A unified approach to systemic risk measures via acceptance set

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Outline

- The classical set up:
  - Coherent and Convex risk measures
  - Degree of acceptability (Quasi-Convexity)
  - General Capital Requirements
- Systemic Risk Measures (SRM)
  - Aggregation functions
  - First aggregate, second add capital

Outline

- In our approach
  - We contemplate both
    - first aggregate and second add capital
    - first add capital and second aggregate
  - We not only allow adding cash but we permit adding random capital
  - We employ multidimensional acceptance sets
  - We allow for degrees of acceptability
- Definitions and properties
- Some classes of SRM in our approach
- Applications
  - Random cash allocation
  - Gaussian financial system
  - Model of borrowing and lending
  - Finite probability space model

Monetary Risk Measures

A monetary risk measure is a map

$$\eta : \mathcal{L}^0(\mathbb{R}) \to \mathbb{R}$$

that can be interpreted as the **minimal capital needed to secure a financial position** with payoff $X \in \mathcal{L}^0(\mathbb{R})$, i.e. the minimal amount $m \in \mathbb{R}$ that must be added to X in order to make the resulting payoff at time T acceptable:

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m \in \mathbb{A}\},$$

where the acceptance set

$$\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$$

is assumed to be monotone, i.e.

$$X \geq Y \in \mathbb{A} \quad \text{implies} \quad X \in \mathbb{A}.$$

Coherent / Convex Risk Measures

Artzner, Delbaen, Eber and Heath (1999); Föllmer and Schied (2002); F. and Rosazza Gianin (2002)

In addition to decreasing monotonicity, the characterizing feature of these monetary maps is the **cash additivity** property:

$$\eta(X+m)=\eta(X)-m, \quad \text{for all } m\in\mathbb{R}.$$

Under the assumption that the set $\mathbb{A}$ is convex (resp. is a convex cone) the maps

$$\eta(X):=\inf\{m\in\mathbb{R}\mid X+m\in\mathbb{A}\},$$

are convex (resp. convex and positively homogeneous) and are called **convex (resp. coherent) risk measures**.

Diversification

The principle that diversification should not increase the risk is mathematically translated not necessarily with the convexity property but with the weaker condition of **quasiconvexity**:

$$\eta(\lambda X + (1-\lambda)Y) \leq \eta(X) \vee \eta(Y).$$

As a result in Cerreia-Vioglio, Maccheroni, Marinacci Montrucchio 2010, the only properties assumed in the definition of a **quasi-convex risk measure** are *decreasing monotonicity and quasiconvexity*.

Such quasi-convex risk measures can always be written as:

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X \in \mathbb{A}^m\}, \tag{1}$$

where each set $\mathbb{A}^m \subseteq \mathcal{L}^0(\mathbb{R})$ is monotone and convex, for each $m \in \mathbb{R}$.

- $\mathbb{A}^m$ is the class of payoffs carrying the same risk level $m \in \mathbb{R}$.

Contrary to the convex cash additive case, where each random variable is binary cataloged as acceptable or as not acceptable, in the quasi-convex case one **admits various degrees of acceptability, described by the risk level** m

- By selecting $\mathbb{A}^m := \mathbb{A} - m$, the convex cash additive risk measure is clearly a particular case of the one in (1).

General Capital Requirement

F. and Scandolo 2006

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m1 \in \mathbb{A}\}, \quad \mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$$

Why should we consider **only** "money" as safe capital ?

One should be more liberal and **permit the use of other financial assets** (other than the bond $:= 1$), in an appropriate set $\mathcal{C}$ of *safe* instruments, **to hedge the position** X.

Definition

The general capital requirement is

$$\eta(X) \triangleq \inf\{\pi(Y) \in \mathbb{R} \mid Y \in \mathcal{C},\, X + Y \in \mathbb{A}\},$$

for some evaluation functional $\pi : \mathcal{C} \to \mathbb{R}$.

This is exactly the approach taken by F-Scandolo 2006.

Systemic Risk Measure

Consider a system of N interacting financial institutions and a vector $\mathbf{X} = (X^1, \ldots, X^N) \in \mathcal{L}^0(\mathbb{R}^N) := \mathcal{L}^0(\Omega, \mathcal{F}; \mathbb{R}^N)$ of associated risk factors (financial positions) at a given future time horizon T.

- In this paper we are interested in **real-valued** systemic risk measures:

$$\rho : \mathcal{L}^0(\mathbb{R}^{\mathbb{N}}) \to \overline{\mathbb{R}}$$

  that evaluates the risk $\rho(\mathbf{X})$ of the complete financial system $\mathbf{X}$.

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  that evaluates the risk $\rho(\mathbf{X})$ of the complete financial system $\mathbf{X}$.
- Many of the SRM in the existing literature are of the form

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})),$$

  where $\eta : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}}$ is a univariate risk measure and

$$\Lambda : \mathbb{R}^N \to \mathbb{R}$$

  is an aggregation rule that aggregates the N-dimensional risk factor $\mathbf{X}$ into a univariate risk factor $\Lambda(\mathbf{X})$ representing the total risk in the system.

Examples of aggregation rule

- $\Lambda(\mathbf{x}) = \sum_{i=1}^{N} x_i, \quad \mathbf{x} = (x_1, ..., x_N) \in \mathbb{R}^N$.
- $$\Lambda(\mathbf{x}) = \sum_{i=1}^{N} -x_i^- \quad \text{or} \quad \Lambda(\mathbf{x}) = \sum_{i=1}^{N} -(x_i - d_i)^-, \; d_i \in \mathbb{R}$$
  takes into account the lack of cross-subsidization between financial institutions
- Eisenberg and Noe: accounts for counterparty contagion.
- $$\Lambda(\mathbf{x}) = \sum_{i=1}^{N} -\exp\left(-\alpha_i x_i\right), \; \alpha_i \in \mathbb{R}_+$$
  $$\Lambda(\mathbf{x}) = \sum_{i=1}^{N} -\exp\left(-\alpha_i x_i^-\right), \; \alpha_i \in \mathbb{R}_+$$

First aggregate, second add capital

cash additive case

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})),$$

If η is a convex (cash additive) risk measure then we can rewrite such ρ as

$$\rho(\mathbf{X}) \triangleq \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A}\}. \quad (2)$$

The SRM is the minimal capital needed to secure the system **after aggregating individual risks**

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If $\Lambda(\mathbf{X})$ is not capital, the risk measure in (2) is some general risk level of the system.

- For an axiomatic approach for this type of SRM see Chen Iyengar Moallemi 2013 and Kromer Overbeck Zilch 2013 and the references therein. Acharya et al. 2010, Adrian Brunnermeier 2011, Cheridito Brunnermeier 2014, Gauthier Lehar Souissi 2010, Hoffmann Meyer-Brandis Svindland 2014, Huang Zhou Zhu 2009, Lehar 2005.

First aggregate, second add capital

The quasi-convex case

Similarly, if η is a quasi-convex risk measure the systemic risk measure then

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X}))$$

can be rewritten as

$$\rho(\mathbf{X}) \triangleq \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) \in \mathbb{A}^m\}.$$

Again one first aggregates the risk factors via the function Λ and in a second step one computes the minimal risk level associated to the total system risk $\Lambda(\mathbf{X})$.

Purpose of this paper

- The purpose of this paper is to specify a general methodological framework that is flexible enough to cover a wide range of possibilities to design systemic risk measures via acceptance sets and aggregation functions and to study corresponding examples.
- We extend the conceptual framework for systemic risk measures via acceptance sets step by step in order to gradually include certain novel key features of our approach.

Key feature 1: First add capital, second aggregate

- A regulator might have the possibility to intervene on the level of the single institutions before contagion effects generate further losses.
- Then it might be more relevant to measure systemic risk as the **minimal capital that secures the aggregated system by injecting the capital into the single institutions before aggregating the individual risks.**

First add capital, second aggregate

$$\rho(\mathbf{X}) \triangleq \inf\{\sum_{i=1}^{N} m_i \in \mathbb{R} \mid \mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N;\ \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}\}\,.$$

- The amount m_i is added to the financial position X^i before the corresponding total loss $\Lambda(\mathbf{X} + \mathbf{m})$ is computed.
- The systemic risk is the minimal total capital $\sum_{i=1}^{N} m_i$ injected into the institutions to secure the system.

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- The systemic risk is the minimal total capital $\sum_{i=1}^{N} m_i$ injected into the institutions to secure the system.
- ρ delivers at the same time a measure of total systemic risk and a potential **ranking** of the institutions in terms of systemic riskiness.

Suppose $\rho(\mathbf{X}) = \sum_{i=1}^{N} m_i^*$.
Then one could argue that the risk factor X_{i_1} that requires the biggest capital allocation $m_{i_1}^*$ corresponds to the riskiest institution, and so on. When the allocation $\mathbf{m}^*$ is not unique, one has to discuss criteria that justify the choice of a specific allocation.

Key feature 2: Random allocation

We allow for the possibility of adding to $\mathbf{X}$ not merely a vector $\mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N$ of cash but a **random vector**

$$\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{L}^0(\mathbb{R}^N)$$

which represents, in the spirit of F. and Scandolo, admissible financial assets that can be used to secure a system by adding $\mathbf{Y}$ to $\mathbf{X}$ component-wise.

To each $\mathbf{Y} \in \mathcal{C}$ we assign a measure $\pi(\mathbf{Y})$ of the cost associated to $\mathbf{Y}$ determined by a monotone increasing map

$$\pi : \mathcal{C} \to \mathbb{R}$$

Hence:

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C};\, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\}.$$

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- If $\rho(\mathbf{X}) = \pi(\mathbf{Y}^*)$ and $\mathbf{Y}^* = (Y_1^*, ..., Y_N^*) \in \mathcal{C}$ is optimal, an ordering of the $Y_1^*, ..., Y_N^*$ may induce a systemic ranking of the institution $(X_1, ..., X_N)$.

Particular case: lender of last resort

$$\mathcal{C} \subseteq \{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\} =: \mathcal{C}_{\mathbb{R}},$$

and set $\pi(\mathbf{Y}) = \sum_{n=1}^{N} Y^n$.

$$\rho(\mathbf{X}) \triangleq \inf\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C};\ \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}\}$$

is the minimal total cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed today to secure the system by distributing the capital at time T among $(X^1, ..., X^N)$.

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is the minimal total cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed today to secure the system by distributing the capital at time T among $(X^1, ..., X^N)$.

- In general the allocation $Y^i(\omega)$ to institution i does not need to be decided today but depends on the scenario ω realized at time T.
- For $\mathcal{C} = \mathbb{R}^N$ the situation corresponds to the previous case where the distribution is already determined today.
- For $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ the distribution can be chosen freely depending on the scenario ω realized in T (including negative amounts, i.e. withdrawals of cash from certain components).

Dependence can be taken into account

Allowing **random** allocations of cash $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ the systemic risk measure will take the dependence structure of the components of $\mathbf{X}$ into account even though acceptable positions might be defined in terms of the marginal distributions of X_i, $i = 1, ..., N$, only.

Example: dependence may be taken into account

$\Lambda(\mathbf{x}) := \sum_{i=1}^{N} -x_i^-$, $\mathbf{x} \in \mathbb{R}^N$,
$\mathbb{A} := \{Z \in \mathcal{L}^0(\mathbb{R}) \mid E[Z] \geq \gamma\}, \gamma \in \mathbb{R}$.
$\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N)$ is acceptable if and only if $\Lambda(\mathbf{Z}) \in \mathbb{A}$, i.e.

$$\sum_{i=1}^{N} -E[Z_i^-] \geq \gamma,$$

which only depends on the marginal distributions of $\mathbf{Z}$.

- If $\mathcal{C} = \mathbb{R}^N$ then $\rho(\mathbf{X})$ will depends on the marginal distributions of $\mathbf{X}$ only.
- If one allows for more general allocations of cash $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ that might differ from scenario to scenario the systemic risk measure will in general depend on the multivariate distribution of $\mathbf{X}$ since it can play on the dependence of the components of $\mathbf{X}$ to minimize the costs.

Multidimensional acceptance set: an example

Consider single univariate monetary risk measures η_i, $i = 1, ..., N$, and set

$$\rho(\mathbf{X}) := \sum_{i=1}^{N} \eta_i(X_i) \, .$$

In general, this systemic risk measure **cannot** be expressed in the form

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C};\ \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\} \, .$$

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If $\mathbb{A}_i \subseteq \mathcal{L}^0(\mathbb{R})$ is the acceptance set of η_i, $i = 1, ..., N$, then it is possible to write such $\rho = \sum_{i=1}^{N} \eta_i(X_i)$ in terms of the multivariate acceptance set $\mathbb{A}_1 \times ... \times \mathbb{A}_N$:

$$\rho(\mathbf{X}) \triangleq \inf\{\sum_{i=1}^{N} m_i \mid \mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N , \mathbf{X} + \mathbf{m} \in \mathbb{A}_1 \times ... \times \mathbb{A}_N\} .$$

Key feature 3: General multidimensional acceptance set

We extend our formulation of systemic risk measures as the minimal cost of admissible asset vectors $\mathbf{Y} \in \mathcal{C}$ that, when added to the vector of financial positions $\mathbf{X}$, makes the augmented financial positions $\mathbf{X} + \mathbf{Y}$ acceptable in terms of a general multidimensional acceptance set

$$\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$$

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C};\, \mathbf{X} + \mathbf{Y} \in \mathcal{A}\}\,. \tag{3}$$

Our previous definition of systemic risk measure

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C};\, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\}\,.$$

is therefore a particular case of (3): just let

$$\mathcal{A} := \left\{\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}) \in \mathbb{A}\right\}$$

Generalized cash invariance

The systemic risk measure

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}; \mathbf{X} + \mathbf{Y} \in \mathcal{A}\}$$

exhibit an extended type of cash invariance:

$$\rho(\mathbf{X} + \mathbf{Y}) = \rho(\mathbf{X}) + \pi(\mathbf{Y})$$

for $\mathbf{Y} \in \mathcal{C}$ such that $\mathbf{Y}' \pm \mathbf{Y} \in \mathcal{C}$ for all $\mathbf{Y}' \in \mathcal{C}$.

Recall the one dimensional case:

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X \in \mathbb{A}^m\}$$

Our general systemic risk measure is defined by:

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\}. \tag{4}$$

We associate to each $\mathbf{Y} \in \mathcal{C}$ a set $\mathcal{A}^{\mathbf{Y}} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ of risk vectors that are acceptable for the given (random) vector $\mathbf{Y}$.
Then $\rho(\mathbf{X})$ represents some minimal aggregated risk level $\pi(\mathbf{Y})$ at which the system $\mathbf{X}$ is acceptable.

General form of Systemic Risk Measure

Key feature 4: Degree of acceptability

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Then $\rho(\mathbf{X})$ represents some minimal aggregated risk level $\pi(\mathbf{Y})$ at which the system $\mathbf{X}$ is acceptable.
The approach in (4) is very flexible and **it includes all previous cases** if we set

$$\mathcal{A}^{\mathbf{Y}} := \mathcal{A} - \mathbf{Y},$$

where the set $\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ represents acceptable risk vectors.

General aggregation functional

Another advantage of the formulation in terms of general acceptance sets $\mathcal{A}^{\mathbf{Y}}$ is the possibility to design systemic risk measures via general aggregation rules. Indeed our last formulation includes the case

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\ \Theta(\mathbf{X},\mathbf{Y}) \in \mathbb{A}\}$$

where $\Theta : \mathcal{L}^0(\mathbb{R}^N) \times \mathcal{C} \to \mathcal{L}^0(\mathbb{R})$ denotes some aggregation function jointly in $\mathbf{X}$ and $\mathbf{Y}$. Just select

$$\mathcal{A}^{\mathbf{Y}} := \left\{ \mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Theta(\mathbf{Z},\mathbf{Y}) \in \mathbb{A} \right\}$$

This formulation includes

- **aggregation before injecting capital** by putting $\Theta(\mathbf{X},\mathbf{Y}) := \Lambda_1(\mathbf{X}) + \Lambda_2(\mathbf{Y})$, where $\Lambda_1 : \mathcal{L}^0(\mathbb{R}^N) \to \mathcal{L}^0(\mathbb{R})$ is an aggregation function and $\Lambda_2 : \mathcal{C} \to \mathcal{L}^0(\mathbb{R})$ could be, for example, $\Lambda_2 = \pi$.
- **injecting capital before aggregation** by putting $\Theta(\mathbf{X},\mathbf{Y}) = \Lambda(\mathbf{X} + \mathbf{Y})$,

Summarizing our approach

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\}\,.$$

- We contemplate both
  - first aggregate and second add capital
  - first add capital and second aggregate
- We not only allow adding cash but we permit adding random capital
- We employ multidimensional acceptance sets
- We allow for degrees of acceptability

Structure of the paper

1. We provide **sufficient condition** on π, $\mathcal{C}$, $\mathcal{A}^{\mathbf{Y}}$ so that the Systemic Risk Measure (SRM)

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\}$$

is **a well defined monotone decreasing and quasi-convex (or convex) map** $\rho : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \overline{\mathbb{R}}$.

2. We provide **four classes of Systemic Risk Measures**, satisfying the conditions in item (1) above, defined via one dimensional **acceptance sets** and **aggregation functions** $\Lambda : \mathcal{L}^0(\mathbb{R}^N) \times \mathcal{C} \rightarrow \mathcal{L}^0(\mathbb{R})$.

3. We study the particular case of **random cash allocations,** i.e. when:

$$\mathcal{C} \subseteq \{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\} =: \mathcal{C}_{\mathbb{R}}.$$

Structure of the paper

4 For a Gaussian financial system, i.e. if $\mathbf{X} \sim N(\mu, Q)$, we study the SRM

$$\rho(\mathbf{X}) \; : \; = \inf\left\{ \sum_{i=1}^{N} Y_i \mid \mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \,,\; \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}_{\gamma} \right\},$$

$$\Lambda(\mathbf{X}) \; : \; = \sum_{i=1}^{N} -(X_i - d_i)^{-},\; d_i \in \mathbb{R},$$

$$\mathbb{A}_{\gamma} \; : \; = \left\{ Z \in \mathcal{L}^0(\mathbb{R}) | E[Z] \geq -\gamma \right\},\; \gamma \in \mathbb{R}_{+}.$$

in the two cases: $\mathcal{C} := \mathbb{R}^{\mathbb{N}}$, and

$$\mathcal{C} := \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^n) | \mathbf{Y} = \mathbf{m} + \boldsymbol{\alpha} I_D,\; \mathbf{m}, \boldsymbol{\alpha} \in \mathbb{R}^N, \sum_{i=1}^{N} \alpha_i = 0 \right\} \subseteq \mathcal{C}_{\mathbb{R}},$$

where $D := \left\{ \sum_{i=1}^{n} X_i \leq d \right\}$, for some $d \in \mathbb{R}$

5 **Model of borrowing and lending**: We apply the above risk measure in case the Gaussian financial vector $\mathbf{X}_t$ is generated by the following dynamics

$$dX_t^i = \left[\sum_{j=1}^{N} p_{i,j} (X_t^j - X_t^i) \right] dt + \sigma^i \left(\rho_i dW_t^0 + \sqrt{1-\rho_i^2} dW_t^i \right),$$

as defined in Carmona Fouque Sun (2015). We analyze:

1. A fully connected symmetric network:
2. A central clearing symmetric network

Structure of the paper

6 Example on a finite probability space with

$$\rho(\mathbf{X}) \; : = \inf \left\{ \sum_{i=1}^{N} Y_i \,|\mathbf{Y} = (Y_1, \cdots, Y_N) \in \mathcal{C}^h, \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}_\gamma \right\}$$

$$\mathbb{A}_\gamma \; : = \left\{ Z \in \mathcal{L}^0(\mathbb{R}) | \mathbb{E}[Z] \geq -\gamma \right\}, \; \gamma \in \mathbb{R}_+$$

$$\Lambda(\mathbf{x}) \; : = \sum_{i=1}^{N} -\exp\left(-\alpha_i x_i\right), \; \alpha_i \in \mathbb{R}_+$$

- The computation of the systemic risk measure reduces to solving a finite-dimensional system of equations.
- Explicit formula for the systemic risk measure and for the (unique) optimal allocation.

Definition

The *systemic risk measure* associated with $\mathcal{C}, \mathcal{A}^{\mathbf{Y}}$ and π is a map $\rho : \mathcal{L}^0(\mathbb{R}^N) \to \overline{\mathbb{R}}$, defined by:

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\},$$

Moreover ρ is called a quasi-convex (resp. convex) systemic risk measure if it is $\succeq$-monotone decreasing and quasi-convex (resp. convex on $\{\rho(\mathbf{X}) < +\infty\}$).

1: Sufficient conditions

1. For all $\mathbf{Y} \in \mathcal{C}$ the set $\mathcal{A}^{\mathbf{Y}} \subset \mathcal{L}^0(\mathbb{R}^N)$ is $\succeq$-monotone.
2. For all $m \in \mathbb{R}$, for all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$ such that $\pi(\mathbf{Y}_1) \leq m$ and $\pi(\mathbf{Y}_2) \leq m$ and for all $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and all $\lambda \in [0,1]$ there exists $\mathbf{Y} \in \mathcal{C}$ such that $\pi(\mathbf{Y}) \leq m$ and $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.
3. For all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$ and all $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and all $\lambda \in [0,1]$ there exists $\mathbf{Y} \in \mathcal{C}$ such that $\pi(\mathbf{Y}) \leq \lambda \pi(\mathbf{Y_1}) + (1-\lambda)\pi(\mathbf{Y_2})$ and $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.

Fact

.

a. *If the systemic risk measure ρ satisfies the properties 1 and 2, then ρ is $\succeq$-monotone decreasing and quasi-convex.*

b. *If the systemic risk measure ρ satisfies the properties 1 and 3, then ρ is $\succeq$-monotone decreasing and convex on $\{\rho(\mathbf{X}) < +\infty\}$.*

1: Alternative sufficient conditions

1. For all $\mathbf{Y} \in \mathcal{C}$ the set $\mathcal{A}^{\mathbf{Y}} \subset \mathcal{L}^0(\mathbb{R}^N)$ is $\succeq$-monotone.

2a For all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0,1]$ there exists $\alpha \in [0,1]$ such that $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\alpha \mathbf{Y}_1 + (1-\alpha)\mathbf{Y}_2}$

3a For all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0,1]$ it holds: $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\lambda \mathbf{Y}_1 + (1-\lambda)\mathbf{Y}_2}$.

4. $\mathcal{C}$ is convex,

5. π is quasi-convex,

6. π is convex.

Fact

.

a *Under the conditions 1, 2a, 4, and 5 the map ρ is a quasi-convex SRM.*

b *Under the conditions 1, 3a, 4 and 6, the map ρ is a convex SRM.*

2: SRM via general aggregation functions and acceptance set

The following three assumptions hold true in the next propositions.

1. The aggregation functions are defined by:

$$\begin{aligned} \Lambda &: \mathcal{L}^0(\mathbb{R}^N) \times \mathcal{C} \rightarrow \mathcal{L}^0(\mathbb{R}), \\ \Lambda_1 &: \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathcal{L}^0(\mathbb{R}), \end{aligned}$$

2. The acceptance family

$$(\mathbb{A}^m)_{m \in \mathbb{R}},$$

   of monotone convex subsets $\mathbb{A}^m \subseteq \mathcal{L}^0(\mathbb{R})$, is supposed an increasing family w.r.to $m \in \mathbb{R}$

3. The monotone and convex acceptance subset is:

$$\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R}).$$

2: Family of SRM

Proposition

If $\mathcal{C}$ is convex, if π is quasi-convex, Λ is concave and $\Lambda(\cdot, \mathbf{Y})$ is $\succeq$-increasing for all $\mathbf{Y} \in \mathcal{C}$ then the map

$$\rho(\mathbf{X}) \triangleq \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X}, \mathbf{Y}) \in \mathbb{A}\}$$

is a quasi-convex SRM; if in addition π is convex then ρ is a convex SRM.

- Notice that such a risk measure may describe both cases:
    - *first aggregate and second add the capital*: for example if $\Lambda(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X}) + \Lambda_2(\mathbf{Y})$, where $\Lambda_2(\mathbf{Y})$ could be interpreted as the discounted cost of $\mathbf{Y}$)
    - *first add and second aggregate:* for example if $\Lambda(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X} + \mathbf{Y})$.

2: Family of SRM

Proposition

If $\mathcal{C}$ is convex, if π is quasi-convex, if Λ_1 is $\succeq$-increasing and concave, if $\theta : \mathcal{C} \to \mathbb{R}$ then

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\ \Lambda_1(\mathbf{X}) \in \mathbb{A}^{\theta(\mathbf{Y})}\}$$

is a (truly) quasi-convex SRM.

- This SRM represents the generalization of the quasi-convex classical one dimensional risk measure

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X \in \mathbb{A}^m\},$$

.

2: Family of SRM

Proposition

Suppose:

1. $\mathbf{0} \in \mathcal{C} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ *is a convex set,* $\mathcal{C} + \mathbb{R}^N_+ \in \mathcal{C}$
2. π *satisfies* $\pi(u) = 1$, *for a fixed* $u \in \mathbb{R}^N_+$ *and*

$$\pi(\alpha_1 Y_1 + \alpha_2 Y_2) = \alpha_1 \pi(Y_1) + \alpha_2 \pi(Y_2)$$

   for all $\alpha_i \in \mathbb{R}_+$ *and* $Y_i \in \mathcal{C}$
3. Λ *is concave and* $\Lambda(\mathbf{X}, \cdot) : \mathcal{C} \to \mathcal{L}^0(\mathbb{R})$ *is increasing for all* $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$.

Then the map

$$\rho(\mathbf{X}) = \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X}, \mathbf{Y}) \in \mathbb{A}^{\pi(\mathbf{Y})}\}$$

is a quasi-convex SRM.

We study the particular case of **random cash allocations,** i.e. when:

$$\mathcal{C} \subseteq \{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\} =: \mathcal{C}_{\mathbb{R}}.$$

3: Random cash allocations

$$\mathcal{C} \subseteq \{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\} =: \mathcal{C}_{\mathbb{R}} \text{ and } \mathcal{C} + \mathbb{R}^N_+ \in \mathcal{C}.$$

$$\rho(\mathbf{X}) := \inf\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} + \mathbf{Y} \in \mathcal{A}^{\theta(\sum Y^n)}\},$$

- If $\theta : \mathbb{R} \to \mathbb{R}$ is increasing then ρ is a (truly) quasi-convex SRM.
- The criteria whether a system is safe or not after injecting a capital vector $\mathbf{Y}$ is given by the (N-dimensional) acceptance set $\mathcal{A}^{\theta(\sum Y^n)}$ which itself depends on the total cash amount $\sum_{n=1}^{N} Y^n$.
- Model an **increasing level of prudence** when defining safe systems for higher amounts of required total capital.

3: A particular case

In the case $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ then every systemic risk measure of this type can be written as a univariate quasi-convex risk measure applied to the sum of the risk factors:
If $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ then

$$\rho(\mathbf{X}) := \inf\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} + \mathbf{Y} \in \mathcal{A}^{\theta(\sum Y^n)}\},$$

is of the form

$$\rho(\mathbf{X}) = \widetilde{\rho}(\sum_{n=1}^{N} X^n)$$

for some quasi-convex risk measure

$$\widetilde{\rho} : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}}.$$

3: Example: Worst case type of risk measures

We compare: injecting capital before or after aggregation

Fix $\mathbb{A} := \mathcal{L}^0_+(\mathbb{R})$,

$$\Lambda(\mathbf{X}) \quad : \quad = \sum_{i=1}^{N} -(X_i)^-,$$

$$\pi(\mathbf{Y}) \quad : \quad = \sum_{i=1}^{n} Y_i$$

Deterministic or random allocations:

$$\mathcal{C} \quad = \quad \mathbb{R}^N$$

$$\mathcal{C}_\gamma \quad : \quad = \{\mathbf{Y} \in \mathcal{C}_\mathbb{R} \mid Y_i \geq \gamma_i\}, \quad \gamma_i \in [-\infty, 0], \ i = 1, ...N$$

(for $\gamma := (-\infty, ..., -\infty)$ this family of subsets includes $\mathcal{C}_\infty = \mathcal{C}_\mathbb{R}$).

3: Continuing the example

We compare: injecting capital before or after aggregation

- We compare: **first aggregate and second injecting capital**:

$$\rho^{ag}(\mathbf{X}) := \inf\{y \in \mathbb{R} \mid \Lambda(\mathbf{X}) + y \in \mathbb{A}\} = \rho_{Worst}(\sum_{i=1}^{N} -(X_i)^-),$$

  and **first injecting capital and second aggregate**
- for both the **deterministic** cash allocations:

$$\rho^{\mathbb{R}^N}(\mathbf{X}) := \inf\left\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathbb{R}^N, \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}\right\},$$

  and the **random** cash allocations:

$$\rho^{\gamma}(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}_{\gamma}, \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}\}.$$

3: Continuing the example

$\rho^{ag}(\mathbf{X})$ first aggregate second add capital
$\rho^{\mathbb{R}^N}(\mathbf{X})$ first add second aggregate: deterministic cash
$\rho^{\gamma}(\mathbf{X})$ first add second aggregate: random cash

Then one obtains:

$$\rho^{\mathbb{R}^N}(\mathbf{X}) = \sum_{i=1}^{N} \rho_{Worst}(X_i) \geq \rho^{ag}(\mathbf{X})$$

$$\rho^{\gamma}(\mathbf{X}) = \rho_{Worst}\left(\sum_{i=1}^{N}(X_i \mathbb{I}_{X_i \leq -\gamma_i} - \gamma_i \mathbb{I}_{X_i \geq -\gamma_i})\right) \leq \rho^{ag}(\mathbf{X}),$$

If $\gamma = \overline{0} := (0, ..., 0)$ then $\rho^{\overline{0}}(\mathbf{X}) = \rho^{ag}(\mathbf{X})$,
If $\gamma = -\overline{\infty} := (-\infty, ..., -\infty)$ then $\rho^{-\overline{\infty}}(\mathbf{X}) = \rho_{Worst}(\sum_{i=1}^{N} X_i)$.

- The interplay between $\mathbb{A}$ and Λ is critical.

3: Example: Expected Shortfall

Same example but with different acceptance set

$$\mathbb{A}^{ES} := \{X \in \mathcal{L}^0(\mathbb{R}) \mid \rho_{ES}(X) \leq 0\}$$

Everything else is as in the previous Example. Then

$$\rho^{ag}(\mathbf{X}) = \rho_{ES}(\sum_{i=1}^{N} -(X_i)^-).$$

For $\rho^{\mathbb{R}^N}$ and ρ^{γ}, however, $\mathbb{A}^{ES}$ gives the same result as $\mathbb{A}^{Worst}$, i.e.

$$\rho^{\mathbb{R}^N}(\mathbf{X}) = \sum_{i=1}^{N} \rho_{Worst}(X_i) \geq \rho^{ag}(\mathbf{X})$$

$$\rho^{\gamma}(\mathbf{X}) = \rho_{Worst}(\sum_{i=1}^{N}(X_i \mathbb{I}_{X_i \leq -\gamma_i} - \gamma_i \mathbb{I}_{X_i \geq -\gamma_i}))\,.$$

4: Gaussian financial system

We now illustrate the application to the Gaussian financial system

4: Gaussian financial system

Consider a Gaussian financial system, i.e. $\mathbf{X} = (X_1, \cdots, X_N)$ is an N-dimensional vector with $\mathbf{X} \sim N(\mu, Q)$, $[Q]_{ii} := \sigma_i^2$, and $[Q]_{ij} := \rho_{i,j}$ for $i \neq j$, and mean vector $\mu := (\mu_1, \cdots, \mu_N)$.

$$\begin{aligned} \rho(\mathbf{X}) \quad &: \quad = \inf \left\{ \sum_{i=1}^{N} Y_i \mid \mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}},\ \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}_\gamma \right\}, \\ \Lambda(\mathbf{X}) \quad &: \quad = \sum_{i=1}^{N} -(X_i - d_i)^-,\ d_i \in \mathbb{R}, \\ \mathbb{A}_\gamma \quad &: \quad = \left\{ Z \in \mathcal{L}^0(\mathbb{R}) | E[Z] \geq -\gamma \right\},\ \gamma \in \mathbb{R}_+. \end{aligned}$$

d_i in the aggregation rule denotes some critical liquidity level of institution i and the risk measure is concerned with the expected total shortfall below these levels in the system

- First, we consider the case: $\mathcal{C} := \mathbb{R}^{\mathbb{N}}$
- Second, we consider a random allocation
- Third we compare the above two cases.

Example: Gaussian case with deterministic allocation

Let $\mathcal{C} = \mathbb{R}^{\mathbb{N}}$ and

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^{N} m_i \mid \mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{R}^{\mathbb{N}}, \ \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}_{\gamma} \right\},$$

By solving the Lagrangian system we obtain that the global minimum point $\mathbf{m}^* = (m_1^*, \cdots, m_N^*)$ is given by

$$m_i^* = d_i - \mu_i - \sigma_i R,$$

where and R solves the equation (for $\Phi(x) := \int_{+\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$):

$$R\Phi(R) + \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{R^2}{2}\right] = \frac{\gamma}{\sum_{i=1}^{N} \sigma_i}.$$

The **unique optimal cash allocation $\mathbf{m}^*$ induces a ranking** of the institutions according to systemic riskiness.

1. $\frac{\partial m_i}{\partial \mu_i} = -1$: the systemic riskiness decreases with increasing mean.
2. $\frac{\partial m_i}{\partial \sigma_i} > 0$: the systemic riskiness increases with increasing volatility.

4: Example: the Gaussian system with random allocation

We allow for different allocations of the total capital depending ω.
Set $D := \{\sum_{i=1}^{n} X_i \leq d\}$, $d \in \mathbb{R}$, and

$$\mathcal{C} := \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^n) \mid \mathbf{Y} = \mathbf{m} + \boldsymbol{\alpha} I_D, \, \mathbf{m}, \boldsymbol{\alpha} \in \mathbb{R}^N, \sum_{i=1}^{N} \alpha_i = 0 \right\} \subseteq \mathcal{C}_{\mathbb{R}},$$

The condition $\sum_{i=1}^{n} \alpha_i = 0$ implies that $\sum_{i=1}^{n} Y_i$ is constant a.s.

- Flexibility: **we let the allocation depend on whether the system at time T is in trouble or not**, represented by the events that $\sum_{i=1}^{n} X_i$ is less or greater than some critical level d, respectively.

The SRM is then:

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^{N} m_i \mid \mathbf{m} + \boldsymbol{\alpha} I_D \in \mathcal{C} \, , \, \Lambda(\mathbf{X} + \mathbf{m} + \boldsymbol{\alpha} I_D) \in \mathbb{A}_\gamma \right\}.$$

4: Solving by Lagrange method

We minimize the objective function $\sum_{i=1}^{N} m_i$ over $(\mathbf{m}, \boldsymbol{\alpha}) \in \mathbb{R}^{2N}$ under the constrains

$$\sum_{i=1}^{N} \alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^{N} E[(X_i + m_i + \alpha_i I_A - d_i)^-] = \gamma .$$

We apply the method of Lagrange multipliers to minimize the function

$$\phi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}, \lambda) = \sum_{i=1}^{N} m_i + \lambda \left(\Psi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}) - \gamma \right),$$

where

$$\Psi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}) := \sum_{i=1}^{N-1} E[(X_i + m_i + \alpha_i I_D - d_i)^-] + E[(X_N + m_N - \sum_{j=1}^{N-1} \alpha_j I_D - d_N)^-].$$

Obtaining explicit formulas for the derivatives.

$$\rho(\mathbf{X}) = \inf\{m_1 + m_2 \mid \mathbf{m} + \boldsymbol{\alpha} I_D \in \mathcal{C}\,,\ \Lambda(\mathbf{X} + \mathbf{m} + \boldsymbol{\alpha} I_D) \in \mathbb{A}_\gamma\}$$

$$\mathbb{A}_\gamma = \left\{Z \in \mathcal{L}^0(\mathbb{R}) \mid E[Z] \geq -\gamma\right\} \qquad D := \{X_1 + X_2 \leq d\}$$

Means of (X_1, X_2): $\mu_1 = \mu_2 = 0$
Standard deviations of (X_1, X_2): $\sigma_1 = 1$, $\sigma_2 = 3$,
The acceptance level $\gamma = 0.7$
The critical level $d = 2$.

$$\rho(\mathbf{X}) = \inf\{m_1 + m_2 \mid \mathbf{m} + \boldsymbol{\alpha} I_D \in \mathcal{C}, \Lambda(\mathbf{X} + \mathbf{m} + \boldsymbol{\alpha} I_D) \in \mathbb{A}_\gamma\}$$
$$\mathbb{A}_\gamma = \{Z \in \mathcal{L}^0(\mathbb{R}) \mid E[Z] \geq -\gamma\} \qquad D := \{X_1 + X_2 \leq d\}$$

Means of (X_1, X_2): $\mu_1 = \mu_2 = 0$
Standard deviations of (X_1, X_2): $\sigma_1 = 1$, $\sigma_2 = 3$,
The acceptance level $\gamma = 0.7$
The critical level $d = 2$.
We compare the sensitivities with respect to the correlation $\rho_{1,2}$ for

- deterministic allocation ($\alpha = 0$)
- random allocation

For highly **positively correlated** banks the random allocation does not change the total capital requirement $(\mathbf{m}_1+\mathbf{m}_2)$.
However, when they are **negatively correlated**, one benefits from random allocation since the total allocation $(\mathbf{m}_1+\mathbf{m}_2)$ is lower.

$\rho_{1,2} \downarrow$		Deterministic	Random
-0.9	m_1	0.5766	0.1151
	m_2	1.7333	1.6614
	$\rho = m_1 + m_2$	2.3099	1.7765
-0.5	m_1	0.5766	0.2908
	m_2	1.7333	1.7776
	$\rho = m_1 + m_2$	2.3099	2.0683
0	m_1	0.5766	0.4490
	m_2	1.7333	1.7796
	$\rho = m_1 + m_2$	2.3099	2.2286
0.5	m_1	0.5766	0.5463
	m_2	1.7333	1.7461
	$\rho = m_1 + m_2$	2.3099	2.2924
0.9	m_1	0.5766	0.5780
	m_2	1.7333	1.7310
	$\rho = m_1 + m_2$	2.3099	2.3090

Table: Sensitivity with respect to correlation.

4: Numerical illustration with two banks (X1,X2)

Sensitivity with respect to standard deviation

$$\rho(\mathbf{X}) \;:\; = \inf\left\{m_1 + m_2 \mid \mathbf{m} + \boldsymbol{\alpha} I_D \in \mathcal{C}\,,\, \Lambda(\mathbf{X} + \mathbf{m} + \boldsymbol{\alpha} I_D) \in \mathbb{A}_\gamma\right\}$$

$$\mathbb{A}_\gamma \;:\; = \left\{Z \in \mathcal{L}^0(\mathbb{R}) \mid E[Z] \geq -\gamma\right\} \qquad D := \{X_1 + X_2 \leq d\}$$

Means of (X_1, X_2): $\mu_1 = \mu_2 = 0$
Standard deviation of X_1 is $\sigma_1 = 1$; Correlation $\rho_{1,2} = -0.5$
The acceptance level $\gamma = 0.7$; The critical level $d = 2$.

$$\rho(\mathbf{X}) \quad : \quad = \inf \{ m_1 + m_2 \mid \mathbf{m} + \boldsymbol{\alpha} I_D \in \mathcal{C} \, , \, \Lambda(\mathbf{X} + \mathbf{m} + \boldsymbol{\alpha} I_D) \in \mathbb{A}_\gamma \}$$
$$\mathbb{A}_\gamma \quad : \quad = \{ Z \in \mathcal{L}^0(\mathbb{R}) | \; E[Z] \geq -\gamma \} \qquad D := \{ X_1 + X_2 \leq d \}$$

Means of (X_1, X_2): $\mu_1 = \mu_2 = 0$
Standard deviation of X_1 is $\sigma_1 = 1$; Correlation $\rho_{1,2} = -0.5$
The acceptance level $\gamma = 0.7$; The critical level $d = 2$.
We compare the sensitivities with respect to σ_2 for

- deterministic allocation ($\alpha = 0$)
- random allocation

For equal marginals ($\sigma_1 = \sigma_2 = 1$) random allocation does not change the total capital requirement.
As σ_2 increases, the systemic risk measure increases and the allocation increases, in agreement with the deterministic case.
Random allocation allows for smaller total capital ($\mathbf{m}_1 + \mathbf{m}_2$).

$\sigma_2 \downarrow$		Deterministic	Random
1	m_1	0.1008	0.1008
	m_2	0.1031	0.1031
	α	0	0.0002
	$\rho = m_1 + m_2$	0.2039	0.2039
5	m_1	0.8168	0.3167
	m_2	4.0816	4.1295
	α	0	3.5987
	$\rho = m_1 + m_2$	4.8984	4.4462
10	m_1	1.1417	0.4631
	m_2	11.3964	11.4333
	α	0	6.9909
	$\rho = m_1 + m_2$	12.5381	11.8963

Table: Sensitivity with respect to standard deviation.

5: Borrowing and lending: Carmona Fouque Sun 2015

Let the Gaussian financial vector $\mathbf{X}_t$ be generated by the following dynamics

$$dX_t^i = \left[\sum_{j=1}^{N} p_{i,j} (X_t^j - X_t^i) \right] dt + \sigma^i \left(\rho_i dW_t^0 + \sqrt{1 - \rho_i^2} dW_t^i \right),$$

The lending-borrowing preferences $p_{i,j}$ are nonnegative and symmetric: $p_{i,j} = p_{j,i}$. $\left(W_t^0, W_t^i, i = 1, \cdots, N \right)$ are independent standard BM and W_t^0 is a common noise.

- The joint distribution of $(X_t^1, ..., X_t^N)$ will be fully characterized by the means μ_i's and by the covariance matrix Q which will depend on the parameters of the model, in particular the preferences $p_{i,j}$ (lending/borrowing) and the individual σ^i.
- Applying the approach just described (in 4), for given parameters, **one computes the marginal means and variances needed in our systemic risk measures, in order to obtain the optimal allocation m** $= (m_i)_{i=1,\cdots,N}$ and a **ranking** of the banks.

5: Two types of networks

$$dX_t^i = \left[\sum_{j=1}^{N} p_{i,j} (X_t^j - X_t^i) \right] dt + \sigma^i \left(\rho_i dW_t^0 + \sqrt{1 - \rho_i^2} dW_t^i \right),$$

We analyze:

1. A fully connected symmetric network:
   $x_0^i = x_0$, $p_{i,j} = p/N$, $\sigma^i = \sigma$, $\rho_i = \rho$
2. A central clearing symmetric network, where bank 1 plays a clearing role and is related to each of the other banks which are not directly related to each other.
   $p_{i,j} = p$ if $i = 1$ or $j = 1$, and $p_{i,j} = 0$ if $i \neq 1$ and $j \neq 1$;
   $\sigma^i = \sigma$ if $i \neq 1$; $\rho_i = \rho$ if $i \neq 1$;
   $x_0^i = x_0$ if $i \neq 1$;

5: Marginals in the fully connected symmetric network

$$\mu_i = \mathbb{E}(X_t^i) = x_0,$$

$$\sigma_i^2 = \sigma^2(1-\rho^2)(1-\frac{1}{N})\left(\frac{1-e^{-2pt}}{2p}\right) + \sigma^2\left(\rho^2 + \frac{1-\rho^2}{N}\right)t.$$

- These marginal distributions depend on the coupled dynamics of the X^i, in particular on the parameters p and ρ.
- Increasing p, that is increasing liquidity, would decrease σ_i^2 (from $\sigma^2 t$ for $p = 0$ to $\sigma^2\left(\rho^2 + \frac{1-\rho^2}{N}\right)t$ for $p = \infty$), and therefore would decrease systemic risk.

5: Central clearing symmetric network

$$\mu_i = \mathbb{E}(X_t^i) = \mathbb{E}(X_t^1) = c_0/N.$$

$$\sigma_1^2 = \sigma^2\rho^2 t$$
$$+ \frac{1}{N}\left[(\sigma^2 + 2\sigma\sigma_c\rho\rho_c - 3\sigma^2\rho^2)t + \frac{2}{p}(\sigma_c^2 - \sigma\sigma_c\rho\rho_c)\right] + \mathcal{O}(\frac{1}{N^2}),$$

$$\sigma_i^2 = \sigma^2(1-\rho^2)\left(\frac{1-e^{-2pt}}{2p}\right) + \sigma^2\rho^2 t$$
$$+ \frac{1}{N}\left[(\sigma^2 + 2\sigma\sigma_c\rho\rho_c - 3\sigma^2\rho^2)t - \sigma^2(1-\rho^2)\left(\frac{1-e^{-2pt}}{2p}\right)\right] + \mathcal{O}(\frac{1}{N^2}),$$

At order one in $1/N$, this variance is the same as the homogeneous model, but they may differ at order $1/N$.
Writing $\sigma^2 + 2\sigma\sigma_c\rho\rho_c - 3\sigma^2\rho^2 = \sigma^2(1-\rho^2) + 2\sigma\rho(\sigma_c\rho_c - \sigma\rho)$, we see **that the sign of $\sigma_c\rho_c - \sigma\rho$ determines which network is most stable (smaller variance)**.

- We assume a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$
  $\Omega = \{\omega_j\}$, $j = 1, \cdots, M$
- Then the computation of optimal cash allocations associated to the systemic risk measure reduces to solving a finite-dimensional system of equations, even for most general random cash allocations, where the allocation can be adjusted scenario by scenario.
- Explicit formula for the systemic risk measure and for the (unique) optimal allocation.

6: Example on a finite probability space

The Systemic Risk Measure is defined by:

$$\rho(\mathbf{X}) \quad : \quad = \inf\left\{\sum_{i=1}^{N} Y_i \mid \mathbf{Y} = (Y_1, \cdots, Y_N) \in \mathcal{C}^h, \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}_\gamma\right\}$$

$$\mathbb{A}_\gamma \quad : \quad = \left\{Z \in \mathcal{L}^0(\mathbb{R}) | \mathbb{E}[Z] \geq -\gamma\right\}, \gamma \in \mathbb{R}_+$$

$$\Lambda(\mathbf{x}) \quad : \quad = \sum_{i=1}^{N} -\exp\left(-\alpha_i x_i\right), \alpha_i \in \mathbb{R}_+$$

6: Random allocations in subgroups

The permitted allocations are defined by:

$$\mathcal{C}^h = \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{i=1}^{h_1} Y_i(\omega_j) = d_1, \sum_{i=h_1+1}^{h_2} Y_i(\omega_j) = d_2, \cdots , \sum_{i=h_{k-1}+1}^{N} Y_i(\omega_j) = d_k \right\} \subseteq \mathcal{C}_{\mathbb{R}}, \quad d_1, ..., d_k \in \mathbb{R}, \ \forall j.$$

- The **regulator is constrained to distribute the capital only within** k **subgroups** that are induced by the partition $h = (h_1, .., h_k)$ with $0 < h_1 < h_2 < \cdots < h_{k-1} < h_k = N$
- **The risk measure is the sum of** k **minimal cash funds** $d_1, ..., d_k$ **determined today**, **that at time** T **can be freely allocated within the** k **subgroups**
- Note that this family includes as two extreme cases
  - deterministic allocations $\mathcal{C} = \mathbb{R}^N$ for $k = N$
  - completely free allocations $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ for $k = 1$.

6: Applying Lagrange method

For a specific subfamily of $\mathcal{C}^h$ we obtain the **explicit formula for** ρ:

$$\rho(\mathbf{X}) = -\beta_{N-r} \log(\frac{\gamma}{\alpha_1 \beta_N d_{N-r}}) - \sum_{j=N-r+1}^{N} \frac{1}{\alpha_j} \log(\frac{\gamma}{\alpha_j \beta_N K_j}),$$

where:

$$\beta_{N-r} = \sum_{i=1}^{N-r} \frac{1}{\alpha_i}$$

$$\beta_N = \sum_{i=1}^{N} \frac{1}{\alpha_i}$$

$$d_{N-r} = \sum_{j=1}^{M} p_j \exp\left[-\frac{1}{\beta_{N-r}} \sum_{i=1}^{N-r} X_i(w_j) - \frac{1}{\beta_{N-r}} \sum_{i=1}^{N-r} \frac{1}{\alpha_i} \log(\frac{\alpha_1}{\alpha_i}) \right].$$

6: Applying Lagrange method

and the **optimal allocations** are given by:

$$y_1^j = \frac{1}{\alpha_1 \beta_{N-r}} \sum_{i=1}^{N-r} X_i(w_j) - X_1(w_j) + \frac{1}{\alpha_1 \beta_{N-r}} \sum_{i=1}^{N-r} \frac{1}{\alpha_i} \log(\frac{\alpha_1}{\alpha_i}) + \frac{1}{\alpha_1 \beta_{N-r}} c_{N-r}$$

for $j = 1, \cdots, M$; by

$$\begin{aligned} y_k^j &= \frac{1}{\alpha_k} \left[\alpha_1 X_1(w_j) - \alpha_k X_k(w_j) - \log(\frac{\alpha_1}{\alpha_k}) + \alpha_1 y_1^j \right] \\ &= \frac{1}{\alpha_k \beta_{N-r}} \sum_{i=1}^{N-r} X_i(w_j) - X_k(w_j) - \frac{1}{\alpha_k} \log(\frac{\alpha_1}{\alpha_k}) \\ &+ \frac{1}{\alpha_k \beta_{N-r}} \sum_{i=1}^{N-r} \frac{1}{\alpha_i} \log(\frac{\alpha_1}{\alpha_i}) + \frac{1}{\alpha_k \beta_{N-r}} c_{N-r} \end{aligned}$$

for all $k = 2, \cdots, N-r-1$ and $j \in 1, \cdots, M$; and by:

$$y_k^j = c_k - c_{k-1} = -\frac{1}{\alpha_k} \log(\frac{\gamma}{\alpha_k \beta_N K_k})$$

for all $k = N-r, \cdots, N$ and $j \in 1, \cdots, M$.

Thank you for your attention