# An Integrated Model of Systemic Risk in Financial Networks

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(joint work with Kerstin Awiszus)

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# Motivation

- Systemic risk: financial system as a whole is susceptible to failures initiated by the characteristics of the system itself
- Local and global interaction channels:

  direct liabilities, bankruptcy costs, cross-holdings and fire sales
- Provide fully integrated model; this is missing in the literature so far
  - Bankruptcy costs are, for example, considered by Rogers & Veraart (2013), Elliott, Golub & Jackson (2014), Elsinger (2009) and Glasserman & Young (2014), cross-holdings e.g. by Elsinger (2009) and Elliott et al. (2014). Cifuentes et al. (2005) incorporate fire sales into the framework of Eisenberg & Noe (2001); their approach is further extended by Gai & Kapadia (2010), Nier, Yang, Yorulmazer & Alentorn (2007), Amini, Filipović & Minca (2013), and Chen, Liu & Yao (2014).
- In numerical case studies we will analyze the number of contagious defaults

# Outline

(i) Comprehensive model of financial network

- direct liabilities, bankruptcy costs, cross-holdings, fire sales
- model is analyzed in Awiszus & W. (2014)

(ii) Existence of equilibrium and algorithm

(iii) Numerical case studies

- Number of contagious defaults: separate and joint effects
- Analysis of both expected quantities and their random fluctuations

# An Integrated Model

# Main Interaction Channels

Single period is interpreted as a snapshot of a banking system that continues to exist afterwards.

Banks are connected to each other via three different channels:

- **Direct liabilities**: Banks have nominal liabilities against each other. These liabilities – due after one period – are promises that will only partially be fulfilled if some of the banks default.
- **Cross-holdings**: Banks may hold shares of each other. In this case, the financial net worths of banks depend on the net worths of other banks due to these cross-holdings.
- **Fire sales**: If the portfolios of the banks contain the same assets, changes in asset prices simultaneously influence the net worths of these banks. For simplicity, our model assumes the existence of a single (representative) illiquid asset.

# Setup and External Assets

## Financial system

- Banks: $\mathcal{N} = \{1, \ldots, n\}$
- Vector of financial net worths of banks: $w \in \mathbb{R}^n_+$
- Realized interbank payments: $p \in \mathbb{R}^n_+$

## External assets

- Cash asset: $r \in \mathbb{R}^n_+$
- Illiquid asset: $s \in \mathbb{R}^n_+$ with price $q$
- Corresponding total value: $r_i + s_i q$, $i \in \mathcal{N}$

## Quantities computed in equilibrium

$$(\mathbf{w}, \mathbf{p}, \mathbf{q})$$

# Liabilities

- Nominal liabilities matrix: $L \in \mathbb{R}^{n \times n}$
  - $L_{ij} \geq 0$ describes the interbank obligation of bank $i$ to bank $j$
- Further liabilities: $l \in \mathbb{R}^n_+$
  - Liabilities to entities outside the banking system
- Vector of total liabilities: $\bar{p} \in \mathbb{R}^n_+$

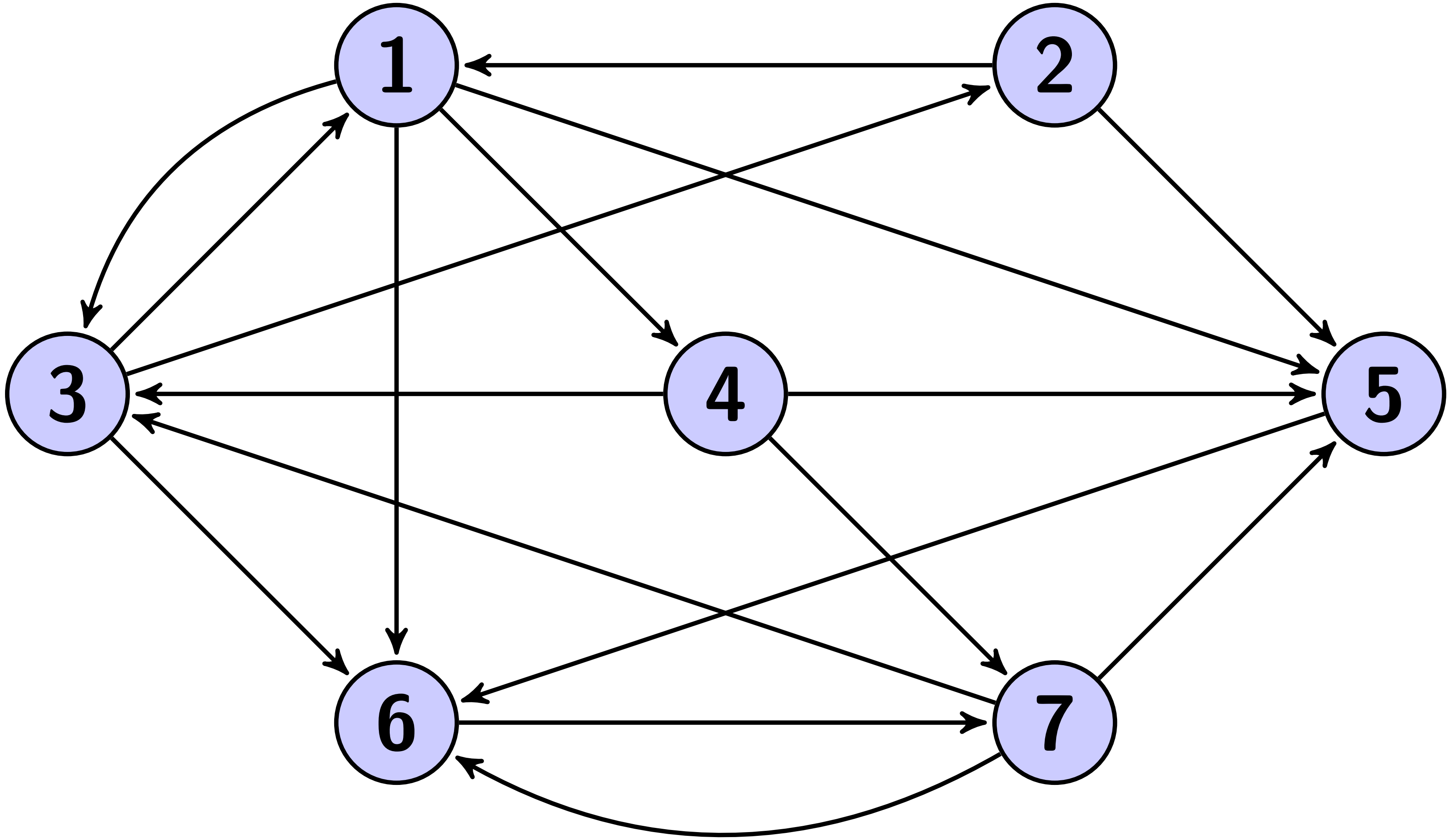
$$\bar{p}_i = \sum\nolimits_{j \in \mathcal{N}} L_{ij} + l_i, \qquad i \in \mathcal{N}.$$

- Realized payments: $p \in \mathbb{R}^n_+$ such that $p_i \leq \bar{p}_i$ for $i \in \mathcal{N}$
- Relative liabilities matrix: $\Pi \in \mathbb{R}^{n \times n}$ by

$$\Pi_{ij} = L_{ij}/\bar{p}_i, \qquad \text{if } \bar{p}_i > 0,$$

and $\Pi_{ij} = 0$, otherwise.

# Network of Liabilities



# Liabilities and Cross-Holdings

**Liabilities**

- Value: $\sum_{j \in \mathcal{N}} \Pi_{ji} p_j.$
  - Realized payments $p_i < \bar{p}_i$ will be distributed proportionally among its creditors according to the size of each creditor's claim

**Cross-holdings**

- Cross-holdings matrix: $C \in \mathbb{R}^{n \times n}$
  - The component $C_{ij}$ denotes the fraction of bank $i$ that is held by bank $j$ with $C_{ij} \geq 0$, $C_{ii} = 0$, $\sum_{j \in \mathcal{N}} C_{ij} < 1$
- Financial value of cross-holdings: $\sum_{j \in \mathcal{N}} C_{ji} w_j.$
- Haircut: We assume that over the time horizon of one period a fraction $\nu_i(p, w)$ of the cross-holdings can be exchanged against cash, but is subject to a haircut of $1 - \lambda$ .

# Financial Net Worths of Banks

Resources are used in the following order to pay for liabilities:

(i) Liquid asset and received liability payments

(ii) Cross-holdings exchanged against cash, subject to haircut

(iii) Illiquid asset

$\Longrightarrow$

- Setting $\mu_i(p,w) := \nu_i(p,w)\lambda + 1 - \nu_i(p,w)$, $i \in \mathcal{N}$, the total (inflated) net worth of bank $i$ is given by

$$\mathbf{w_i = r_i + s_i q + \sum_{j \in \mathcal{N}} \Pi_{ji} p_j + \mu_i(p,w) \sum_{j \in \mathcal{N}} C_{ji} \max(w_j, 0) - \bar{p}_i.}$$

- The bank is *in default*, if it cannot cover its liabilities, i.e. if $\boldsymbol{w_i < 0}$.

# Financial Net Worths of Banks (cont.)

For $p \in \mathbb{R}^n_+$, $q \in \mathbb{R}_+$, an **essential net worths vector** is a fixed-point vector $w^*(p,q) \in \mathbb{R}^n$ such that

$$\mathbf{w^*(p,q) = \Psi(w^*(p,q)),}$$

where the function $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathbf{\Psi(w) := r + sq + \Pi^T p + diag(\mu(p,w))C^T(w \vee 0) - \bar{p}.}$$

- Essential net worths vector always exists and is unique.
- The function $w^*$ is increasing in both $p$ and $q$.
- The sum of its components does not equal the “real value” of the banking system, but is inflated by cross-holdings; the “real value” can be computed as the sum of the holdings of outside equity investors. Adjusted case-studies are work in progress.

# Price of the Illiquid Asset

- Suppose now that only $p$ is exogenously given
- Inverse demand function: $q = f(\theta(p,q))$,
  - $\theta$ denotes the quantity of the illiquid asset that is liquidated in the market; the rest is marked to market
- Price is fixed-point: $q = f(\theta(p,q))$ such that

$$\theta(p,q) := \sum_{i \in \mathcal{N}} \min\left( \frac{\max[\bar{p}_i - r_i - \sum_{j \in \mathcal{N}} \Pi_{ji} p_j - \lambda \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^*(p,q),0),0]}{q}, s_i \right)$$

  - Banks fulfill as much of their obligations as they can by selling first the liquid asset and using received payments, then cross-holdings and afterwards selling the illiquid asset.

# Bankruptcy Costs

- Banks that cannot fulfill their total obligations are bankrupt
- Legal or administrative expenses may be incurred that reduce the amount of net worths that can be used to fulfill liabilities
- Following Rogers & Veraart (2013), we introduce two new parameters:
  - Frictional cost for external assets: $0 \leq 1 - \alpha \leq 1$
  - Frictional cost for internal assets: $0 \leq 1 - \beta \leq 1$

# Price-Payment Equilibrium

A *price-payment equilibrium* is a pair $(p^*, q^*) \in [\mathbf{0}, \bar{p}] \times [q_{\min}, q_0] \subseteq \mathbb{R}^{n+1}$, consisting of a *clearing payment vector* $p^*$ and a *clearing price* $q^*$, such that

$$(p^*, q^*) = \Phi(p^*, q^*), \tag{1}$$

where $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the function defined by

$$\Phi_i(p, q) := \begin{cases} \chi_i(p, q), & \text{for } i = 1, \ldots, n, \\ f(\theta(p, q)), & \text{for } i = n + 1, \end{cases}$$

$$\chi_i(p, q) := \begin{cases} \bar{p}_i, & \text{if } r_i + s_i q + \eta_i(p, q) \geq \bar{p}_i, \\ \alpha[r_i + s_i q] + \beta\,[\eta_i(p, q)]\,, & \text{otherwise,} \end{cases}$$

$$\eta_i(p, q) := \sum_{j \in \mathcal{N}} \Pi_{ji} p_j + \mu_i(p, q) \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^*(p, q), 0),$$

$$\mu_i(p, q) = \nu_i(p, q)\lambda + 1 - \nu_i(p, q),$$

$$\nu_i(p, q) = \min\left( \frac{\max(\bar{p}_i - r_i - \sum_{j \in \mathcal{N}} \Pi_{ji} p_j, 0)}{\lambda \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^*(p, q), 0)}, 1 \right),$$

$$\theta(p, q) := \sum_{i \in \mathcal{N}} \min\left( \frac{\max(\bar{p}_i - r_i - \sum_{j \in \mathcal{N}} \Pi_{ji} p_j - \lambda \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^*(p, q), 0), 0)}{q}, s_i \right).$$

# Price-Payment Equilibrium (2)

The price-payment equilibrium provides a solution concept for an *integrated financial system* which is characterized by

$$(\Pi, \bar{p}, r, s, \alpha, \beta, \lambda, C, f).$$

- Our integrated financial system admits a joint analysis of a network of liabilities, bankruptcy costs, cross-holdings, and fire sales as well as an analysis of models that incorporate only some of these effects.
- Namely, by choosing $\alpha = \beta = 1$, $s = \mathbf{0}$, or $C$ as the zero $n \times n$ matrix, we can simply exclude the corresponding extensions from our system. This shows that the models of e.g. Eisenberg & Noe (2001), Rogers & Veraart (2013), Cifuentes et al. (2005) and Elsinger (2009) are special cases of our integrated financial system.

# Price-Payment Equilibrium – Existence

**Theorem 1** *There exist a unique largest and a unique smallest price-payment equilibrium, $(p^+, q^+)$ and $(p^-, q^-)$.*

*That is, $(p^+, q^+)$ and $(p^-, q^-)$ are price-payment equilibria and for every price-payment equilibrium $(p^*, q^*)$:*

$$(p^-, q^-) \leq (p^*, q^*) \leq (p^+, q^+).$$

- Awiszus & W. (2014) extend the algorithm of Rogers & Veeraart (2013) to this case. The largest and smallest price-payment equilibrium can be computed in at most $n + 1$ iterations of the respective algorithm.
- The set of equilibria is not necessarily connected.

# Numerical Examples: Number of Defaults

# Simulation Parameters

Letting $f(x) = \exp(-\gamma x)$, the integrated financial system is characterized by

$$(\Pi, \bar{p}, r, s, \alpha, \beta, \lambda, C, \gamma).$$

- $\Pi$ will be generated according to a random mechanism encoded by parameters $c_\Pi$ and $d_\Pi$
- $r$ and $s$ will depend on $\Pi$ according to parameters $\delta$ and $\rho$
- $C$ will be generated according to a random mechanism encoded by parameters $c$ and $d$

Setting $n = 100$ and $\bar{p} = \mathbf{1}$, the following parameters govern the simulation model:

$$(c_\Pi, d_\Pi, \delta, \rho, \alpha, \beta, \lambda, c, d, \gamma).$$

# Relative Liabilities Matrix $\Pi$

$\Pi$ will not be specified as a deterministic quantity, but is simulated randomly on the basis of Erdös-Rényi random networks:

- level of integration $c_\Pi \in [0, 1]$,
- level of diversification $d_\Pi \in [0, n-1]$

(i) Construct an adjacency matrix $A \in \mathbb{R}^{n \times n}$ by letting $A_{ij}$, $i \neq j \in \mathcal{N}$, be i.i.d. Bernoulli random variables, taking the value 1 with probability $d_\Pi/(n-1)$, and 0 with probability $1 - d_\Pi/(n-1)$. Set $A_{ii} = 0$ for all $i \in \mathcal{N}$.

(ii) For all banks $i \in \mathcal{N}$, set $d_i^{out} = \sum_{j \in \mathcal{N}} A_{ij}$, and let

$$\Pi_{ij} = c_\Pi / d_i^{out}$$

if $A_{ij} = 1$, otherwise $\Pi_{ij} = 0$ for all $j \in \mathcal{N}$.

# External Assets

Given parameters $\delta$ and $\rho$, $r$ and $s$ are calculated as follows:

(i) Compute the random vector of the minimal value of assets that are necessary in order to keep the banks from defaulting (not considering cross-holdings): $h := (\bar{p} - \Pi^T \bar{p}) \vee \mathbf{0}$.

(ii) Given a capital buffer $\delta > 0$, set the overall external assets to $e := (1 + \delta)h$.

(iii) Given a proportion $\rho \in [0, 1]$ of the illiquid asset, let $r = (1 - \rho)e$ and $s = \rho e$.

# Cross-Holdings Matrix $C$

$C$ is specified according to the same mechanism as $\Pi$ with parameters:

- level of integration $c \in [0, 1)$,
  - i.e. the fraction of net worth that banks sell as cross-holdings to other banks;
- level of diversification $d \in [0, n-1]$,
  - i.e. the expected number of shareholders within the interbank market.

# Simulation Methodology

(i) Fix the parameters of the model.

(ii) $\Pi$ and $C$ are randomly sampled.

(iii) The derived random quantities $r$ and $s$ are computed from the samples.

(iv) One bank $i \in \mathcal{N}$ is uniformly sampled at random; its external asset holdings $r_i$ and $s_i$ are set to zero. This corresponds to a local shock to a single bank.

(v) For the resulting scenario, the greatest price-payment equilibrium and the corresponding number of defaulting firms is calculated.

(vi) The simulation is repeated a large number of times, and sample averages and standard deviations are computed.

# Remarks

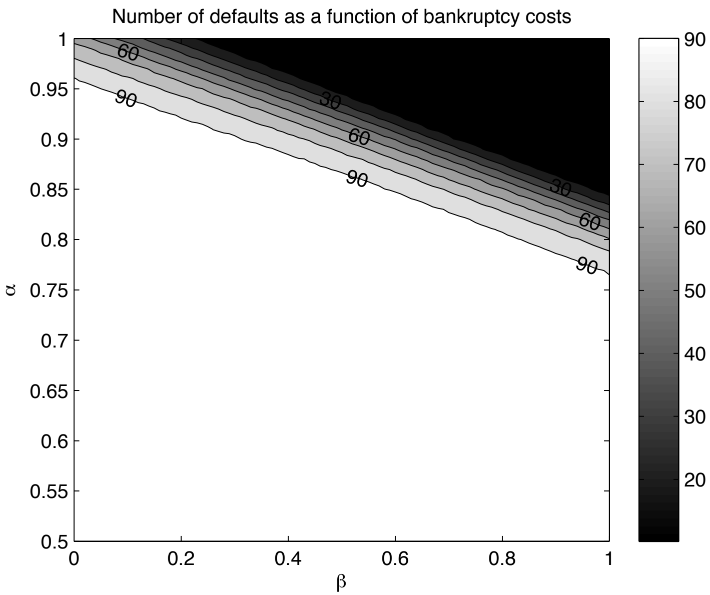
- The qualitative effect of varying $\lambda$ is similar to the effect that can be observed when $c$ is varied. For this reason, we focus on $\lambda = 1$.
- Our paper also investigates core-periphery networks of liabilities with parameter choices derived from real-world data. These provide a realistic financial network structure.

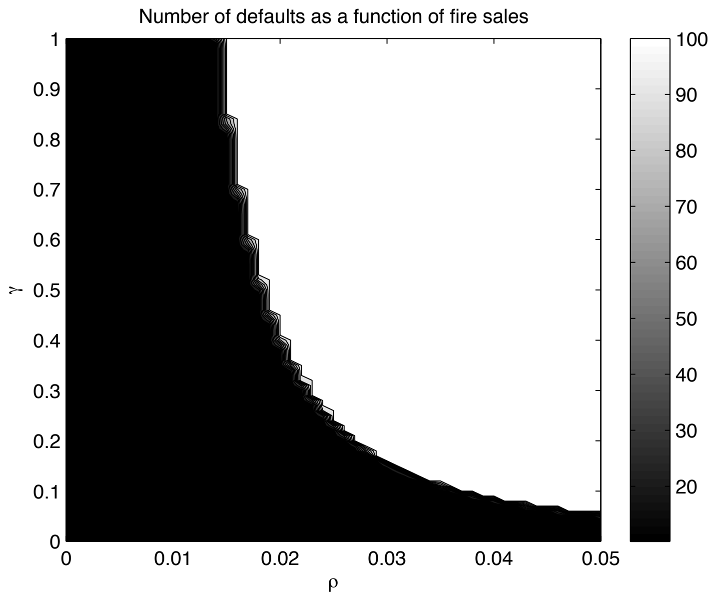
  $\Longrightarrow$

  The qualitative effects are still very similar to those observed for the Erdös-Rényi random networks.

  For simplicity, we thus focus in this talk on case studies in these simpler networks.

# Separate Effects:<br>Bankruptcy Costs and Fire Sales




Contour plots of the number of defaults for $n = 100$ banks as a function of (a) bankruptcy costs and (b) fire sales, averaged over 1000 simulations of $\Pi$.

# Separate Effects: Bankruptcy Costs and Fire Sales (2)

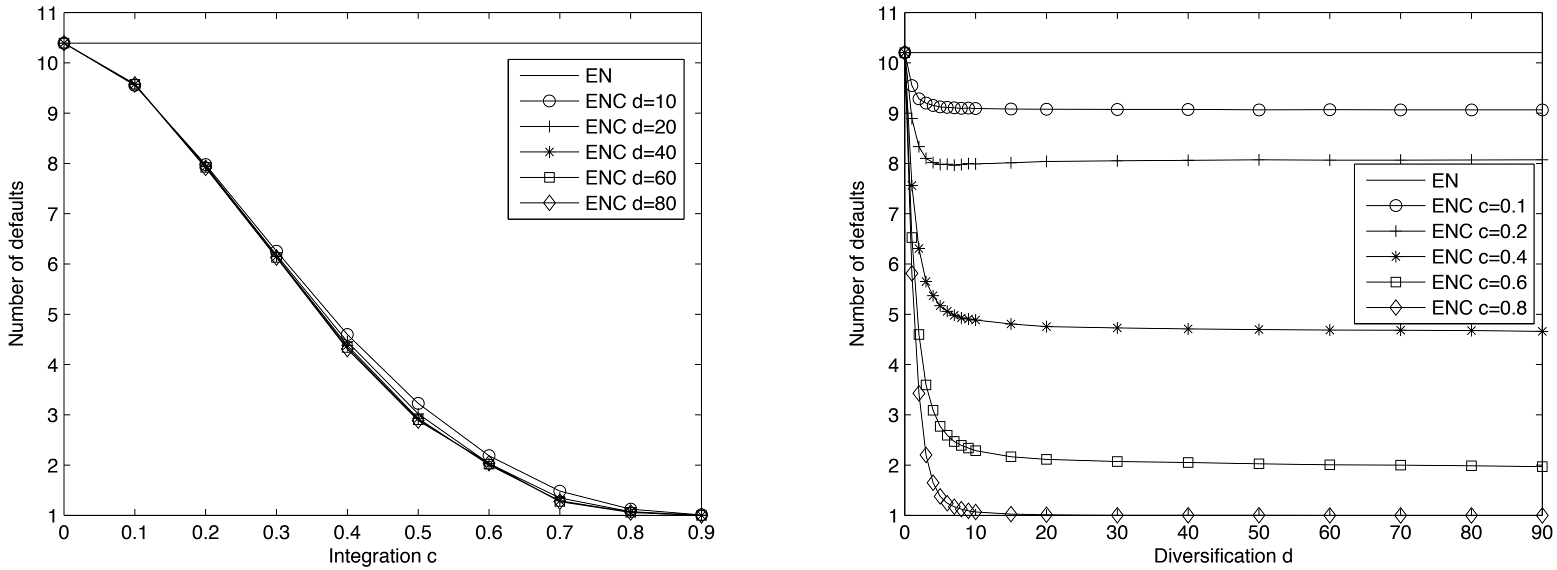
The threshold curve can approximately be described by the following power-law function:

$$\rho = \exp(-4.3183) \cdot \gamma^{-0.4528}.$$

$\Longrightarrow$

For each $\gamma$ characterizing the inverse demand function a corresponding proportion $\rho$ of the illiquid asset can be computed, beyond which the breakdown of the system occurs.

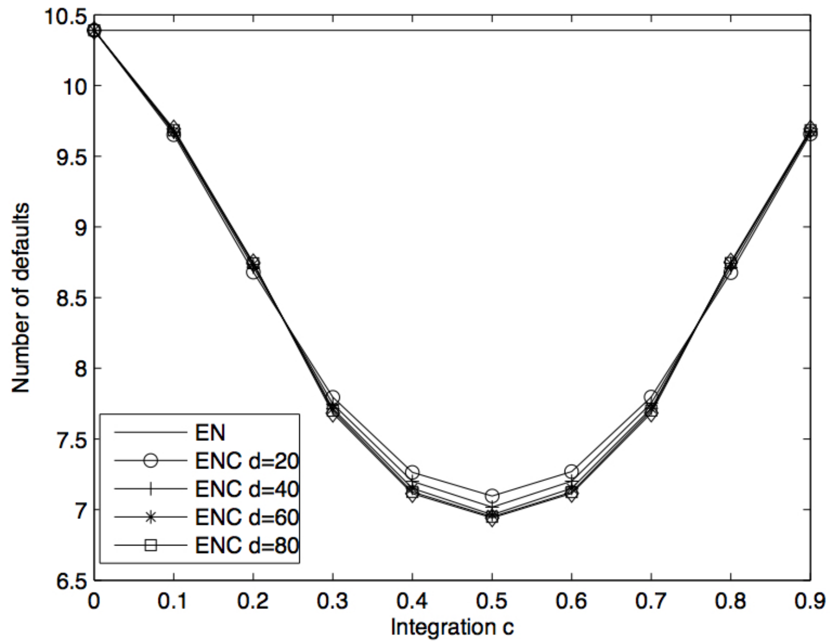
# Separate Effects: Cross-Holdings




Number of defaults for $n = 100$ banks as a function of (a) integration and (b) diversification of the cross-holdings matrix $C$, averaged over 100 simulations of $\Pi$, each averaged over 100 simulations of $C$.

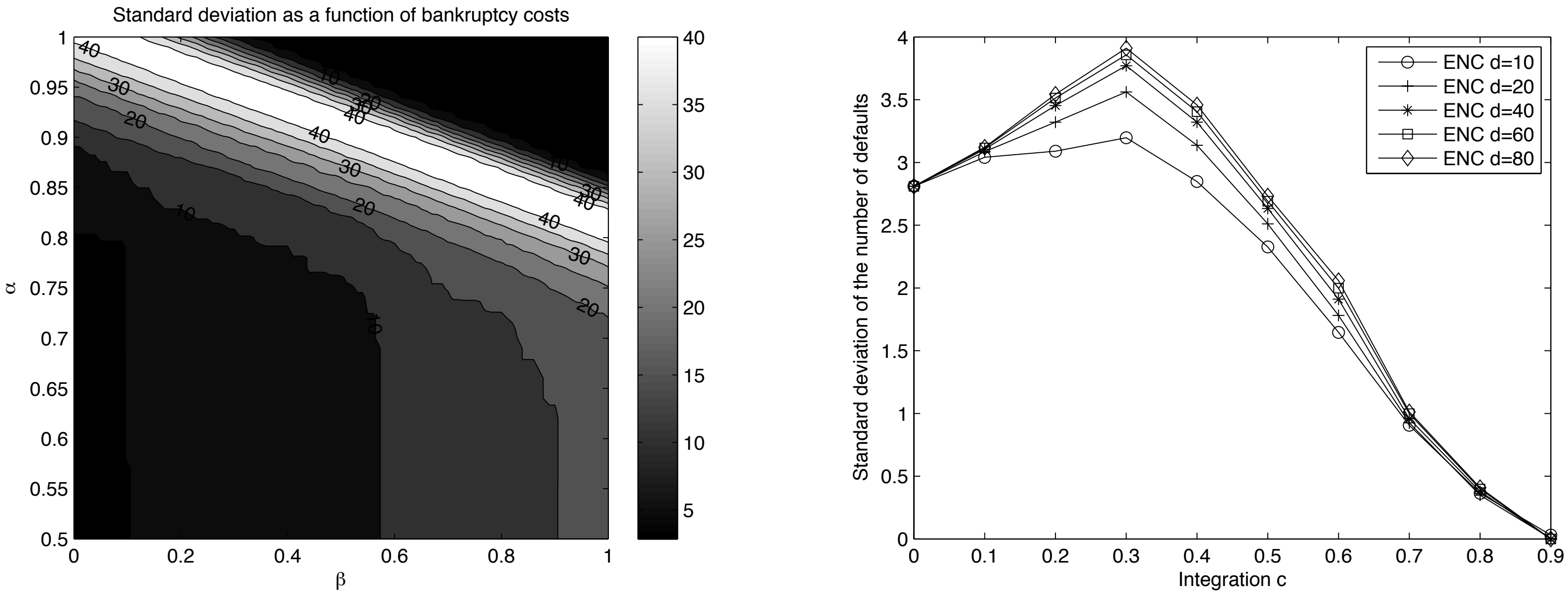
# Separate Effects: Cross-Holdings (2)

## Preliminary results on market value of cross-holdings



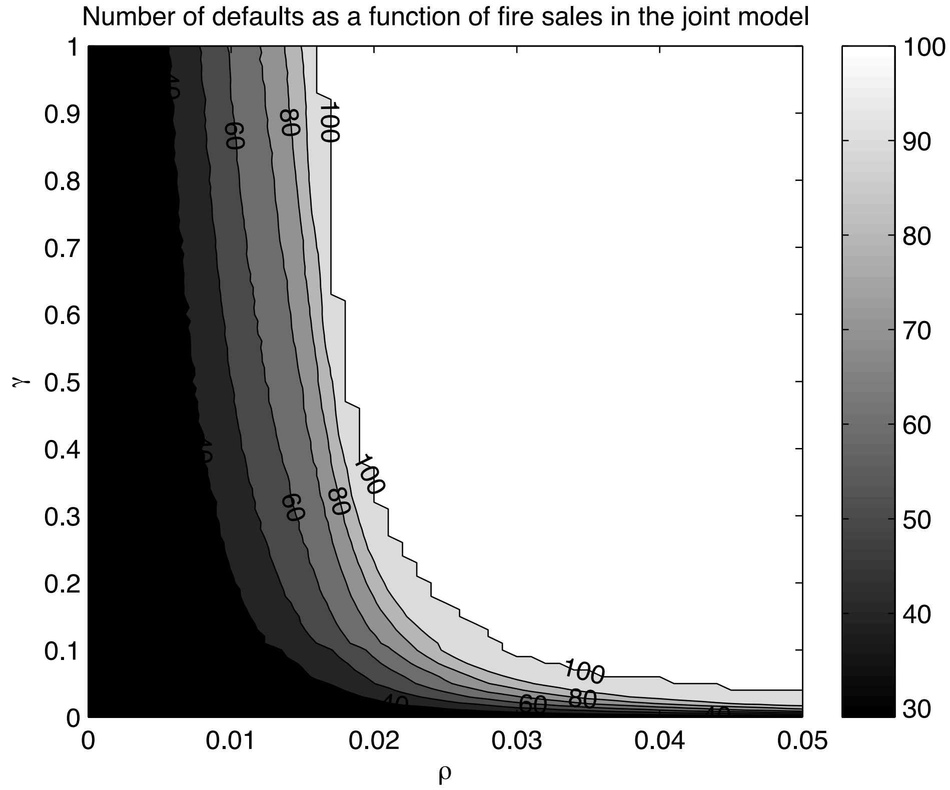

Number of defaults for $n = 100$ banks as a function of integration in a modified model, averaged over 100 simulations of $\Pi$, each averaged over 100 simulations of $C$.

# Separate Effects: Standard Deviation



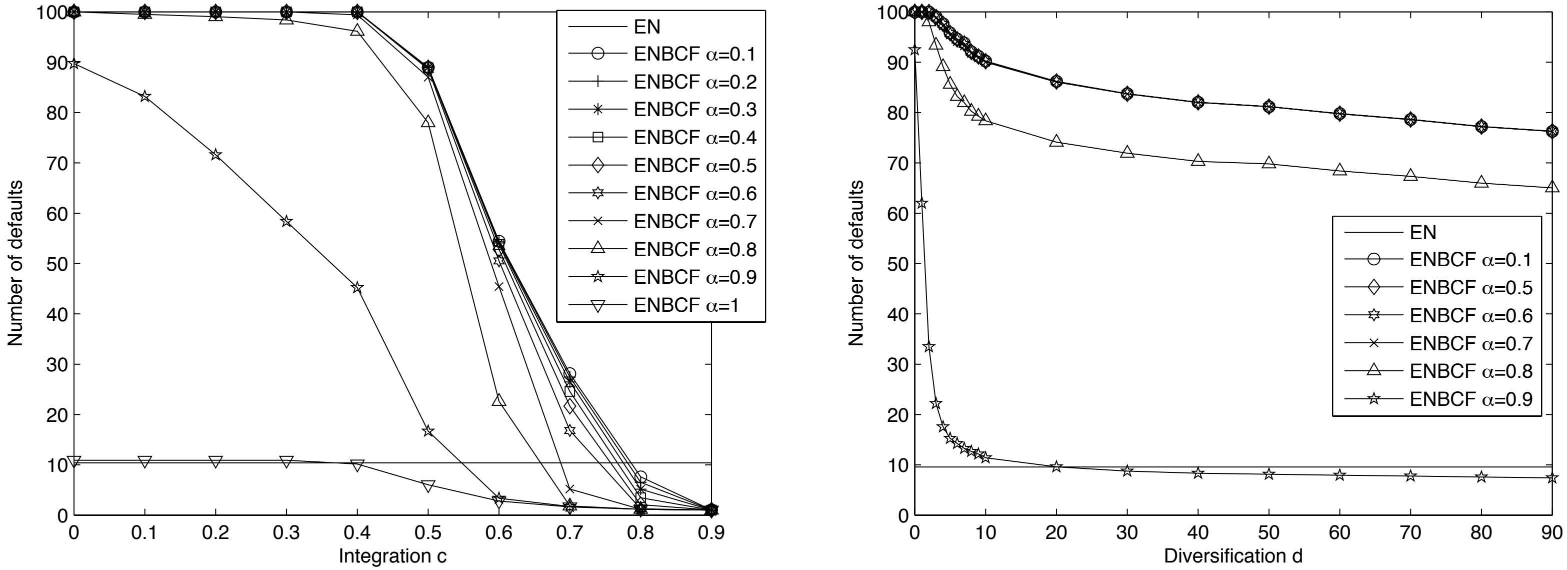

Standard deviation of the number of defaults for $n = 100$ banks (a) as a function of both parameters of bankruptcy costs for 1000 simulations of $\Pi$, and (b) as a function of integration of the cross-holdings matrix $C$ for 100 simulations of $\Pi$, each averaged over 100 simulations of $C$.

# Joint Effects: ENBCF



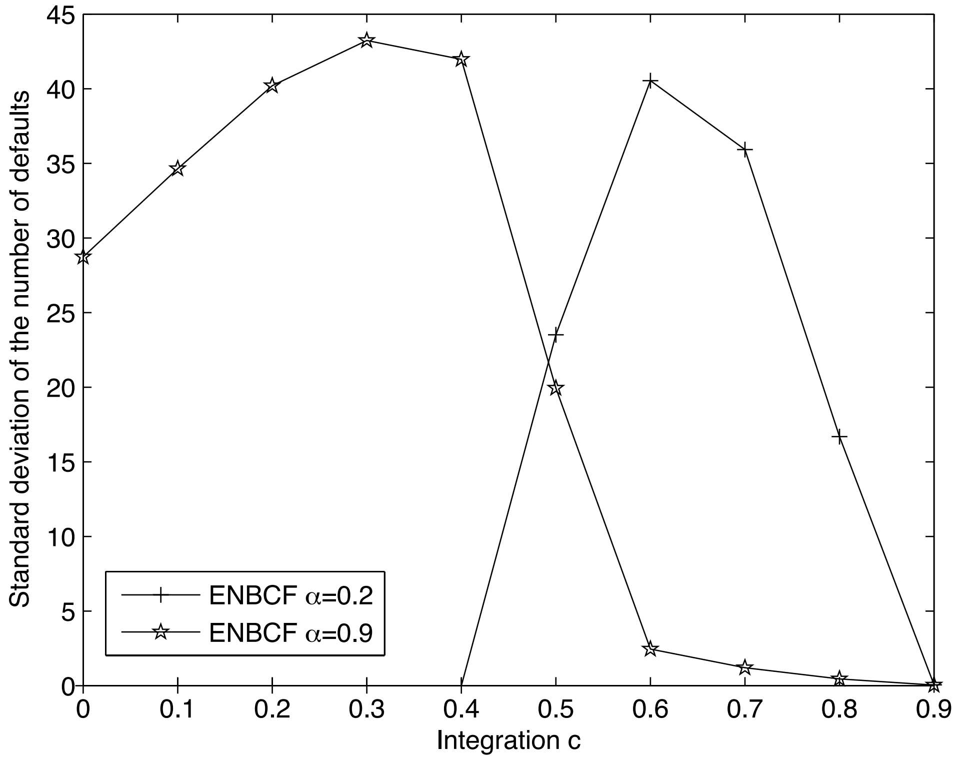

Contour plot of the number of defaults for $n = 100$ banks as a function of both parameters of fire sales, $\rho$ and $\gamma$, for $\alpha = 0.7$, $\beta = 0.9$, $c = 0.5$, $d = 10$, averaged over 1000 simulations of $\Pi$.

# Joint Effects: ENBCF (2)




Average number of defaults for $n = 100$ banks (a) as a function of integration $c$ for $d = 10$, and (b) as a function of diversification $d$ for $c = 0.5$ of the cross-holdings matrix $C$ for different recovery rates $\alpha$ in an integrated financial system with $\beta = 0.9$, $\gamma = 0.2$, $\rho = 0.02$, averaged over 100 simulations of $\Pi$, each averaged over 100 simulations of $C$.

# Joint Effects: ENBCF (3)




Standard deviation of the number of defaults for $n = 100$ banks as a function of integration $c$ of the cross-holdings matrix $C$ for $\alpha \in \{0.2, 0.9\}$, $d = 10$, $\beta = 0.9$, $\gamma = 0.2$, $\rho = 0.02$, for 100 simulations of $\Pi$, each averaged over 100 simulations of $C$.

# Conclusion

- Bankruptcy costs increase the risk of contagion in a non-linear way.
- The introduction of an illiquid external asset (i.e. fire sales) increases the risk of contagion.
  - Dependence is non-linear and exhibits a threshold boundary.
  - The critical boundary can approximately be described by a power law.
  - The transition region becomes larger if bankruptcy costs and cross-holdings are present.

# Conclusion (2)

- Cross-holdings reduce the risk of contagion, if they can be exchanged against the liquid asset.
  - They inflate the total net worth of the financial system and are effectively a mechanism to increase capital by providing outside investors with shares of equity.
  - Increasing diversification of cross-holdings alone can only decrease the average number of defaults to a certain level.
  - Increasing integration can significantly mitigate contagion risk in our model.
  - Increasing diversification is most efficient for low levels of diversification. If no bankruptcy costs are present, increasing diversification beyond a certain threshold does almost not have an impact anymore on the average number of defaults.

# Conclusion (3)

- Random fluctuations around averages, quantified by the standard deviation of the number of defaults,
  - are typically stronger in regimes with a medium size average number of defaults, indicating significant systemic risk.
  - In contrast, the standard deviation is small for either a very large or a very small average number of defaults.

# Further Research

(i) Adjusted net worth computation without inflating effect of cross-holdings (work in progress)

(ii) Analysis of the role and design of CCPs in financial markets

(iii) Shocks to multiple institutions, deterministic or random

(iv) Other metrics like systemic risk measures,

see e.g. Chen, Iyengar & Moallemi (2013), Kromer, Overbeck & Zilch (2014), Hoffmann, Meyer-Brandis & Svindland (2014), Brunnermeier & Cheridito (2013),

and the recent work on multi-variate systemic risk measures by Feinstein, Rudloff & W. (2014)