1} : \operatorname{Im} f_u \to \mathbb{R}$ is the unique inverse function of $f_u : \mathbb{R} \to \mathbb{R}; x \mapsto u(x\mathbf{1}_d)$
- $g_\mathcal{G} : L^\infty(\mathcal{G}) \to L^\infty(\mathcal{G})$ is antitone, strongly sensitive, local, and fulfills the Lebesgue property
- In particular, $g_\mathcal{G} = f_{\rho_\mathcal{G}}$ for all $\mathcal{G} \in \mathcal{E}$.

Strong consistency and risk-consistency

Corollary: Let $(\rho_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ be a family of conditionally law-invariant, strongly consistent CSRMs. Under the assumptions from above, the risk-consistent decomposition $\rho_{\mathcal{G}} = \eta_{\mathcal{G}} \circ \Lambda_{\mathcal{G}}$ is given by a stochastic certainty equivalent $\eta_{\mathcal{G}}$ of the form

$$\eta_{\mathcal{G}}(F) = -U_{\mathcal{G}}^{-1}\left(\mathbb{E}_{\mathbb{P}}\left(U_{\mathcal{G}}(F)\,|\,\mathcal{G}\right)\right), \quad F \in L^{\infty}(\mathcal{F}),$$

where $U_{\mathcal{G}} := f_u \circ \widetilde{g}_{\mathcal{G}}^{-1} : L^{\infty}(\mathcal{F}) \to L^{\infty}(\mathcal{F})$ and $\widetilde{g}_{\mathcal{G}}$ is a certain extension of $g_{\mathcal{G}}$ from $L^{\infty}(\mathcal{G})$ to $L^{\infty}(\mathcal{F})$, and the aggregation

$$\Lambda_{\mathcal{G}} := \widetilde{g}_{\mathcal{G}} \circ f_u^{-1} \circ u.$$

Further, the CBRM $\eta_{\mathcal{G}}$ and the aggregation $\Lambda^{\mathcal{G}}$ are related to each other by

$$U_{\mathcal{G}}(\Lambda^{\mathcal{G}}(x)) = u(x) \quad \forall\, x \in \mathbb{R}^d,\, \mathcal{G} \in \mathcal{E}.$$

Strong consistency and risk-consistency

Corollary: Let CBRMs $(\eta_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ and CAFs $(\Lambda_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ be given such that $(\eta_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ are strongly consistent and $(\rho_{\mathcal{G}} := \eta_{\mathcal{G}} \circ \Lambda_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ are conditionally law-invariant, strongly consistent CSRMs. Then the CBRMs $(\eta_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ must be certainty equivalents of the form

$$\eta_{\mathcal{G}}(F) := u^{-1}\big(\mathbb{E}_{\mathbb{P}}\,(\,u(F)\,|\,\mathcal{G})\,\big), \quad F \in L^{\infty}(\mathcal{F}),$$

where $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous, and the CAFs $(\Lambda_{\mathcal{G}})_{\mathcal{G}\in\mathcal{E}}$ must be of the form

$$\Lambda_{\mathcal{G}} = f\left(\alpha_{\mathcal{G}} \cdot \Lambda(x) + \beta_{\mathcal{G}}\right),$$

for some deterministic AF Λ, $\mathcal{G}$-constants $\alpha_{\mathcal{G}}, \beta_{\mathcal{G}} \in L^{\infty}(\mathcal{G})$, and some strictly increasing and continuous $f : \mathbb{R} \to \mathbb{R}$.

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