Risk-Consistent Conditional Systemic Risk Measures

Thilo Meyer-Brandis
University of Munich

joint with:
H. Hoffmann and G. Svindland, University of Munich

Workshop on Systemic Risk and Financial Networks
IPAM, UCLA
March 24, 2015
Motivation

Let

\[ X = (X_1, \ldots, X_d) \in L^{\infty}_d(\mathcal{F}) \]

represent (monetary) risk factors associated to a system of \( d \) interacting financial institutions.
Let

\[
X = (X_1, \ldots, X_d) \in L_\infty^d(\mathcal{F})
\]

represent (monetary) risk factors associated to a system of \(d\) interacting financial institutions.

**Traditional approach to risk management**: Measuring stand-alone risk of each institution

\[
\eta(X_i)
\]

for some univariate risk measure \(\eta : L^\infty(\mathcal{F}) \to \mathbb{R}\).
Axiomatic characterization of (univariate) risk measures:

A map $\eta : L^\infty(\mathcal{F}) \rightarrow \mathbb{R}$ is a 

**monetary risk measure**, if it is

- **Monotone**: $X_1 \geq X_2 \Rightarrow \eta(X_1) \leq \eta(X_2)$.
- **Cash-invariant**: $\eta(X + m) = \eta(X) - m$ for all $m \in \mathbb{R}$.
  
  (Constant-on-constants: $\eta(m) = -m$ for all $m \in \mathbb{R}$.)

A monetary risk measure is called **convex**, if it is

- **Convex**: $\eta(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \eta(X_1) + (1 - \lambda)\eta(X_2)$ for $\lambda \in [0, 1]$.
  
  (Quasi-convex: $\eta(\lambda X_1 + (1 - \lambda)X_2) \leq \max\{\eta(X_1), \eta(X_2)\}$.)

A convex risk measure is called **coherent**, if it is

- **Pos. homogeneity**: $\eta(\lambda X) = \lambda \eta(X)$ for $\lambda \geq 0$. 

Risk-Consistent Conditional Systemic Risk Measures
However,

- **Financial crisis:** traditional approach to regulation and risk management insufficiently captures

  **Systemic risk:** risk that in case of an adverse (local) shock substantial parts of the system default.

- **Question:** Given the system $X = (X_1, \ldots, X_d)$, what is an appropriate risk measure

  $$\rho : L_\infty^d(\mathcal{F}) \to \mathbb{R}$$

  of systemic risk?
Most proposals of systemic risk measures in the post-crisis literature are of the form

\[ \rho(X) := \eta(\Lambda(X)) \]

for some univariate risk measure

\[ \eta : L^\infty(\mathcal{F}) \to \mathbb{R} \]

and some aggregation function

\[ \Lambda : \mathbb{R}^d \to \mathbb{R}. \]
**Motivation**

*Some examples:*

- Appropriate aggregation to reflect systemic risk?

**Systemic Expected Shortfall**

[Acharya et al., 2011]

\[
\rho(X) := ES_q \left( \sum_{i=1}^{d} X_i \right)
\]

where \( ES_q \) is the univariate Expected Shortfall at level \( q \in [0, 1] \).
Deposit Insurance [Lehar, 2005], [Huang, Zhou & Zhu, 2011]

\[ \rho(X) := E \left( \sum_{i=1}^{d} -X_{i}^{-} \right) \]

SystRisk [Brunnermeier & Cheridito, 2013]

\[ \rho(X) := \eta_{SystRisk} \left( \sum_{i=1}^{d} -\alpha_{i}X_{i}^{-} + \beta_{i}(X_{i}^{+} - v_{i}) \right) \]

where \( \eta_{SystRisk} \) is some utility-based univariate risk measure.
Contagion model

- $\Pi_{ji}$ denotes the proportion of the total liabilities $L_j$ of bank $j$ which it owes to bank $i$.

- $X_i$ represents the capital endowment of bank $i$.

- If a bank $i$ defaults, i.e. $X_i < 0$, this generates further losses in the system by contagion.

- Systemic risk concerns the total loss in the network generated by a profit/loss profile $X = (X_1, \ldots, X_d)$. 
Motivation

[Chen, Iyengar & Moallemi, 2013], [Eisenberg, Noe, 2001]

Total loss in the system induced by some initial loss profile $x \in \mathbb{R}^d$:

$$\Lambda(x) := \min_{y,b \in \mathbb{R}^d} \sum_{i=1}^{d} -y_i - \gamma b_i$$

subject to

$$y_i = x_i - b_i + \sum_{j=1}^{d} \Pi_{ji} y_j \quad \forall i$$

- Bank $i$ decreases its liabilities to the remaining banks by $y_i$.
- This decreases the equity values of its creditors, which can result in a further default.
- A regulator injects $b_i$ (weighted by $\gamma > 1$) into bank $i$. 
Various extensions of the Eisenberg & Noe framework to include further channels of contagion, see e.g.

- [Amini, Filipovic & Minca, 2013]
- [Awiszus & Weber, 2015]
- [Cifuentes, Ferrucci & Shin, 2005]
- [Gai & Kapadia, 2010]
- [Rogers & Veraart, 2013]
Motivation

- Conditional risk measuring interesting in systemic risk
  → identification of systemic relevant structures

**CoVar** and **CoES** [Adrian, Brunnermeier, 2011]

\[
\rho(X) := \text{VaR}_q \left( \sum_{i=1}^{d} X_i \mid X_j \leq -\text{VaR}_q(X_j) \right)
\]

[Acharya et al., 2011]

\[
\text{ES}_q \left( X_j \mid \sum_{i=1}^{d} X_i \leq -\text{VaR}_q \left( \sum_{i=1}^{d} X_i \right) \right)
\]
Motivation

→ conditional risk measure

\[ \eta : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G}) \]

for some sub-\(\sigma\)-algebra \(\mathcal{G} \subset \mathcal{F}\)

▶ Broad literature on conditional risk measures in a dynamic setup (e.g. [Detlefsen & Scandolo, 2005], [Föllmer & Schied, 2011], [Frittelli & Maggis, 2011], [Tutsch, 2007]).

▶ In the context of systemic risk, see also [Föllmer, 2014] and [Föllmer & Klüppelberg, 2014]:

Risk-Consistent Conditional Systemic Risk Measures
But also conditional aggregation $\Lambda(x, \omega)$ appears naturally; f.ex.:

- The way of aggregation might depend on macroeconomic factors:
  - Countercyclical regulation: more severe when economy is in good shape than in times of financial distress
  - Stochastic discounting: $\Lambda(x, \omega) = \tilde{\Lambda}(x)D(\omega)$, where $D$ is some stochastic discount factor.

- Liability matrix $\Pi = \Pi(\omega)$ in contagion model above might be stochastic (derivative exposures between banks)
Objective of this presentation: Structural analysis of conditional systemic risk measures $\rho_G : L^\infty_d(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$, $\mathcal{G} \subset \mathcal{F}$, of the form

$$\rho_G(X) := \eta_G(\Lambda_G(X))$$

where $\eta_G$ is some univariate conditional risk measure and $\Lambda_G$ some conditional aggregation function.
Motivation

Structure of remaining presentation:

1. *Axiomatic characterization of conditional systemic risk measures of this type*

2. *Examples and simulation study*

3. *Strong consistency and risk-consistency*
AXIOMATIC CHARACTERIZATION OF RISK-CONSISTENT CONDITIONAL SYSTEMIC RISK MEASURES
Aim: Axiomatic characterization of conditional systemic risk measures of the form $\eta(\Lambda(X))$ in terms of properties on constants and risk-consistent properties.

- Risk-consistent properties ensure a consistency between local - that is $\omega$-wise - risk assessment and the measured global risk.

For deterministic risk measures [Chen, Iyengar & Moallemi, 2013], f. ex.:

- Risk-monotonicity: if for given risk vectors $X$ and $Y$ we have $\rho(X(\omega)) \geq \rho(Y(\omega))$ in a.a. states $\omega$, then $\rho(X) \geq \rho(Y)$. 
In the deterministic case:

- [Chen, Iyengar & Moallemi, 2013] on finite probability space
- [Kromer, Overbeck & Zilch, 2013] general probability space

Our contribution:

- Conditional setting
- More comprehensive structural analysis
- More flexible aggregation and axiomatic setting
**Notation:** Given \((\Omega, \mathcal{F}, \mathbb{P})\) and a sub-\(\sigma\)-algebra \(\mathcal{G} \subset \mathcal{F}\):

- A realization of a function
  \[ \rho_{\mathcal{G}} : L^\infty_d(\mathcal{F}) \rightarrow L^\infty(\mathcal{G}) \]
  is a function
  \[ \rho_{\mathcal{G}}(\cdot, \cdot) : L^\infty_d(\mathcal{F}) \times \Omega \rightarrow \mathbb{R} \]
  such that \(\rho_{\mathcal{G}}(X, \cdot) \in \rho_{\mathcal{G}}(X)\) for all \(X \in L^\infty_d(\mathcal{F})\).

- A realization \(\rho_{\mathcal{G}}(\cdot, \cdot)\) has **continuous path** if \(\rho_{\mathcal{G}}(\cdot, \omega) : \mathbb{R}^d \rightarrow \mathbb{R}\)
  is continuous for all \(\omega \in \Omega\).

- We denote the constant random vector \(X_{\widehat{\omega}}, \widehat{\omega} \in \Omega\), by
  \[ X_{\widehat{\omega}}(\omega) := X(\widehat{\omega}) \quad \forall \omega \in \Omega \]
**Definition:** A function \( \rho_G : L^\infty_d(\mathcal{F}) \rightarrow L^\infty(\mathcal{G}) \) is called \textit{risk-consistent conditional systemic risk measure} (CSRM), if it is

**Monotone on constants:** \( x, y \in \mathbb{R}^d \) with \( x \geq y \Rightarrow \rho_G(x) \leq \rho_G(y) \)

and if there exists a realization \( \rho_G(\cdot, \cdot) \) with \textit{continuous paths} such that \( \rho_G \) is

**Risk-monotone:** For all \( X, Y \in L^\infty_d(\mathcal{F}) \)

\[
\rho_G(X_\omega, \omega) \geq \rho_G(Y_\omega, \omega) \text{ a.s.} \Rightarrow \rho_G(X, \omega) \geq \rho_G(Y, \omega) \text{ a.s.}
\]

**(Risk-) regular:** \( \rho_G(X, \omega) = \rho_G(X_\omega, \omega) \text{ a.s.} \forall X \in L^\infty_d(\mathcal{G}) \)
Further risk-consistent properties: We say that $\rho_G$ is

**Risk-convex:** If for $X, Y, Z \in L_d^\infty(\mathcal{F})$, $\alpha \in L^\infty(\mathcal{G})$ with $0 \leq \alpha \leq 1$

$$
\rho_G (Z_\omega, \omega) = \alpha(\omega) \rho_G (X_\omega, \omega) + (1-\alpha(\omega)) \rho_G (Y_\omega, \omega) \ a.s.
$$

$$
\Rightarrow \rho_G (Z, \omega) \leq \alpha(\omega) \rho_G (X, \omega) + (1-\alpha(\omega)) \rho_G (Y, \omega) \ a.s.
$$

**Risk-quasiconvex:** If for $X, Y, Z \in L_d^\infty(\mathcal{F})$, $\alpha \in L^\infty(\mathcal{G})$, $0 \leq \alpha \leq 1$

$$
\rho_G (Z_\omega, \omega) = \alpha(\omega) \rho_G (X_\omega, \omega) + (1-\alpha(\omega)) \rho_G (Y_\omega, \omega) \ a.s.
$$

$$
\Rightarrow \rho_G (Z, \omega) \leq \rho_G (X, \omega) \lor \rho_G (Y, \omega) \ a.s.
$$

**Risk-pos.-homogeneous:** If for $X, Y \in L_d^\infty(\mathcal{F})$, $0 \leq \alpha \in L^\infty(\mathcal{G})$

$$
\rho_G (Y_\omega, \omega) = \alpha(\omega) \rho_G (X_\omega, \omega) \ a.s.
$$

$$
\Rightarrow \rho_G (Y, \omega) \leq \alpha(\omega) \rho_G (X, \omega) \ a.s.
$$
Further properties on constants: We say that $\rho_G$ is

**Convex on constants:** If for all $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$

$$\rho_G(\lambda x + (1 - \lambda)y) \leq \lambda \rho_G(x) + (1 - \lambda)\rho_G(y)$$

**Positively homogeneous on constants:** If for all $x \in \mathbb{R}^d$ and $\lambda \geq 0$

$$\rho_G(\lambda x) = \lambda \rho_G(x)$$
Proposition: Let \( \rho_G : L^\infty_d(\mathcal{F}) \to L^\infty(\mathcal{G}) \) be a function which has a realization with continuous paths. Further suppose that

\[
\rho_G(x) = \sum_{i=1}^{s} a_i(x) \mathbb{I}_{A_i}, \ x \in \mathbb{R}^d,
\]

where \( a_i(x) \in \mathbb{R} \) and \( A_i \) are pairwise disjoint sets such that \( \Omega = \bigcup_{i=1}^{s} A_i \) for \( s \in \mathbb{N} \cup \{\infty\} \). Define

\[
k : \Omega \to \mathbb{N}; \ \omega \mapsto i \text{ such that } \omega \in A_i.
\]

Then \( \rho_G \) is risk-monotone if and only if

\[
\rho_G(X_\omega) \mathbb{I}_{A_{k(\omega)}} \geq \rho_G(Y_\omega) \mathbb{I}_{A_{k(\omega)}} \text{ for a.a. } \omega \Rightarrow \rho_G(X) \geq \rho_G(Y).
\]

Also the remaining risk-consistent properties can be expressed in a similar way without requiring a particular realization of \( \rho_G \).
Proposition: The following holds for a risk-consistent CSRM $\rho_G$: 

- If $\rho_G$ is risk-monotone and monotone on constants, then $\rho_G$ is monotone.
- If $\rho_G$ is risk-quasiconvex and convex on constants, then $\rho_G$ is quasiconvex.
- If $\rho_G$ is risk-convex and convex on constants, then $\rho_G$ is convex;
- $\rho_G$ is risk positively homogeneous and positively homogeneous on constants iff $\rho_G$ is positively homogeneous;
Definition: A function $\Lambda_G : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a conditional aggregation function (CAF), if

1. $\Lambda_G (x, \cdot) \in L^\infty (\mathcal{G})$ for all $x \in \mathbb{R}^d$.

2. $\Lambda_G (\cdot, \omega)$ is continuous for all $\omega \in \Omega$.

3. $\Lambda_G (\cdot, \omega)$ is monotone (increasing) for all $\omega \in \Omega$.

Furthermore, $\Lambda_G$ is called concave (positively homogeneous) if $\Lambda_G (\cdot, \omega)$ is concave (positively homogeneous) for all $\omega \in \Omega$. 
Given a CAF $\Lambda_G$, we extend the aggregation from deterministic to random vectors in the following way:

$$\tilde{\Lambda}_G : L^\infty_d(\mathcal{F}) \to L^\infty(\mathcal{F})$$
$$X(\omega) \mapsto \Lambda_G(X(\omega), \omega)$$

Notice that the aggregation of random vectors is $\omega$-wise in the sense that given a certain state $\omega \in \Omega$, in that state we aggregate the sure payoff $X(\omega)$. 

Definition: Let $F, G \in L^\infty(\mathcal{F})$. A function $\eta_G : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{G})$ is a \textit{conditional base risk measure} (CBRM), if it is

\textbf{Monotone:} $F \geq G \Rightarrow \eta_G(F) \leq \eta_G(G)$.

\textbf{Constant on $G$-constants:} $\eta_G(\alpha) = -\alpha \quad \forall \alpha \in L^\infty(\mathcal{G})$. 
Additional properties of CBRMs:

**Convexity:** For all $\alpha \in L^\infty(\mathcal{G})$ with $0 \leq \alpha \leq 1$

$$\eta_{\mathcal{G}}(\alpha F + (1 - \alpha)G) \leq \alpha \eta_{\mathcal{G}}(G) + (1 - \alpha)\eta_{\mathcal{G}}(G)$$

**Quasiconvexity:** For all $\alpha \in L^\infty(\mathcal{G})$ with $0 \leq \alpha \leq 1$

$$\eta_{\mathcal{G}}(\alpha F + (1 - \alpha)G) \leq \eta_{\mathcal{G}}(F) \lor \eta_{\mathcal{G}}(G)$$

**Positive homogeneity:** For all $\alpha \in L^\infty(\mathcal{G})$ with $\alpha \geq 0$

$$\eta_{\mathcal{G}}(\alpha F) = \alpha \eta_{\mathcal{G}}(F)$$
Theorem: (Decomposition of conditional systemic risk measures)

A map \( \rho_G : L^\infty_d (\mathcal{F}) \to L^\infty (\mathcal{G}) \) is a risk-consistent CSRM if and only if there exists a CBRM \( \eta_G : L^\infty (\mathcal{F}) \to L^\infty (\mathcal{G}) \) and a CAF \( \Lambda_G \) such that

\[
\rho_G (X) = \eta_G \left( \tilde{\Lambda}_G (X) \right) \quad \forall X \in L^\infty_d (\mathcal{F}),
\]

where \( \tilde{\Lambda}_G (X) := \Lambda_G (X(\omega), \omega) \).
Theorem cont’: Furthermore, the following equivalences hold:

- $\rho_G$ is risk-convex iff $\eta_G$ is convex;
- $\rho_G$ is risk-quasiconvex iff $\eta_G$ is quasiconvex;
- $\rho_G$ is risk-positive homogeneous iff $\eta_G$ is positive homogeneous;

and

- $\rho_G$ is convex on constants iff $\Lambda_G$ is concave;
- $\rho_G$ is positive homogeneous on constants iff $\Lambda_G$ is positive homogeneous.
EXAMPLES AND SIMULATION STUDY
CoVar (analogue CoES) [Adrian & Brunnermeier, 2011]:

Let

\[ \eta_G(F) := \text{VaR}_\lambda(F|G) := -\text{essinf}_{G \in L^\infty(G)} \{ \mathbb{P}(F \leq G \mid G) > \lambda \}, \]

where \( F \in L^\infty(\mathcal{F}) \), \( \lambda \in L^\infty(\mathcal{G}) \), and \( 0 < \lambda < 1 \).

For a fixed \( j \in \{1, \ldots, d\} \) and \( q \in (0, 1) \), define the event \( A := \{ X_j \leq -\text{VaR}_q(X_j) \} \) and let \( \mathcal{G} := \sigma(A) \).

Define \( \Lambda(x) := \sum x_i, x \in \mathbb{R} \).

Then the CoVar is represented by \( \eta_G(\Lambda(X)) \), which is a positively homogeneous CSRM.
Example: Extended contagion model

- $\Pi_{ji}$ denotes the proportion of the total interbank liabilities $L_j$ of bank $j$ which it owes to bank $i$.
- $X_i$ represents the capital endowment of bank $i$.
- If a bank $i$ defaults, i.e. $X_i < 0$, this generates further losses in the system by contagion.
- Systemic risk concerns the total loss in the network generated by a profit/loss profile $X = (X_1, \ldots, X_d)$. 
More realistic contagion aggregation:

- Total aggregated loss in the system for loss profile $x \in \mathbb{R}^d$:

$$\Lambda(x) := \min_{y,b \in \mathbb{R}^d} \sum_{i=1}^{d} \left( x_i + b_i + \left( \Pi^\top y \right)_i \right)^- + \gamma b_i$$

subject to \quad \begin{align*}
y_i &= \max \left( x_i + b_i + \sum_{j=1}^{d} \Pi_{ji} y_j, -L_i \right) \quad \forall i \\
\text{and} \quad y &\leq 0, b \geq 0.
\end{align*}

- Bank $i$ decreases its liabilities to the remaining banks by $y_i$.

- This decreases the equity values of its creditors, which can result in a further default.

- A regulator injects $b_i$ (weighted by $\gamma > 1$) into bank $i$.  

Examples

Conditional contagion aggregation:

- Let the proportional liability matrix $\Pi_{ji}(\omega) \in L^\infty(\mathcal{G})$ be stochastic.

- Then the corresponding CAF becomes

$$\Lambda(x, \omega) := \min_{y, b \in \mathbb{R}^d} \sum_{i=1}^d \left( x_i + b_i + \left( \Pi^\top(\omega)y \right)_i \right)^- + \gamma b_i$$

subject to $y_i = \max \left( y_i + b_i \leq x_i + \sum_{j=1}^d \Pi_{ji}(\omega)y_j , -L_i \right)$ $\forall i$

and $y \leq 0, b \geq 0$. 
Numerical case study

- $d = 10$ financial institutions;
- One realization of an Erdős-Rényi graph with success probability $p = 0.35$ and with half-normal distributed weights/liabilities;
- Initial capital endowment proportional to the total interbank assets;
- 0.5 correlated normally distributed shocks on this initial capital;
Examples

Risk-Consistent Conditional Systemic Risk Measures
Statistics for $\Lambda$ for 30000 shock scenarios:

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>1.6</th>
<th>2.6</th>
<th>'∞'</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>78.26</td>
<td>115.69</td>
<td>195.27</td>
</tr>
<tr>
<td>5% Quantile</td>
<td>333.09</td>
<td>541.41</td>
<td>977.79</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>121.79</td>
<td>198.17</td>
<td>337.98</td>
</tr>
<tr>
<td>$\sum b_i$</td>
<td>38.54</td>
<td>30.45</td>
<td>0.00</td>
</tr>
<tr>
<td>Initially defaulted banks</td>
<td>2.78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Defaulted banks without regulator</td>
<td></td>
<td>3.83</td>
<td></td>
</tr>
<tr>
<td>Defaulted banks with regulator</td>
<td>2.93</td>
<td>3.33</td>
<td>3.83</td>
</tr>
</tbody>
</table>
Systemic ranking by CoVar:

<table>
<thead>
<tr>
<th></th>
<th>CoVaR$_{0.1}^j$</th>
<th>FI j</th>
<th>CoVaR$_{0.1}^j$</th>
<th>FI j</th>
<th>-VaR$_{0.1}(x_j)$</th>
<th>FI j</th>
<th>$L_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>266.94</td>
<td>2</td>
<td>297.28</td>
<td>3</td>
<td>298.49</td>
<td>6</td>
<td>308.61</td>
</tr>
<tr>
<td>8</td>
<td>397.73</td>
<td>4</td>
<td>419.11</td>
<td>3</td>
<td>423.18</td>
<td>7</td>
<td>459.33</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>471.81</td>
<td>6</td>
<td>473.61</td>
<td>1</td>
<td>481.40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>548.21</td>
<td>1</td>
<td>563.60</td>
<td>8</td>
<td>601.09</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td></td>
<td>5</td>
<td></td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td></td>
<td>4</td>
<td></td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td></td>
<td>8</td>
<td></td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Systemic importance ranking based on CoVaR$_{0.1}^j$. 
STRONG CONSISTENCY AND RISK-CONSISTENCY
We now consider CSRM\(\rho_G: L^\infty_d(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})\) fulfilling:

**Monotonicity:** \(X \geq Y \implies \rho_G(X) \leq \rho_G(Y)\)

**Strong Sensitivity:** \(X \geq Y\) and \(\mathbb{P}(X > Y) > 0\)

\[\implies \mathbb{P}(\rho_G(X) < \rho_G(Y)) > 0\]

**Locality:** \(\rho_G(XI_A + YI_{A^c}) = \rho_G(X)I_A + \rho_G(Y)I_{A^c} \quad \forall A \in \mathcal{G}\)

**Lebesgue property:** For any uniformly bounded sequence \((X_n)_{n \in \mathbb{N}}\) in \(L^\infty_d(\mathcal{F})\) such that \(X_n \rightarrow X\) \(\mathbb{P}\)-a.s.

\[\rho_G(X) = \lim_{n \rightarrow \infty} \rho_G(X_n) \quad \mathbb{P}\)-a.s.
On \((\Omega, \mathcal{F}, \mathbb{P})\) let \(\mathcal{E}\) be a family of sub-\(\sigma\)-algebras of \(\mathcal{F}\) such that \(\{\emptyset, \Omega\}, \mathcal{F} \in \mathcal{E}\).

**Definition:** A family \(\left(\rho_G\right)_{G \in \mathcal{E}}\) of conditional systemic risk measures (CSRM)

\[
\rho_G : L^\infty_d(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})
\]

is (strongly) consistent if for all \(G, H \in \mathcal{E}\) with \(G \subseteq H\) and \(X, Y \in L^\infty_d(\mathcal{F})\)

\[
\rho_H(X) \leq \rho_H(Y) \quad \text{a.s.} \implies \rho_G(X) \leq \rho_G(Y) \quad \text{a.s.}
\]
Remark: In case $\rho_G$ and $\rho_H$ are univariate monetary risk measures that are constant-on-constants, strong consistency is equivalent to

$$\rho_G(X) = \rho_G(\rho_H(X)).$$
**Definition:** Given a CSRM $\rho_G$ we define

$$f_{\rho_G} : \mathbb{L}^\infty(G) \to \mathbb{L}^\infty(G); \alpha \mapsto \rho_G(\alpha 1_d)$$

and its corresponding inverse function (which is well-defined)

$$f_{\rho_G}^{-1} : \text{Im } f_{\rho_G} \to \mathbb{L}^\infty(G).$$
Lemma: $f_{\rho_G}$ and $f_{\rho_G^{-1}}$ are antitone, strongly sensitive, local, and fulfill the Lebesgue property. Further

$$\rho_G(L_d^\infty(F)) = f_{\rho_G}(L^\infty(G)).$$
Lemma: The following statements are equivalent

1. \((\rho_g)_{g \in \mathcal{E}}\) is strongly consistent;

2. \(\rho_g(X) = \rho_g\left(f^{-1}_{\rho_H}(\rho_H(X))1_d\right) \quad \forall X \in L^\infty_d(\mathcal{F}), \mathcal{G} \subseteq \mathcal{H}\)
Lemma: The following statements are equivalent

1. \((\rho_G)_{G \in \mathcal{E}}\) is strongly consistent;

2. \(\rho_G(X) = \rho_G\left(f_{\rho_H}^{-1}(\rho_H(X))\mathbf{1}_d\right)\) \(\forall X \in L^\infty_d(\mathcal{F}), G \subseteq H\)

Remark: In case \(\rho_G\) and \(\rho_H\) are univariate risk measures that are constant-on-constants, \(f_{\rho_H}^{-1} = -\text{id}\) and 2. reduces to

2. \(\rho_G(X) = \rho_G(\rho_H(X))\)
Strong consistency and risk-consistency

Remark: Let \((\rho_g)_{g \in \mathcal{E}}\) be a family of CRMs and define the normalized family

\[
(\tilde{\rho}_g)_{g \in \mathcal{E}} := (\rho_g^{-1} \circ \rho_g)_{g \in \mathcal{E}}
\]

Then \((\tilde{\rho}_g)_{g \in \mathcal{E}}\) is a family of CRMs which is consistent iff \((\rho_g)_{g \in \mathcal{E}}\) is consistent.

Further,

\[
f_{\tilde{\rho}_g} = -id
\]

which can be considered as a vector generalization of the constant-on-constants property for univariate risk measures.
Strong consistency and risk-consistency

**Theorem:** Let \((\rho_G)_{G \in \mathcal{E}}\) be **strongly consistent**. Assume there exists a continuous realizations \(\rho_G(\cdot, \cdot)\) for all \(G \in \mathcal{E}\) and that

\[
f_{\rho_{\mathcal{F}}}^{-1} \circ \rho_{\mathcal{F}}(x) \in \mathbb{R} \quad \forall x \in \mathbb{R}^d.
\]

Then for all \(G \in \mathcal{E}\) the risk measure \(\rho_G\) is **risk-consistent**, i.e. there exists an CAF \(\Lambda_G\) and a CBRM \(\eta_G\) such that

\[
\rho_G = \eta_G \circ \Lambda_G.
\]
Strong consistency and risk-consistency

**Theorem cont’**: Further, the CAF are strongly consistent in the sense that for all \( G \subseteq H \)

\[
\Lambda_H(X) \leq \Lambda_H(Y) \implies \Lambda_G(X) \leq \Lambda_G(Y) \quad (X, Y \in L^\infty_d(\mathcal{F})),
\]

which is equivalent to

\[
f_{\Lambda_G}^{-1}(\Lambda_G(X)) = f_{\Lambda_F}^{-1}(\Lambda_F(X)), \text{ for all } X \in L^\infty_d(\mathcal{F}).
\]
Law-invariant, consistent systemic risk measures:

**Definition:** A CSRM is called *conditional law-invariant* if

\[ \mu_X(\cdot | G) = \mu_Y(\cdot | G) \implies \rho_G(X) = \rho_G(Y), \]

where \( \mu_X(\cdot | G) \) and \( \mu_Y(\cdot | G) \) are the \( G \)-conditional distributions of \( X, Y \in L^\infty(F) \) resp.

**Assumption:** There exists \( H \in \mathcal{E} \) such that \( (\Omega, H, \mathbb{P}) \) is atomless, and \( (\Omega, F, \mathbb{P}) \) is conditionally atomless given \( H \).
**Theorem** [Föllmer, 2014]: Let \((\rho_G)_{G \in \mathcal{E}}\) be a family of univariate, conditionally law-invariant, monetary risk measures. Then \((\rho_G)_{G \in \mathcal{E}}\) is strongly consistent iff the \(\rho_G\) are **certainty equivalents** of the form

\[
\rho_G(F) = u^{-1}(\mathbb{E}_P (u(F) | \mathcal{G})), \quad \forall F \in L^\infty(\mathcal{F})
\]

where \(u : \mathbb{R} \to \mathbb{R}\) is strictly increasing and continuous.
Strong consistency and risk-consistency

**Theorem:** Let \((\rho_G)_{G \in \mathcal{E}}\) be a family of conditionally law-invariant CSRMs. Then \((\rho_G)_{G \in \mathcal{E}}\) is strongly consistent iff each \(\rho_G\) is of the form

\[
\rho_G(X) = g_G\left( f_u^{-1}\left( \mathbb{E}_{\mathbb{P}}\left( u(X) | G \right) \right) \right), \quad \forall X \in L^\infty_d(\mathcal{F}),
\]

where

- \(u : \mathbb{R}^d \to \mathbb{R}\) is strictly increasing and continuous
- \(f_u^{-1} : \text{Im } f_u \to \mathbb{R}\) is the unique inverse function of \(f_u : \mathbb{R} \to \mathbb{R}; x \mapsto u(x \mathbf{1}_d)\)
- \(g_G : L^\infty(\mathcal{G}) \to L^\infty(\mathcal{G})\) is antitone, strongly sensitive, local, and fulfills the Lebesgue property
- In particular, \(g_G = f_{\rho_G}\) for all \(G \in \mathcal{E}\).
**Corollary:** Let \((\rho_{\mathcal{G}})_{\mathcal{G} \in \mathcal{E}}\) be a family of conditionally law-invariant, strongly consistent CSRM s. Under the assumptions from above, the risk-consistent decomposition \(\rho_{\mathcal{G}} = \eta_{\mathcal{G}} \circ \Lambda_{\mathcal{G}}\) is given by a stochastic certainty equivalent \(\eta_{\mathcal{G}}\) of the form

\[
\eta_{\mathcal{G}}(F) = -U_{\mathcal{G}}^{-1}(\mathbb{E}_{\mathbb{P}}(U_{\mathcal{G}}(F) | \mathcal{G})), \quad F \in L^\infty(\mathcal{F}),
\]

where \(U_{\mathcal{G}} := f_u \circ \tilde{g}_{\mathcal{G}}^{-1} : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F})\) and \(\tilde{g}_{\mathcal{G}}\) is a certain extension of \(g_{\mathcal{G}}\) from \(L^\infty(\mathcal{F})\) to \(L^\infty(\mathcal{F})\), and the aggregation

\[
\Lambda_{\mathcal{G}} := \tilde{g}_{\mathcal{G}} \circ f_u^{-1} \circ u.
\]

Further, the CBRM \(\eta_{\mathcal{G}}\) and the aggregation \(\Lambda_{\mathcal{G}}\) are related to each other by

\[
U_{\mathcal{G}}(\Lambda_{\mathcal{G}}(x)) = u(x) \quad \forall x \in \mathbb{R}^d, \mathcal{G} \in \mathcal{E}.
\]
Corollary: Let CBRMs \((\eta_G)_{G \in \mathcal{E}}\) and CAFs \((\Lambda_G)_{G \in \mathcal{E}}\) be given such that \((\eta_G)_{G \in \mathcal{E}}\) are strongly consistent and \((\rho_G := \eta_G \circ \Lambda_G)_{G \in \mathcal{E}}\) are conditionally law-invariant, strongly consistent CSRMs. Then the CBRMs \((\eta_G)_{G \in \mathcal{E}}\) must be certainty equivalents of the form

\[
\eta_G(F) := u^{-1}(\mathbb{E}_\mathbb{P}(u(F) \mid G)), \quad F \in L^\infty(\mathcal{F}),
\]

where \(u: \mathbb{R} \to \mathbb{R}\) is strictly increasing and continuous, and the CAFs \((\Lambda_G)_{G \in \mathcal{E}}\) must be of the form

\[
\Lambda_G = f(\alpha_G \cdot \Lambda(x) + \beta_G),
\]

for some deterministic AF \(\Lambda\), \(G\)-constants \(\alpha_G, \beta_G \in L^\infty(\mathcal{G})\), and some strictly increasing and continuous \(f: \mathbb{R} \to \mathbb{R}\).
References


Artzner, Delbaen, Eber, Heath: Coherent measures of risk, Math. Fin., 1999

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., and Montrucchio, L.,
Riskmeasures: rationality and diversication, Mathematical Finance, 21, 743 - 774
(2011)

systemic risk. Management Science 59(6), 13731388.

Cifuentes, R., G. Ferrucci & H. S. Shin (2005), Liquidity risk and contagion,
Journal of the European Economic Association 3 (2-3), 556566]

measures. Finance and Stochastics 9(4), 539561..

Follmer, H., Spatial Risk Measures and their Local Specification: The Locally

time (3rd ed.). De Gruyter.


